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A CLASS OF EVOLUTIONARY OPERATORS AND ITS APPLICATIONS TO ELECTROSEISMIC WAVES IN ANISOTROPIC, INHOMOGENEOUS MEDIA

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Abstract. In the framework of a comprehensive theory for a new class of evolutionary problems wellposedness of associated initial boundary value problems is considered. The dynamic linear model for electroseismic waves in anisotropic, inhomogeneous, time-shift invariant media is used as an illustration of the theory.

1. Introduction

In [4, 6] it is shown that a canonical form for a large class of evolutionary problems from mathematical physics is

$$\begin{aligned} \partial_0 V + AU &= g \text{ on } \mathbb{R}_{>0}, \\ V(0+) &= \Phi, \end{aligned}$$

where ∂_0 denotes the time-derivative and A is skew-selfadjoint[1] in a suitable Hilbert space setting. Here however we shall prefer to consider problems on the whole real time-line and we shall assume – without loss of generality – that $\Phi = 0$. This results in the problem

$$\partial_0 V + AU = g \text{ on } \mathbb{R}. \tag{1.1}$$

This evolutionary problem is now completed by a "material law", which we assume to be time-translation-invariant and of the form

$$V = M\left(\partial_0^{-1}\right) U, \tag{1.2}$$

where $z \mapsto M(z)$ is uniformly bounded as a mapping from $\mathbb{C}$ into the Banach space of continuous linear mappings and analytic in an open ball $B_{\mathbb{C}}(r,r)$ in $\mathbb{C}$ centered at $r \in \mathbb{R}_{>0}$ with radius r. The interpretation of $M\left(\partial_0^{-1}\right)$ in the sense of a functional calculus rests on establishing ∂_0 as a normal operator. The operator class in question has

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[1] Such skew-selfadjoint operators A occur in applications frequently in the simple block operator matrix form (Hamiltonian form)

$$A = \begin{pmatrix} 0 & C \\ -C^* & 0 \end{pmatrix},$$

where C is a closed, densely defined, linear operator, see e.g. [5].

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been demonstrated to cover all the standard initial boundary value problems of mathematical physics and has proven to be useful for describing coupling between different physical effects. While the applications previously considered are of considerable complexity, we consider here a particular model for which the suitability of our operator class is far from obvious. The model describes the interaction of elastic, acoustic and electromagnetic effects in seismic activities.

Based on Frenkel's pioneering work from 1944, republished as [1], Pride [9] developed a system of equations, which can be considered to be the standard model for the description of time-harmonic electroseismic phenomena in isotropic, homogeneous media.

Later this was generalized to the anisotropic case by Pride and Haartsen [8]. Transcribing this system into the underlying time-dependent system with ∂_0 denoting the derivative with respect to time, we obtain:

$$
\begin{aligned}
\rho\,\partial_0^2 u + \rho_f\,\partial_0^2 w - \operatorname{Div} T &= F \\
B &= \mu H \\
D &= \varepsilon E \\
\partial_0 w &= L\left(\partial_0^{-1}\right) E + \frac{k\left(\partial_0^{-1}\right)}{\eta}\left(-\operatorname{grad} P - \rho_f^* \partial_0^2 u + f\right) \\
J &= \sigma_0\left(\partial_0^{-1}\right) E + L\left(\partial_0^{-1}\right)\left(-\operatorname{grad} P - \rho_f^* \partial_0^2 u + f\right) \\
P + W \operatorname{Grad} u + M \operatorname{div} w &= 0 \\
\partial_0 B + \operatorname{curl} E &= 0 \\
\partial_0 D - \operatorname{curl} H &= -J - J_0 \\
T &= \mathscr{C}\,\operatorname{Grad} u + W^*\,\operatorname{div} w
\end{aligned}
$$

Here grad, div, curl are the familiar vector analytic operations. In contrast Grad denotes the symmetric part of the derivative of vector fields and Div denotes the divergence operator applied to symmetric-matrix-valued functions in three space dimensions:

$$
\operatorname{Grad}\left(u_k\right)_{k=1,2,3} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)_{i,j=1,2,3},
$$

$$
\operatorname{Div}\left(T_{ij}\right)_{i,j=1,2,3} = \left(\sum_{j=1}^{3} \partial_j T_{ij}\right)_{i=1,2,3}.
$$

Other notations are standard, see [9, 8]. The aim of this paper is to establish from this formal set of equations a well-posed evolutionary problem describing the propagation of electromagnetic waves associated with seismic waves. This system however, obtained via a number of heuristic variational arguments, is not of a form suitable for understanding the wave nature of the phenomena being modelled.

In the section 2 we shall formally transform the above system into an appropriate evolutionary form and only in section 3 we shall endeavour to establish this formal system as a rigorous evolutionary problem in a suitable Hilbert space framework. In the next section, however, we describe the operator class in which we will be able to embed this standard model of electroseismic waves.

