Closed form solution for $\mathfrak{p}$-curves in SO(4)

Carol Linton, William Holderbaum, James Biggs

Abstract—**This paper describes the solution for $\mathfrak{p}$-curves in $SO(4)$ and gives its closed form. The rotational symmetry was exploited in order to simplify the algebra. The relationship between the Casimir invariant functions and Lax operator is provided, along with its use as part of a Lax pair. The double cover by $SU(2) \times SU(2)$ enables two simpler problems to be found and integrated using Philip Hall coordinates and the solutions are then projected onto $SO(4)$. The methodology is generic and can be applied to other problems.**

Index Terms—**Motion planning, $\mathfrak{p}$-curves in $SO(4)$, Casimir invariants and Lax operators, double cover isomorphism**

I. Introduction

Planning the input controls necessary to move from an initial configuration to a target configuration is important in many fields: satellites and underwater vehicles [1], interacting spin particles [2], visual scanning (by moving the eyeball), robotic arms [3] and many others. The ideal trajectory plan is in closed form so that the expected configuration can be calculated directly, avoiding the inherent inaccuracies and processing time of an iterative process. Controls are then applied to bring the moving body back on track. Such solutions are difficult to find in 6 dimensions because of the number of conservation laws that interact.

$SO(4)$ is the rotation group in 4D Euclidean space. It is used in quantum control [2] but it can also be used to approximate $SE(3)$ [4] which is used to represent a rigid body translating and rotating. The symmetry of $SO(4)$ can be exploited as this paper will show. $SU(2) \times SU(2)$ provides a double cover which enables the problem to be split into two simpler problems, which can be integrated.

The associated Lie algebra, $\mathfrak{so}(4)$, can be subdivided into two sets, denoted by $\mathfrak{p}$ and $\mathfrak{k}$. $\mathfrak{p}$-curves are sub-Riemannian curves which use only the 3 elements of $\mathfrak{p}$ to achieve any 6 coordinate configuration from any other configuration. The system has just 3 input controls, but is fully controllable.

Section II provides a brief introduction to some other technicalities, notation and terms used later in the analysis. The structure of $SO(4)$ provides the Casimir functions, Lax operators and the coordinate equations. This information allows the problem to be simplified by rotation and adjustment of the time parameter. The generalized $\mathfrak{p}$-curve problem in $SO(4)$ is stated in section III. By expressing the problem as a Lax operator and Hamiltonian vector, in section V the isomorphism to $\mathfrak{su}(2)\times\mathfrak{su}(2)$ easily subdivides the problem into 2 simpler problems. An understanding of the Lax theorem identifies a further simplification in section VI. Using a Philip Hall basis in section VII, an analytic integration down to the $SU(2)$ group is made. The two solutions are rotated back in section VIII and then combined into the analytic solution in $SO(4)$ in section IX. Because this is in closed form, it is possible to algebraically check against the original problem statement. The approximation to $SE(3)$ is used to illustrate the solution in section X.

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II. Background

The structure of the rotation group SO(4) enables the definition of the $\mathfrak{p}$-curves problem which is the subject of this paper, and provides the symmetry required for its solution.

The Lie algebra $\mathfrak{so}(4)$ can be subdivided into two sets denoted by $\mathfrak{p}$ and $\mathfrak{k}$. Three controls belonging to just the $\mathfrak{p}$ set are able to drive the system to any target even through those are in 6 dimensions. This makes the control system simpler from a practical and a mathematical point of view. To understand this, the Cartan decomposition for $\mathfrak{so}(4)$ is described below.

Fact 1. *A basis for the Lie algebra $\mathfrak{so}(4)$ is $\{e_i\}$ for $i \in \{1,2,3,4,5,6\}$ as shown in the expression*

$$\sum_{i=1}^{3} p_i e_i + k_i e_{i+3} = \begin{bmatrix} 0 & -p_1 & -p_2 & -p_3 \\ p_1 & 0 & -k_3 & k_2 \\ p_2 & k_3 & 0 & -k_1 \\ p_3 & -k_2 & k_1 & 0 \end{bmatrix}$$

The coordinates using this basis are therefore $\{p_i, k_i\}$ for $i \in \{1,2,3\}$.