2. A Class of Evolutionary Operators and its Solution Theory

As in [6] we recall from [3], section 1.2, that by introducing an exponential weight function $t \mapsto \exp(-\nu t)$, $\nu \in \mathbb{R}$, we may construct a weighted L^2-space $H_{\nu,0}$ generated by completion of $\mathring{C}_\infty(\mathbb{R})$ with respect to the inner product

$$\langle \cdot | \cdot \rangle_{\nu,0} : (\varphi, \psi) \mapsto \int_{\mathbb{R}} \varphi(t)^* \psi(t) \exp(-2\nu t) \, dt.$$

The associated norm will be denoted by $|\cdot|_{\nu,0}$. The time-derivative operator ∂_0 has a realization as a normal linear operator in $H_{\nu,0}$. The composition $\mathscr{L}_\nu := \mathscr{F} \exp(-\nu m)$ of the Fourier transform $\mathscr{F}$ with the multiplication operator $\exp(-\nu m)$ given by

$$(\exp(-\nu m)\varphi)(x) = \exp(-\nu x)\varphi(x), \; \nu, x \in \mathbb{R},$$

is a spectral representation for the imaginary part of ∂_0. We shall refer to the unitary mapping $\mathscr{L}_\nu : H_{\nu,0} \to H_{0,0}$ as the Fourier-Laplace transform, where $\nu \in \mathbb{R}$ is a parameter, which for our purposes is always to be chosen as positive.

We shall use again the notation ∂_0 for the canonical extension of the time derivative to Hilbert-space-valued generalized functions. This is done in a standard manner by extending the operators ∂_0 to the tensor product spaces $H_{\nu,0} \otimes H$, where H is some other Hilbert space, by interpreting ∂_0 henceforth as the operator $\partial_0 \otimes 1_H$ with $1_H : H \to H$ being the identity operator in H. Moreover, we need to extend the Fourier-Laplace transform to $H_{\nu,0} \otimes H$. We shall re-use the notation $\mathscr{L}_\nu$ and the name Fourier-Laplace transform for the unique unitary extension of

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}) \underset{a}{\otimes} H \subseteq H_{\nu,0} \otimes H &\to H_{0,0} \otimes H \\ \varphi \otimes w &\mapsto (\mathscr{L}_\nu \varphi) \otimes w \end{aligned}$$

to $H_{\nu,0} \otimes H$. With this extended Fourier-Laplace transform we will be able to describe the class of material laws.

Let $(M(z))_{z \in B_{\mathbb{C}}(r,r)}$ be a holomorphic family of uniformly bounded linear operators on H. We define, for $\nu > \frac{1}{2r}$,

$$M\left(\partial_0^{-1}\right) := \mathscr{L}_\nu^* M\left(\frac{1}{\mathrm{i} m_0 + \nu}\right) \mathscr{L}_\nu,$$

where $m_0 := m \otimes 1_H$. Note that for $r \in \mathbb{R}_{>0}$

$$\begin{aligned} B_{\mathbb{C}}(r,r) &\to \mathrm{i}\mathbb{R} + \mathbb{R}_{>1/(2r)} \\ z &\mapsto z^{-1} \end{aligned}$$

is a bijection.

As with ∂_0, we also need to extend the operator A to the tensor product spaces $H_{\nu,0} \otimes H$ by interpreting A as $1_{H_{\nu,0}} \otimes A$ where $1_{H_{\nu,0}} : H_{\nu,0} \to H_{\nu,0}$ is the identity operator in $H_{\nu,0}$.

It is in the space $H_{\nu,0} \otimes H$ that we pose the problem (1.1) - (1.2). Our aim is to be able to find $U, V \in H_{\nu,0} \otimes H$ with $\nu \geqslant \frac{1}{2r}$ such that for a given $g \in H_{\nu,0} \otimes H$ we have

$$\partial_0 V + AU = g \tag{2.1}$$

and

$$V = M\left(\partial_0^{-1}\right) U, \tag{2.2}$$

where $(M(z))_{z \in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H.

The solution theory for this class of evolutionary problems has been fully developed in [6]. We recall the following main result from there.

THEOREM 1. (Solution Theory) *Let* $(M(z))_{z \in B_{\mathbb{C}}(r,r)}$ *be a holomorphic family of uniformly bounded linear operators on a Hilbert space* H*, satisfying*

$$\bigvee_{c \in \mathbb{R}_{>0}} \bigwedge_{z \in B_{\mathbb{C}}(r,r)} \mathfrak{Re}\left(z^{-1} M(z)\right) := \frac{1}{2}\left(z^{-1} M(z) + (z^*)^{-1} M(z)^*\right) \geqslant c. \tag{2.3}$$

and let A *be a skew-selfadjoint operator in* H*. Then, for* $\nu \geqslant \frac{1}{2r}$ *and every* $g \in H_{\nu,0} \otimes H$*, there exists a unique solution* $U \in H_{\nu,0} \otimes H$ *of the problem*

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right) U = g. \tag{2.4}$$

Moreover, the solution depends continuously on the data in $H_{\nu,0} \otimes H$ *and the solution operator* $\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)^{-1}$ *is causal in the sense that, for every* $a \in \mathbb{R}$*, if* g *vanishes as an* H*-valued function on* $\mathbb{R}_{<a}$ *then so does* $\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)^{-1} g$*.*

REMARK. As a matter of notational convenience, we have written here $\partial_0 M\left(\partial_0^{-1}\right) + A$ instead of $\overline{\partial_0 M\left(\partial_0^{-1}\right) + A}$, which strictly speaking is only rigorously justified in the Sobolev lattice sense.