It is easy to show by matrix multiplication that the Lie algebra of $SO(4)$ can be subdivided into two sets, $\mathfrak{p}$ and $\mathfrak{k}$, satisfying the Cartan decomposition:

$$\mathfrak{so}(4) = \mathfrak{p} \oplus \mathfrak{k}, [\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, [\mathfrak{p},\mathfrak{k}] \subseteq \mathfrak{p}, [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$$

[5] where $e_i \in \mathfrak{k}$ when $i \in \{4,5,6\}$ and $e_i \in \mathfrak{p}$ when $i \in \{1,2,3\}$. Since $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$, the span of the basis generated by the elements of $\mathfrak{p}$ is the whole of $\mathfrak{so}(4)$.

Definition 2. If C and F are any differentiable functions on the dual of the Lie algebra, then Casimir invariant functions C are defined (see p132 in [6]) such that, for all F,

$$\{C, F\} = \frac{dC}{dt} = 0$$

The Casimir functions represent conservation laws for the Lie group. The Casimir invariant functions for SO(4) are (ref p458 in [7])

$$C_2 = \sum_{i=1}^{3} \left(p_i^2 \left(t \right) + k_i^2 \left(t \right) \right)$$

$$C_3 = \sum_{i=1}^{3} p_i \left(t \right) k_i \left(t \right) \qquad (1)$$

Definition 3. If L and B are any vectors belonging to the Lie algebra, then Casimir operators L are defined (see [8]) such that, for all B,

$$[L, B] = \frac{dL}{dt} = 0 \qquad (2)$$

The cooresponding Lax operator for the SO(4) used in this paper is (ref [9] which uses a matrix formulation)

$$L_2 = \sum_{i=1}^{3} k_i e_{i+3} + p_i e_i \qquad (3)$$

III. Problem statement and methodology

This section defines the $\mathfrak{p}$-curves problem which is the subject of this paper. Rotations of the spherical space are used to align the rotations of the curves with the base vectors, which eliminates variables and greatly simplifies the algebra.

Problem 4. The control problem solved in this paper is to find the $\mathfrak{p}$-curves, which are defined as the solutions of

$$\dot{g}(t) = g(t) \nabla H(t)$$

at the identity where

$$\nabla H \left(t \right) = \sum_{i=1}^{3} p_i \left(t \right) e_i$$

and $g \left(t \right) \in SO(4)$ and $g \left(0 \right) = I_4$, the identity.

Any 2 elements of $SO(4)$ are accessible from each other by $\mathfrak{p}$-curves, using results from [6]. The system is also assured to be controllable [10].

It is easy to show that ∇H is derived from the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{3} p_i^2 \qquad (4)$$

which arises from the three controls belonging to $\mathfrak{p}$.

In Figure 1, the tangent space for $SO(4)$ is shown diagrammatically by a sphere rotating on another sphere, with interactions between the two sets of rotations. This very approximate representation illustrates the simplification used in this paper. The rotation R aligns the $\mathfrak{k}$ tangent space along the e_4 unit vector and is described in Section IV. The problem is then split into two using the double cover isomorphism from $\mathfrak{so}(4)$ to $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and is covered in Section V. The homomorphism from $SU(2)$ to $SO(3)$ is two-to one so $\mathfrak{su}(2)$ is almost isomorphic to $\mathfrak{so}(3)$ (ref p18 in [11]) and $\mathfrak{su}(2)$ is represented by a sphere in 3 dimensional space. A rotation r in $\mathfrak{su}(2)$ aligns each sub-problem with the e_1' unit base vector - see Section VI.