The proof of the theorem rests on the simple fact that strictly positive definite operators have a continuous inverse, which is densely defined, if the kernel of the adjoint is trivial. The adjoint, however, is given by $\partial_0^* M\left(\partial_0^{-1}\right)^* - A$ and enjoys the same strict positive definiteness as the original operator.

Although the result presented here is expressed for the time-translation-invariant case, we would like to mention that more general situations, such as time-varying coefficients and nonlinear dependence, are accessible via perturbation techniques. This is, however, beyond the scope of this paper.

In the next section we will establish that the standard model for electroseismic phenomena is at least formally compatible with the problem class of theorem 1 and only in section 3 do we endeavour to obtain a realization of this formal system as a rigorous evolutionary problem in a suitable Hilbert space framework.

3. The Formal System Equations

To begin this process we first note that the initial set of equations can be rephrased in the following way by letting $R := \partial_0 w$, $v := \partial_0 u$ and $\mathscr{E} := \operatorname{Grad} u$:

$$\left.\begin{aligned}
\rho\,\partial_0 v + \rho_f\,\partial_0 R - \operatorname{Div} T &= F\\
\partial_0 w - R &= 0\\
\partial_0 u - v &= 0\\
P + W\,\mathscr{E} + M\operatorname{div} w &= 0\\
\eta\, k\left(\partial_0^{-1}\right)^{-1} R + \rho_f^{*}\partial_0 v + \operatorname{grad} P - \eta\, k\left(\partial_0^{-1}\right)^{-1} L\left(\partial_0^{-1}\right) E &= f\\
\partial_0 \mu H + \operatorname{curl} E &= 0\\
\partial_0 \varepsilon E + \sigma\left(\partial_0^{-1}\right) E + L\left(\partial_0^{-1}\right)\eta\, k\left(\partial_0^{-1}\right)^{-1} R - \operatorname{curl} H &= J_0\\
\mathscr{E} - \operatorname{Grad} u &= 0\\
T &= C\mathscr{E} - W^{*}M^{-1}P
\end{aligned}\right\} \tag{3.1}$$

Here $C := \mathscr{C} - W^{*}M^{-1}W$, $\sigma\left(\partial_0^{-1}\right) := \sigma_0\left(\partial_0^{-1}\right) - L\left(\partial_0^{-1}\right)\eta\, k\left(\partial_0^{-1}\right)^{-1} L\left(\partial_0^{-1}\right)$. Note that by letting

$$\begin{aligned}
W &= \omega_0 \operatorname{trace}\\
W^{*} &= \operatorname{trace}^{*}\omega_0
\end{aligned}$$

for ω_0 real and $\operatorname{trace} : \mathbb{C}^{3\times 3} \to \mathbb{C}$ denoting the matrix trace operation $(T_{ij})_{i,j=1,2,3} \mapsto \sum_{j=1}^{3} T_{jj}$, the system derived in [9] is recovered. The operator $\operatorname{trace}^{*} : \mathbb{C} \to \mathbb{C}^{3\times 3}$ is given by

$$\operatorname{trace}^{*}\omega_0 = \begin{pmatrix} \omega_0 & 0 & 0\\ 0 & \omega_0 & 0\\ 0 & 0 & \omega_0 \end{pmatrix}.$$

Next, we attempt to adjust for the proposed asymptotic behavior of

$$\begin{aligned}
1/k\left(\partial_0^{-1}\right) &= \frac{1}{k_0}\left(\left(1 + \partial_0 \frac{4}{m_0\omega_\tau}\right)^{1/2} + \frac{1}{\omega_\tau}\partial_0\right)\\
&= \partial_0 \frac{1}{k_0}\left(\left(\partial_0^{-2} + \partial_0^{-1}\frac{4}{m_0\omega_\tau}\right)^{1/2} + \frac{1}{\omega_\tau}\right) \sim \partial_0
\end{aligned}$$

and

$$L\left(\partial_0^{-1}\right) = L_0\left(1 + \frac{1}{\omega_\tau}\partial_0\right)^{-1/2} \tag{3.2}$$

due to [9] or rather the variants in [10], [7], compare also the transciption to the transient case in Lu and Hanyga [2] for the poro-elastic part. Here the physical parameters m_0, ω_τ, k_0 are non-zero and

$$L_0 \in \mathbb{R}_{\leqslant 0}.$$

Indeed, we shall not need the particular form of L and k, but shall only be concerned with their general form. It turns out that, with $\eta\, k\left(\partial_0^{-1}\right)^{-1} := \partial_0 \kappa_0 + \partial_0^{1/2}\kappa\left(\partial_0^{-1}\right)$ and $L\left(\partial_0^{-1}\right) := -\partial_0^{-1/2}\lambda_0 - \partial_0^{-1}\lambda_1\left(\partial_0^{-1}\right)$, we have (with the abbreviations m_0, m_+, m_-)

$$
\begin{aligned}
&\eta\, k\left(\partial_0^{-1}\right)^{-1} L\left(\partial_0^{-1}\right) \\
&\quad = -\left(\partial_0\, \kappa_0 + \partial_0^{1/2}\left(\kappa_1 + \partial_0^{-1/2}\kappa_2\left(\partial_0^{-1}\right)\right)\right)\left(\partial_0^{-1/2}\lambda_0 + \partial_0^{-1}\lambda_1\left(\partial_0^{-1}\right)\right) \\
&\quad = -\partial_0^{1/2}\kappa_0\lambda_0 - \left(\kappa_1 + \partial_0^{-1/2}\kappa_2\left(\partial_0^{-1}\right)\right)\lambda_0 \\
&\qquad - \kappa_0\lambda_1\left(\partial_0^{-1}\right) - \partial_0^{-1/2}\left(\kappa_1 + \partial_0^{-1/2}\kappa_2\left(\partial_0^{-1}\right)\right)\lambda_1\left(\partial_0^{-1}\right) \\
&\quad = -\partial_0^{1/2}\kappa_0\lambda_0 - \kappa_1\lambda_0 - \kappa_0\lambda_1\left(\partial_0^{-1}\right) \\
&\qquad - \partial_0^{-1/2}\left(\kappa_1\lambda_1\left(\partial_0^{-1}\right) + \kappa_2\left(\partial_0^{-1}\right)\lambda_0\right) - \partial_0^{-1}\kappa_2\left(\partial_0^{-1}\right)\lambda_1\left(\partial_0^{-1}\right) \\
&\quad =: -\partial_0^{1/2}\kappa_0\lambda_0 - m_+\left(\partial_0^{-1}\right) \\
&L\left(\partial_0^{-1}\right)\eta\, k\left(\partial_0^{-1}\right)^{-1} \\
&\quad = -\partial_0^{1/2}\lambda_0\kappa_0 - \lambda_0\kappa_1 - \lambda_1\left(\partial_0^{-1}\right)\kappa_0 \\
&\qquad - \partial_0^{-1/2}\left(\lambda_1\left(\partial_0^{-1}\right)\kappa_1 + \lambda_0\kappa_2\left(\partial_0^{-1}\right)\right) - \partial_0^{-1}\lambda_1\left(\partial_0^{-1}\right)\kappa_2\left(\partial_0^{-1}\right) \\
&\quad =: -\partial_0^{1/2}\lambda_0\kappa_0 - m_-\left(\partial_0^{-1}\right) \\
&L\left(\partial_0^{-1}\right)\eta\, k\left(\partial_0^{-1}\right)^{-1} L\left(\partial_0^{-1}\right) \\
&\quad = \lambda\left(\partial_0^{-1}\right)\kappa_0\lambda\left(\partial_0^{-1}\right) + \partial_0^{-1/2}\lambda\left(\partial_0^{-1}\right)\left(\kappa_1 + \partial_0^{-1/2}\kappa_2\left(\partial_0^{-1}\right)\right)\lambda\left(\partial_0^{-1}\right) \\
&\quad = \left(\lambda_0 + \partial_0^{-1/2}\lambda_1\left(\partial_0^{-1}\right)\right)\kappa_0\left(\lambda_0 + \partial_0^{-1/2}\lambda_1\left(\partial_0^{-1}\right)\right) + \\
&\qquad + \partial_0^{-1/2}\left(\lambda_0 + \partial_0^{-1/2}\lambda_1\left(\partial_0^{-1}\right)\right)\left(\kappa_1 + \partial_0^{-1/2}\kappa_2\left(\partial_0^{-1}\right)\right)\left(\lambda_0 + \partial_0^{-1/2}\lambda_1\left(\partial_0^{-1}\right)\right) \\
&\quad = \lambda_0\,\kappa_0\lambda_0 + \partial_0^{-1/2}\left(\lambda_1\left(\partial_0^{-1}\right)\kappa_0\lambda_0 + \lambda_0\,\kappa_0\,\lambda_1\left(\partial_0^{-1}\right)\right) + \\
&\qquad + \partial_0^{-1/2}\left(\lambda_0 + \partial_0^{-1/2}\lambda_1\left(\partial_0^{-1}\right)\right)\left(\kappa_1 + \partial_0^{-1/2}\kappa_2\left(\partial_0^{-1}\right)\right)\left(\lambda_0 + \partial_0^{-1/2}\lambda_1\left(\partial_0^{-1}\right)\right) \\
&\quad =: \lambda_0\,\kappa_0\lambda_0 + \partial_0^{-1/2} m_0\left(\partial_0^{-1}\right).
\end{aligned}
$$