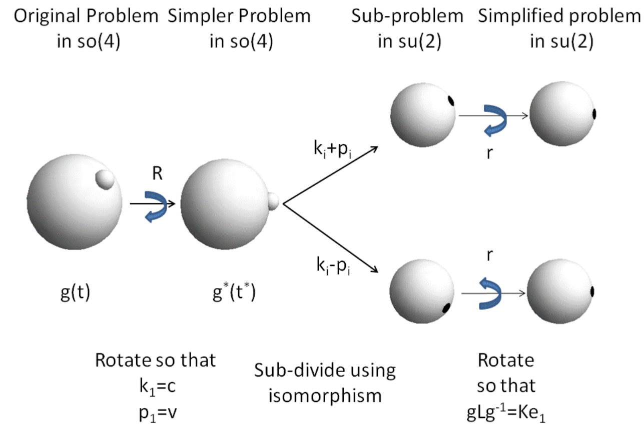

Figure 1. Simplifying the problem

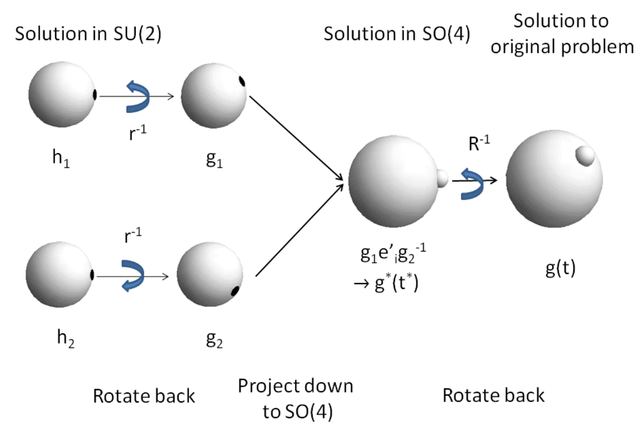

Figure 2. Projecting the solution down to $SO\left(4\right)$

The two simplified problems are integrated in Section VII to give curves in $SU(2)$. Figure 2 illustrates the reversal of this process that enabled the problem to be solved analytically. The rotation in $SU(2)$ is reversed in Section VIII. The two solutions are projecting onto $SO(4)$ in Section IX. A final rotation R^{-1} in $SO(4)$ gives the solution to the general problem.

The final solution was checked against all the constraints to show that methodology is valid.

IV. First Rotation

This section describes how the problem of finding the $\mathfrak{p}$-curves (Problem 4) is simplified by eliminating two parameters and setting one initial condition to zero. The system space is rotated so that the rotation in the $\mathfrak{k}$ tangent space is only about the e_4 unit vector. Either of the other two unit vectors in the $\mathfrak{k}$ set could have been chosen. Also the time scale is set so that initially there is no rotation about the e_3 unit vector, by choice.

From the definition of the Poisson bracket and the Hamiltonian (see p133 in [6]) and using equation (4), the coordinate equations for coordinates p_i are given by

$$\dot{k}_i = 0 \tag{5}$$

for $i \in \{1,2,3\}$ and

$$\dot{p}_i = k_j p_k - k_k p_j \tag{6}$$

permuting i,j,k over $\{1,2,3\}$. Thus $\{k_1,k_2,k_3\}$ are constants. Equations (1) become

$$C_2 = p_1^2(t) + p_2^2(t) + p_3^2(t) + k_1^2 + k_2^2 + k_3^2 \tag{7}$$
$$C_3 = p_1(t)\,k_1 + p_2(t)\,k_2 + p_3(t)\,k_3$$

The space $SO(4)$ is rotated by R defined by

$$R \exp\left[\sum_{i=1}^{3} k_i e_{i+3}\right] R^{-1} = \exp\left[c e_4\right] \tag{8}$$

so that $g = R^{-1} g^* R$. This rotation R can be found using $SU(2)$ and the Philip Hall basis from Definition (9). This allows the simplification

$$k_1^* = c = \sqrt{k_1^2 + k_2^2 + k_3^2},\ k_2^* = k_3^* = 0$$

and hence the Casimir is thus simplified to

$$C_3^* = c p_1^* = cv \tag{9}$$

Since c is a constant, p_1^* is also a constant which will be written as $p_1^* = v$.