Substituting these calculations and the modified Hooke's law into the remaining relations of (3.1) and differentiating the fourth and eighth equation with respect to time yields

$$
\begin{aligned}
\partial_0 u - v &= 0 \\
\rho\,\partial_0 v + \partial_0\left(\rho_f R\right) - \operatorname{Div} T &= 0 \\
\partial_0\left(C^{-1}T + C^{-1}W^* M^{-1} P\right) - \operatorname{Grad} v &= 0\,. \\
\partial_0\left(\left(M^{-1} + M^{-1}W C^{-1}W^* M^{-1}\right)P + M^{-1}W C^{-1}T\right) + \operatorname{div} R &= 0 \\
\partial_0\left(\kappa_0 R\right) + \partial_0^{1/2}\kappa\left(\partial_0^{-1}\right)R + \rho_f^*\partial_0 v + \left(\partial_0^{1/2}\kappa_0\lambda_0 + m_+\left(\partial_0^{-1}\right)\right)E + \operatorname{grad} P &= f
\end{aligned}
$$

$$\begin{aligned}\partial_0 w - R &= 0\\ \partial_0 \varepsilon E + \sigma\left(\partial_0^{-1}\right) E - \left(\partial_0^{1/2}\lambda_0\kappa_0 + m_-\left(\partial_0^{-1}\right)\right) R - \operatorname{curl} H &= J_0\\ \partial_0 \mu H + \operatorname{curl} E &= 0\end{aligned}$$

To emphasize the composition of the equations from the Biot system (which is itself a composition of the elastic and acoustic wave equation) and Maxwell's equations, we write the spatial derivatives in the following nested block form:

$$\widetilde{A} = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix}0&0&0\\0&0&-\operatorname{Div}\\0&-\operatorname{Grad}&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \\ \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&\operatorname{div}&0\\ \operatorname{grad}&0&0\\0&0&0\end{pmatrix}\end{pmatrix} & \begin{pmatrix}\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\end{pmatrix} \\ \begin{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\end{pmatrix} & \begin{pmatrix}\begin{pmatrix}0&-\operatorname{curl}\\ \operatorname{curl}&0\end{pmatrix}\end{pmatrix}\end{pmatrix}$$

in order to write the system at least formally in the form

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + \widetilde{A}\right) U = g$$

to make it accessible to the theory developed in [4]. The block structure of $\widetilde{A}$ shows that

$$U = \begin{pmatrix}\begin{pmatrix}\begin{pmatrix}u\\v\\T\end{pmatrix}\\ \begin{pmatrix}P\\R\\w\end{pmatrix}\end{pmatrix}\\ \begin{pmatrix}E\\H\end{pmatrix}\end{pmatrix}$$

is a corresponding block column vector arrangement of the unkowns. Although we have in the above

$$g = \begin{pmatrix}\begin{pmatrix}\begin{pmatrix}0\\F\\0\end{pmatrix}\\ \begin{pmatrix}0\\f\\0\end{pmatrix}\end{pmatrix}\\ \begin{pmatrix}-J_0\\0\end{pmatrix}\end{pmatrix}$$

we may obviously allow for more general source terms. Concerning $M\left(\partial_0^{-1}\right)$, we see that

$$M(0)=\begin{pmatrix}\begin{pmatrix}\begin{pmatrix}1&0&0\\0&\rho&0\\0&0&C^{-1}\end{pmatrix}&\begin{pmatrix}0&0&0\\0&\rho_f&0\\C^{-1}W^*M^{-1}&0&0\end{pmatrix}\\\begin{pmatrix}0&0&M^{-1}WC^{-1}\\0&\rho_f^*&0\\0&0&0\end{pmatrix}&\begin{pmatrix}M^{-1}+M^{-1}WC^{-1}W^*M^{-1}&0&0\\0&\kappa_0&0\\0&0&1\end{pmatrix}\end{pmatrix}&\begin{pmatrix}\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\end{pmatrix}\\\begin{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\end{pmatrix}&\begin{pmatrix}\begin{pmatrix}\varepsilon&0\\0&\mu\end{pmatrix}\end{pmatrix}\end{pmatrix}.$$

To complete the formal transcription we need to add

$$M_{(*)}\left(\partial_0^{-1}\right):=M\left(\partial_0^{-1}\right)-M(0)$$

in order to represent the lower order time derivative terms. The above system form suggests

$$M_{(*)}\left(\partial_0^{-1}\right)=\partial_0^{-1/2}M_{(1/2)}+\partial_0^{-1}M_1\left(\partial_0^{-1}\right),$$

where

$$M_{(1/2)}:=\begin{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&\kappa_1&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0\\\kappa_0\lambda_0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&-\lambda_0\kappa_0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\end{pmatrix},$$

$$M_1\left(\partial_0^{-1}\right):=\begin{pmatrix}\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&\kappa_2\left(\partial_0^{-1}\right)&0\\0&-1&0\end{pmatrix}&\begin{pmatrix}0&0\\m_+\left(\partial_0^{-1}\right)&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&-m_-\left(\partial_0^{-1}\right)&0\\0&0&0\end{pmatrix}&\begin{pmatrix}\sigma\left(\partial_0^{-1}\right)&0\\0&0\end{pmatrix}\end{pmatrix}.$$

These reformulations allow us to establish a proper evolutionary system modelling electro-seismic waves.

4. The Evolutionary System of Electroseismic Waves

Our above considerations were purely formal and we need to establish a proper evolutionary system in a suitable Hilbert space framework in order to discuss well-posedness issues. Our aim is to establish the above formal system as an evolutionary equation

$$\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)U=g$$

in the spirit of the theoretical framework developed in [6] or rather in the slightly simpler form of [4]. As explained in the introduction, for this we need to establish that A is a skew-selfadjoint realization of $\widetilde{A}$ and that the material operator $M\left(\partial_0^{-1}\right)$ is indeed defined by an analytic operator family $\left(M\left(z\right)\right)_{z\in B_{\mathbb{C}}(r,r)}$ for some $r\in\mathbb{R}_{>0}$ and satisfies a positive-definiteness condition

$$\mathfrak{Re}\left(\partial_0 M\left(\partial_0^{-1}\right)\right)\geqslant c_0>0.$$

To ensure that A becomes skew-selfadjoint we specify a suitable domain for the formal operator $\widetilde{A}$. Obviously this can be done in many ways. In order to be specific, let us for example assume that v and P satisfy generalized homogeneous Dirichlet boundary conditions and E satisfies the generalized homogeneous electric boundary condition. The encoding of these generalized boundary conditions is standard and can be achieved by the following recipe (compare [6, 4]). The formal operators grad, Grad and curl are initially established as mappings in matching L^2-type spaces defined on smooth elements with compact support within the underlying non-empty, open set $\Omega\subseteq\mathbb{R}^3$:

$$\begin{aligned}
\operatorname{grad}|_{\mathring{C}_\infty(\Omega)} &: \mathring{C}_\infty\left(\Omega\right)\subseteq L^2\left(\Omega\right)\to\left(L^2\left(\Omega\right)\right)^3,\\
\operatorname{Grad}|_{\left(\mathring{C}_\infty(\Omega)\right)^3} &: \left(\mathring{C}_\infty\left(\Omega\right)\right)^3\subseteq\left(L^2\left(\Omega\right)\right)^3\to\left(L^2\left(\Omega\right)\right)^{3\times3}_{\mathrm{sym}},\\
\operatorname{curl}|_{\left(\mathring{C}_\infty(\Omega)\right)^3} &: \left(\mathring{C}_\infty\left(\Omega\right)\right)^3\subseteq\left(L^2\left(\Omega\right)\right)^3\to\left(L^2\left(\Omega\right)\right)^3.
\end{aligned}$$

Here the Hilbert space $L^2\left(\Omega\right)$ may be understood as the completion of the space $\mathring{C}_\infty\left(\Omega\right)$ of smooth functions with compact support in Ω with respect to the norm $|\cdot|_{L^2(\Omega)}$ given by

$$\varphi\mapsto\sqrt{\int_\Omega\left|\varphi\left(x\right)\right|^2dx}.$$

Correspondingly, for $\left(L^2\left(\Omega\right)\right)^3$ the norm is given by

$$\left(u_k\right)_{k=1,2,3}\mapsto\sqrt{\int_\Omega\left|u_1\left(x\right)\right|^2dx+\int_\Omega\left|u_2\left(x\right)\right|^2dx+\int_\Omega\left|u_3\left(x\right)\right|^2dx}$$

and $\left(L^2\left(\Omega\right)\right)^{3\times3}_{\mathrm{sym}}$ may like-wise be considered as the completion of $\left(\mathring{C}_\infty\left(\Omega\right)\right)^{3\times3}_{\mathrm{sym}}$, the symmetric matrices with entries in $\mathring{C}_\infty\left(\Omega\right)$ with respect to the natural norm

$$\left(T_{ij}\right)_{i,j=1,2,3}\mapsto\sqrt{\sum_{i,j=1}^{3}\int_\Omega\left|T_{ij}\left(x\right)\right|^2dx}.$$

The mappings $\operatorname{grad}|_{\mathring{C}_\infty(\Omega)}$, $\operatorname{Grad}|_{\left(\mathring{C}_\infty(\Omega)\right)^3}$, have the analogously defined $-\operatorname{div}|_{\left(\mathring{C}_\infty(\Omega)\right)^3}$, $-\operatorname{Div}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3\times3}_{\mathrm{sym}}}$ as their formal adjoint and $\operatorname{curl}|_{\left(\mathring{C}_\infty(\Omega)\right)^3}$ is formally selfadjoint. As the so-called weak derivatives associated with these differentiation operators we define

$$\operatorname{grad}:=-\left(\operatorname{div}|_{\left(\mathring{C}_\infty(\Omega)\right)^3}\right)^*,$$

$$\operatorname{Grad} := -\left(\operatorname{Div}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3\times 3}_{\mathrm{sym}}}\right)^*,$$

$$\operatorname{curl} := \left(\operatorname{curl}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3}}\right)^*,$$

$$\operatorname{Div} := -\left(\operatorname{Grad}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3}}\right)^*,$$

$$\operatorname{div} := -\left(\operatorname{grad}|_{\mathring{C}_\infty(\Omega)}\right)^*.$$