From equations (7), it is obvious that $C_2^* - v^2 - c^2 = A^2$ where A is a constant, so that

$$A^2 = p_2^2(t) + p_3^2(t)$$

Equation (6) confirms that this has solution

$$p_2 = A\cos[ct+\theta] \text{ and } p_3 = A\sin[ct+\theta] \tag{10}$$

To eliminate the phasing constant θ, the time parameter is adjusted to

$$t^* = t + \theta/c \tag{11}$$

Following this rotation, the original problem of finding the $\mathfrak{p}$-curves (Problem 4) has been simplified to the following:

Problem 5. The simplified problem is to find $g^*(t)$ such that

$$\frac{\partial g^*}{\partial t^*}(t^*) = g^*(t^*)\,\nabla H(t^*) \tag{12}$$

at the identity where

$$\nabla H(t^*) = v e_1 + A\cos[ct^*]\,e_2 + A\sin[ct^*]\,e_3 \tag{13}$$

The Casimir invariant functions for the new problem are

$$C_2^* = v^2 + A^2 + c^2 \text{ and } C_3^* = vc \tag{14}$$

The Lax operator becomes, using equation (3)

$$L_2^* = c e_4 + v e_1 + A\cos[ct^*]\,e_2 + A\sin[ct^*]\,e_3 \tag{15}$$

From hereon, the superscript * will be dropped. The analytic solution found in this paper refers to this simplified problem. The time scale adjustment and the rotation need to be reversed for the general solution.

V. Projecting Problem to $\mathfrak{su}(2) \times \mathfrak{su}(2)$

In this section, the isomorphism from $\mathfrak{so}(4)$ to $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is applied to problem 5 above to create two simpler problems which can be integrated separately.

A basis $\{e_i'\}$for $SU(2)$ is shown by

$$\sum_{i=1}^{3} w_i e_i' = \frac{1}{2}\begin{bmatrix} w_1 i & w_2 + i w_3 \\ -w_2 + i w_3 & -w_1 i \end{bmatrix} \tag{16}$$

Theorem 6. *$\mathfrak{so}(4)$ is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$ where an element $A \in \mathfrak{so}(4)$ is associated with the elements $(V_1, V_2) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ via the mapping*

$$A \mapsto (V_1, V_2)$$
$$\sum_{i=1}^{3} p_i e_i + k_i e_{i+3} \mapsto \left\{\sum_{i=1}^{3} (n p_i + k_i)\, e_i'\right\} \text{ for } n \in \{-1,1\} \tag{17}$$

Proof: See pg 174 in [7] where a matrix formulation is used. ■

Using the isomorphism, problem 5 in $SO(4)$ is subdivided into 2 problems in $SU(2)$.

Problem 7. The problem in $SU(2)$ is to find the trajectory $g(t)$ such that

$$\dot{g} = g\nabla H$$

with

$$\nabla H = n v e_1' + nA\cos[ct]\,e_2' + nA\sin[ct]\,e_3' \tag{18}$$

$$L_2 = (c + nv)\,e_1 + nA\cos[ct]\,e_2 + nA\sin[ct]\,e_3 \tag{19}$$

The Lax Pair theorem (see [12] for the original version) states that, given that the inverse of g exists and that g and L are differentiable, then

$$\dot{g} = g\nabla H \text{ and } \dot{L} = [L, \nabla H]$$

if and only if

$$g(t)\,L(t)\,g^{-1}(t) = L(0) \tag{20}$$

with $g(0) = I_2$, the identity.

VI. Align the Lax operator in $\mathfrak{su}(2)$

As a final simplification before integrating to find the solution, the Lax operator in $\mathfrak{su}(2)$ (see equation (19)) is aligned to the e_1' unit base matrix, in order to reduce the number of variables involved. The following theorem proves that this is a valid simplification.