The adjoint of these operators can be utilized to formulate the desired boundary conditions. With

$$\begin{aligned}
\mathring{\operatorname{grad}} &:= -\operatorname{div}^* = \overline{\operatorname{grad}|_{\mathring{C}_\infty(\Omega)}},\\
\mathring{\operatorname{Grad}} &:= -\operatorname{Div}^* = \overline{\operatorname{Grad}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3}}},\\
\mathring{\operatorname{curl}} &:= \operatorname{curl}^* = \overline{\operatorname{curl}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3}}},\\
\mathring{\operatorname{Div}} &:= -\operatorname{Grad}^* = \overline{\operatorname{Div}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3\times 3}_{\mathrm{sym}}}},\\
\mathring{\operatorname{div}} &:= -\operatorname{grad}^* = \overline{\operatorname{div}|_{\left(\mathring{C}_\infty(\Omega)\right)^{3}}}
\end{aligned}$$

we can now in particular formulate the stated generalized boundary conditions as v being in the domain of $\mathring{\operatorname{Grad}}$, P in the domain of $\mathring{\operatorname{grad}}$ and E being in the domain of $\mathring{\operatorname{curl}}$. That is, we take A to be the block matrix operator

$$A := \begin{pmatrix}
\begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\operatorname{Div} \\ 0 & -\mathring{\operatorname{Grad}} & 0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0 & \operatorname{div} & 0 \\ \mathring{\operatorname{grad}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{pmatrix} &
\begin{pmatrix} \begin{pmatrix} 0&0\\0&0\\0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0\\0&0\\0&0 \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0 \end{pmatrix} \end{pmatrix} &
\begin{pmatrix} \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix} \end{pmatrix}
\end{pmatrix},$$

where the entries are now the operators we have just established. Due to its structure this operator can be seen to be skew-selfadjoint, see [6, 4], with respect to the Hilbert space

$$\left(\left(L^2(\Omega)\right)^3 \times \left(L^2(\Omega)\right)^3 \times \left(L^2(\Omega)\right)^{3\times 3}_{\mathrm{sym}}\right) \times \left(L^2(\Omega) \times \left(L^2(\Omega)\right)^3 \times \left(L^2(\Omega)\right)^3\right) \\ \times \left(\left(L^2(\Omega)\right)^3 \times \left(L^2(\Omega)\right)^3\right)$$

with its natural norm, which we shall abbreviate simply as H. This operator has a canonical skew-selfadjoint extension, which we shall also denote by A, to the space

$H_{\nu,0,0}$ of H-valued generalized functions given as the completion of the linear space generated by H-valued functions of the special form

$$t \mapsto \varphi(t)\, w =: \; (\varphi \otimes w)(t)$$

with $\varphi \in \mathring{C}_\infty(\mathbb{R})$, $w \in H$, with respect to the norm

$$u \mapsto \sqrt{\int_{\mathbb{R}} |u(t)|_H^2 \exp(-2\nu t)\, dt},$$

with $\nu \in \mathbb{R}_{>0}$.

Diagonalization of $M_0 := M(0)$ by a symmetric Gauss elimination yields

$$\begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & C^{-1} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} M^{-1} & 0 & 0 \\ 0 & \kappa_0 - \rho_f\, \rho^{-1} \rho_f^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \\ \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

and so for this to lead to the **regular** case in the sense of [6] we require that

$$\rho, C = \mathscr{C} - W^* M^{-1} W, M, \varepsilon, \mu, \kappa_0 - \rho_f\, \rho^{-1} \rho_f^*$$

are strictly positive definite as continuous linear operators in their respective $L^2(\Omega)$-type spaces.

For the solution theory to hold, i.e. to have well-posedness, we need the leading coefficient λ_0 in the expansion of L to be zero. Thus, we have the following constraints on k and L:

Hypothesis 1:

$$\left(k(z)^{-1}\right)_{z \in B_{\mathbb{C}}(r,r)}$$

must be analytic and the uniform operator limits

$$\lim_{z \to 0} z\, \eta\, k(z)^{-1} =: \; \kappa_0$$

$$\lim_{z \to 0} z^{-1/2} \left(z\, \eta\, k(z)^{-1} - \kappa_0\right) =: \; \kappa_1$$

exist with κ_0 selfadjoint, continuous, strictly positive definite and κ_1 selfadjoint, continuous, nonnegative such that

$$\left(\eta\, k(z)^{-1} - z^{-1} \kappa_0 - z^{-1/2} \kappa_1\right)_{z \in B_{\mathbb{C}}(r,r)}$$

is uniformly bounded, and

Hypothesis 2:

$$(L(z))_{z\in B_{\mathbb{C}}(r,r)}$$

must be such that

$$\left(\frac{L(z)}{z}\right)_{z\in B_{\mathbb{C}}(r,r)}$$

is analytic and uniformly bounded.

For the conductivity coefficient σ_0 we assume that $(\sigma_0(z))_{z\in B_{\mathbb{C}}(r,r)}$ is an analytic and uniformly continuous family of linear operators.