Theorem 8. *The general problem in $SU(2)$*

$$\dot{g} = g\nabla H \text{ with } g^{-1}(t) L(0) g(t) = L(t)$$

with $g(0) = I_2$ can be realigned as

$$\dot{h}(t) = h(t)\nabla H \text{ with } h^{-1}(t) e_i' h(t) = L(t)/K$$

where

$$h(t) = h(0)\,g(t)$$

$$h\left(0\right)L\left(0\right)h^{-1}\left(0\right)=Ke_i' \text{ and } K^2=-4tr(L^2\left(t\right))$$

Proof: By the Lax theorem, $L\left(t\right)$ forms a conjugate class with

$$g\left(t\right)L\left(t\right)g^{-1}\left(t\right)=L\left(0\right)$$

if $g\left(0\right)=I_2$. So

$$g^{-1}\left(t\right)L\left(0\right)g\left(t\right)=L\left(t\right)$$

which gives, for any rotation r,

$$\left(g^{-1}\left(t\right)r^{-1}\right)\left(rL\left(0\right)r^{-1}\right)\left(rg\left(t\right)\right)=L\left(t\right)$$

Writing $h\left(t\right)=rg\left(t\right)$ gives $r=h\left(0\right)$ since $g\left(0\right)=I_2$ and

$$h^{-1}\left(rL\left(0\right)r^{-1}\right)h=L\left(t\right)$$

The rotation required is r such that

$$rL\left(0\right)r^{-1}=Ke_i'$$

A property of a conjugate class is that the trace of $L\left(t\right)$ and $L^j\left(t\right)$ are constants, so squaring the above equation gives

$$rL\left(0\right)r^{-1}rL\left(0\right)r^{-1}=Ke_i'Ke_i'$$

Simplifying and taking the trace of this gives

$$tr(L^2\left(t\right))=tr(K^2\left(e_i'\right)^2)=-\frac{1}{4}K^2$$

■

Here $K^2=-4tr(L^2\left(t\right))=\left(\left(nA\right)^2+\left(c+nv\right)^2\right)$.

VII. Integrate the simplified problem

The problem of finding $\mathfrak{p}$-curves in $SO(4)$ has now been simplified and split into two sub-problems in $SU(2)$. The integration uses a Phillip Hall basis and is stated in the form of a theorem below.

Definition 9. A Phillip Hall basis $H=\{B_i\}$ (see [13]) is an ordered set of Lie products. The number of products is $l\geq n$, the dimension of the Lie algebra. A global solution of

$$\frac{dg(t)}{dt}=g(t)X \text{ where } X=\sum_{i=1}^{n}a_ie_i'$$

can be represented by

$$g(t)=\exp\left[h_l(t)B_l\right]\ldots\exp\left[h_1(t)B_1\right]$$

The first few base matrices are the base matrices of the Lie algebra $\{e_i'\}$, followed by the first order Lie brackets $\left\{\left[e_i',e_j'\right]\right\}$ and then higher order Lie brackets.

Theorem 10. *In $SU(2)$, a solution $h(t)$ of*

$$\dot{h}(t)=h(t)\nabla H \tag{21}$$

$$L(t)=Kh^{-1}e_1'h \tag{22}$$

with

$$\nabla H=nve_1'+nA\cos\left[ct\right]e_2'+nA\sin\left[ct\right]e_3' \tag{23}$$

$$L_2=\left(c+nv\right)e_1'+nA\cos\left[ct\right]e_2'+nA\sin\left[ct\right]e_3' \tag{24}$$

is

$$h(t)=\begin{bmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{bmatrix} \tag{25}$$

with $\alpha,\beta\in\mathbb{C}$ and

$$\alpha=\exp\left[\frac{i}{2}\left(\frac{\pi}{2}-ct+Kt\right)\right]\sqrt{\frac{K+\left(c+nv\right)}{2K}}$$

$$\beta=\exp\left[\frac{i}{2}\left(\frac{3\pi}{2}+ct+Kt\right)\right]\sqrt{\frac{K-\left(c+nv\right)}{2K}}$$

$$K^2=A^2+\left(c+nv\right)^2$$

Proof: $h(t)$ is written using the Philip Hall basis defined above as

$$h(t)=\ldots e^{de_1'}e^{je_3'}e^{be_2'}e^{ae_1'} \tag{26}$$

Substituting this and equation (24) into $L(t)=Kh^{-1}e_1'h$ expands to

$$\begin{pmatrix}\frac{1}{2}i(c+nv) & \frac{i}{2}An\exp[ict]\\ -\frac{i}{2}An\exp[-ict] & -\frac{1}{2}i(c+nv)\end{pmatrix}=$$

$$\begin{pmatrix}\frac{1}{2}iK\cos[b]\cos[j] & \frac{1}{2}ie^{-ia}K\begin{pmatrix}\cos[j]\sin[b]\\ +i\sin[j])\end{pmatrix}\\ \frac{1}{2}e^{ia}K(\begin{matrix}i\cos[j]\sin[b]\\ +\sin[j]\end{matrix}) & -\frac{1}{2}iK\cos[b]\cos[j]\end{pmatrix}$$

Setting $j=0$ simplifies this and component a_{11} of the matrix gives

$$\frac{c+nv}{K}=\cos[b]$$

which is a constant.

$$\sin[b]=\sqrt{1-\left(\frac{c+nv}{K}\right)^2}=\frac{An}{K}$$

Also component a_{12} of the matrix gives

$$i\frac{An}{K}\exp\left[ict\right]=i\exp\left[-ia\right])\sin\left[b\right]$$

$$i\frac{An}{K}\exp\left[ict\right]=i\exp\left[-ia\right])\frac{An}{K}$$

$$\exp\left[ict\right]=\exp\left[-ia\right]$$

$$a=\frac{\pi}{2}-ct$$

To find d, the proposed formula for $h(t)$ in equation (26) is differentiated.

$$\begin{aligned}\frac{dh(t)}{dt}&=\frac{d}{dt}\left(e^{de_1'}e^{je_3'}e^{be_2'}e^{ae_1'}\right)\\ &=\dot{d}e_1'e^{de_1'}e^{je_3'}e^{be_2'}e^{ae_1'}\\ &+\dot{j}e^{de_1'}e_3'e^{je_3'}e^{be_2'}e^{ae_1'}\\ &+\dot{b}e^{de_1'}e^{je_3'}e^{be_2'}e_2'e^{ae_1'}\\ &+\dot{a}e^{de_1'}e^{je_3'}e^{be_2'}e^{ae_1'}e_1'\end{aligned}$$

and substituted into (21) $\nabla H = h^{-1}(t)\frac{dh(t)}{dt}$ to give, with (23),

$$\begin{pmatrix} inv & iAn\exp[ict] \\ -iAn\exp[-ict] & -inv \end{pmatrix} =$$

$$\begin{pmatrix} i(\dot{a}+\dot{d}\cos[b]) & e^{-ia}(\dot{b}+i\dot{d}\sin[b]) \\ -e^{ia}(\dot{b}-i\dot{d}\sin[b]) & -i(\dot{a}+\dot{d}\cos[b]) \end{pmatrix}$$

Equating coefficients, for component a_{11}, gives

$$inv = i\left(\dot{d}\cos[b]+\dot{a}\right) = i\left(\dot{d}\frac{c+nv}{K}-c\right)$$

$$\dot{d} = K$$

and so $d = Kt + d_0$ with the integration constant $d_0 = 0$ which later cancels from the equations.

These values can be substituted into equation (26) to give the result in the theorem. ■

VIII. Realign with original Constraint in SU(2)

The two solutions found above are now used to find the solution to the original $\mathfrak{p}$-curves problem as illustrated in Figure 2. Firstly the solutions are aligned back with the original constraint $L(0)$. This is done by multiplying by $h^{-1}(0)$ as in Theorem 8 as follows:

$$\begin{aligned} g(t) &= h^{-1}(0)\,h(t) \\ &= \left[\begin{pmatrix} e^{\frac{i\pi}{4}}\cos\left[\frac{b}{2}\right] & e^{-\frac{i\pi}{4}}\sin\left[\frac{b}{2}\right] \\ -e^{\frac{i3\pi}{4}}\sin\left[\frac{b}{2}\right] & e^{\frac{i3\pi}{4}}\cos\left[\frac{b}{2}\right] \end{pmatrix}\right]^{-1} h(t) \\ &= \begin{pmatrix} a_{11} & a_{12} \\ -\bar{a}_{12} & \bar{a}_{11} \end{pmatrix} \end{aligned} \quad (27)$$