Under these assumptions we obtain a material law operator of the form

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1/2} M_{1/2} + \partial_0^{-1} M_1\left(\partial_0^{-1}\right)$$

with M_0 bounded selfadjoint and strictly positive definite, $M_{1/2}$ bounded selfadjoint and nonnegative, the family $(M_1(z))_{z\in B_{\mathbb{C}}(r,r)}$, $\nu \geqslant \frac{1}{2r}$, of bounded linear operators is holomorphic and uniformly bounded in H. Thus from the general theory of section 1, we obtain the following result.

THEOREM 2. *Let*

$$\rho,\, C = \mathscr{C} - W^* M^{-1} W,\, M,\, \varepsilon,\, \mu,\, \kappa_0 - \rho_f\, \rho^{-1}\, \rho_f^*$$

be strictly positive definite, continuous, selfadjoint mappings in their respective $L^2(\Omega)$ -type spaces and let L and k satisfy the above hypotheses 1 and 2. Then the initial boundary value problem of electroseismic waves is well-posed. Moreover, the solution operator $\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)^{-1} : H_{\nu,0,0} \to H_{\nu,0,0}$ is causal in the sense that if $g \in H_{\nu,0,0}$ vanishes on $\mathbb{R}_{<a}$ then $\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)^{-1} g$ also vanishes on $\mathbb{R}_{<a}$ for every $a \in \mathbb{R}$.

Proof. Due to the general framework, we only need to confirm that condition (2.3) is satisfied. The above assumptions ensure that M_0 and

$$M_{1/2} = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&\kappa_1&0\\0&0&0 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0&0\\0&0\\0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0\\0&0\\0&0 \end{pmatrix} \end{pmatrix} \\ \begin{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0\\0&0&0 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0&0\\0&0 \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

are bounded, selfadjoint and non-negative operator matrices. For

$$M_0 \geqslant m_0 > 0,\, M_{1/2} \geqslant 0,$$

for some constant $m_0 \in \mathbb{R}_{>0}$, we get

$$\begin{aligned}\mathfrak{Re}\, z^{-1}M(z) &= \mathfrak{Re}\, z^{-1}M_0 + \mathfrak{Re}\, z^{-1/2}M_{1/2} + \mathfrak{Re}\, M_1(z) \\ &\geqslant \frac{1}{2r} m_0 - \|M_1(z)\|\end{aligned}$$

for all $z \in B_{\mathbb{C}}(r,r)$. Indeed, noting that for such z, $\frac{1}{z} = \mathrm{i}\lambda + s$, $\lambda \in \mathbb{R}$, $s \in \mathbb{R}_{>\frac{1}{2r}}$, and with $\varphi = \arctan\left(\frac{\lambda}{s}\right)$ we obtain

$$\begin{aligned}\mathfrak{Re}\sqrt{\frac{1}{z}} &= \left(s^2+\lambda^2\right)^{1/4}\cos(\varphi/2) \\ &= \left(s^2+\lambda^2\right)^{1/4}\sqrt{\frac{1+\cos(\varphi)}{2}}\frac{1}{\sqrt{1+\tan(\varphi/2)^2}} \\ &= \left(s^2+\lambda^2\right)^{1/4}\sqrt{\frac{1+\frac{1}{\sqrt{1+\tan(\varphi)^2}}}{2}} \\ &= \left(s^2+\lambda^2\right)^{1/4}\sqrt{\frac{1+\frac{1}{\sqrt{1+\left(\frac{\lambda}{s}\right)^2}}}{2}} \\ &= \sqrt{\frac{\sqrt{s^2+\lambda^2}+s}{2}} \geqslant \sqrt{s} > \frac{1}{\sqrt{2r}}.\end{aligned}$$

Thus choosing

$$r < \frac{2}{\sup\left\{\|M_1(z)\| \,\middle|\, z \in B_{\mathbb{C}}(r,r)\right\}},$$

which corresponds to

$$\nu = \frac{1}{2r} > \sup\left\{\|M_1(z)\| \,\middle|\, z \in B_{\mathbb{C}}(r,r)\right\},$$

we have the desired estimate. That $(M_1(z))_{z\in B_{\mathbb{C}}(r,r)}$ is uniformly bounded follows also from the stated assumptions.

REMARK. The solution result is purely an L^2-type result, which is all that can be expected, given that there are hardly any regularity constraints on the model (boundary, data and material properties). It would certainly be interesting to establish suitable (local or global) regularity results. This, however, cannot be expected without imposing severe regularity constraints on the model data.

In conclusion, we have established the well-posedness of the time-dependent (but time-translation-invariant) model of electroseismic waves. There are, however, two important issues, which were not apparent from Pride's original ideas. The first is the implicit smallness constraints on W and ρ_f needed for well-posedness, which do not seem to have been noticed before. Secondly, there is a stronger decay constraint at ∞ for the coefficient L required to establish well-posedness. Since the original model was time-harmonic, such a constraint could hardly be noticed but it is clearly needed in the time-dependent case discussed here.

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