where

$$K = \sqrt{A^2 + (c+nv)^2}$$

$$a_{11} = e^{-\frac{1}{2}ict}\left(\cos\left[\frac{Kt}{2}\right] + i\frac{(c+nv)}{K}\sin\left[\frac{Kt}{2}\right]\right)$$

$$a_{12} = e^{\frac{ict}{2}}\frac{An}{K}\sin\left[\frac{Kt}{2}\right]$$

IX. Projecting back onto $g(t) \in SO(4)$

In the last section, two curves were found in $SU(2)$ with $n = 1$ and $n = -1$. These two curves, g_1 and g_2, are now projected back into one curve $g(t)$ in $SO(4)$. As a preliminary, the 4 columns of the matrix representing $g(t)$ are labeled $\hat{X}_i$ for $i \in \{1,2,3,4\}$. Each column $\hat{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathbb{R}^4$ has 4 elements and a matrix is constructed from those elements.

$$X = \begin{bmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{bmatrix}$$

There are thus four matrices X_i which are determined using the next theorem.

Theorem 11. *A sub-Riemannian curve $g(t) \in SO(4)$ that solves the optimal control problem (1) is*

$$g(t) = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \hat{X}_3 & \hat{X}_4 \end{bmatrix}$$

where

$$X_1(t) = \frac{1}{2}g_1(t)I_2g_2^{-1}(t)$$

$$X_2(t) = g_1(t)e_1'g_2^{-1}(t)$$

$$X_3(t) = g_1(t)e_2'g_2^{-1}(t)$$

$$X_4(t) = g_1(t)e_3'g_2^{-1}(t)$$

where $g_1(t)$, $g_2(t) \in SU(2)$ are given analytically by equation (27) and the basis is defined by (16).

Proof: This follows from the homomorphism $\Phi : SU(2)\times SU(2) \to SO(4)$ which is defined [14] through the equivalent group action

$$g(t)\hat{z} = \hat{x}$$

for $g(t) \in SO(4)$ if and only if

$$g_1(t)Zg_2^{-1} = X$$

where $g_1(t)$, $g_2(t) \in SU(2)$. ■

Applying this theorem to equation (27) gives the solution $g(t)$ in $SO(4)$ of the simplified Problem 5. It will be remembered that the problem was simplified by rotating the axes (equation (7)) so that all rotation in the $\mathfrak{p}$-subspace was about the e_4 axis and the time parameter was adjusted (11) so that there was no initial rotation about the e_3 axis.

The curve in $SO(4)$ is given by

$$g(t) = \begin{bmatrix} Col1 & Col2 & Col3 & Col4 \end{bmatrix}$$

with

$$Col1 =$$

$$\begin{pmatrix} C_1C_2 - S_1S_2\frac{\left(A^2-c^2+v^2\right)}{K_1K_2} \\ C_2S_1\frac{(c+v)}{K_1} + C_1S_2\frac{(-c+v)}{K_2} \\ C_2S_1\frac{A}{K_1} + C_1S_2\frac{A}{K_2} \\ S_1S_2\frac{2Ac}{K_1K_2} \end{pmatrix}$$

$$Col2 =$$

$$\begin{pmatrix} -C_1S_2\frac{(-c+v)}{K_2} - C_2S_1\frac{(c+v)}{K_1} \\ C_1C_2 + S_1S_2\frac{\left(A^2+c^2-v^2\right)}{K_1K_2} \\ -S_1S_2\frac{2Av}{K_1K_2} \\ -C_2S_1\frac{A}{K_1} + C_1S_2\frac{A}{K_2} \end{pmatrix}$$

$$Col3 =$$

$$\begin{pmatrix} \frac{A}{K_1K_2} & \begin{pmatrix} -(K_1C_1S_2+K_2C_2S_1)\cos[ct] \\ -2cS_1S_2\sin[ct] \end{pmatrix} \\ \frac{A}{K1\,K2} & \begin{pmatrix} -2vS_1S_2\cos[ct] \\ +(-K_2C_2S_1+K_1C_1S_2)\sin[ct] \end{pmatrix} \\ \frac{1}{K1\,K2} & \begin{pmatrix} K_1K_2C_1C_2\cos[ct] \\ -K_1K_2S_1S_2\cos[ct] \\ +(c+v)K_2C_2S_1\sin[ct] \\ +(c-v)K_1C_1S_2\sin[ct] \end{pmatrix} \\ \frac{1}{K1\,K2} & \begin{pmatrix} C_2S_1K_2(c+v)\cos[ct] \\ +S_2C_1K_1(c-v)\cos[ct] \\ -K_1K_2C_1C_2\sin[ct] \\ -S_1S_2\left(A^2-c^2+v^2\right)\sin[ct] \end{pmatrix} \end{pmatrix}$$

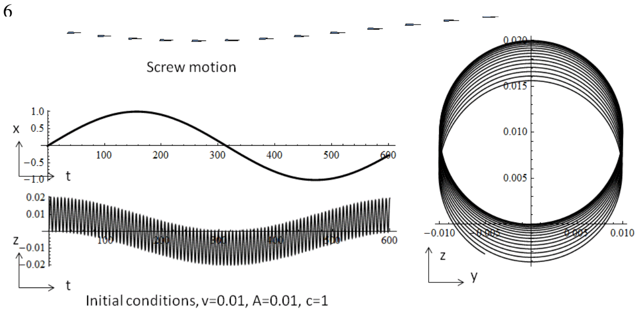

Figure 3. Trajectory approximated to 3D Euclidean space

$$Col4 =$$

$$\begin{pmatrix} \frac{A}{K1\,K2} & \begin{pmatrix} 2cS_1S_2\cos[ct] \\ -(K_1C_1S_2+K_2C_2S_1)\sin[ct] \end{pmatrix} \\ \frac{A}{K1\,K2} & \begin{pmatrix} (K_2C_2S_1-K_1C_1S_2)\cos[ct] \\ -2vS_1S_2\sin[ct] \end{pmatrix} \\ \frac{1}{K1\,K2} & \begin{pmatrix} C_1S_2K_1(-c+v)\cos[ct] \\ -K_2S_1C_2(c+v)\cos[ct] \\ +C_1C_2K_1K_2\sin[ct] \\ -S_1S_2\left(A^2+c^2-v^2\right)\sin[ct] \end{pmatrix} \\ \frac{1}{K1\,K2} & \begin{pmatrix} K_1K_2C_1C_2\cos[ct] \\ -S_1S_2\left(A^2-c^2+v^2\right)\cos[ct] \\ +K_2C_2S_1(c+v)\sin[ct] \\ +K_1C_1S_2(c-v)\sin[ct] \end{pmatrix} \end{pmatrix}$$

where

$$K_1^2 = A^2 + (c+v)^2 \text{ and } K_2^2 = A^2 + (c-v)^2$$

$$S_1 = \sin\left[K_1\,t\right],\, C1_1 = \cos\left[K_1\,t\right]$$

$$S_2 = \sin\left[K_2\,t\right],\, C_2 = \cos\left[K2_2\,t\right]$$

c= fixed rotation of e_4 component
v= fixed rotation of e_1 component
A= initial rotation of e_2 component
Mathematica enables this analytic solution to be check against the original problem statement.

X. Trajectories

To envisage the curve or trajectory found in the previous section, the rotations in the $\mathfrak{k}$ tangent space are considered as rotations of about the centre of the vessel. Small rotations in the $\mathfrak{p}$ tangent space are considered as linear dispacements in Euclidean space.

The initial motion approximates to a the screw motion about an axis in the x:z plane. The vessel rocks through a small angle but loses most of the initial rotation to the screw motion. The screw has a small constant amplitude (approximately A) and starts at the origin, slowly moving away from the axis as shown in the $y-z$ plot in Figure 3.

Longer term, the rotation about the main axis completes a full cycle of the spherical space from $x=-1$ to 1.

XI. Conclusion

By using the symmetry of the 4D rotation group $SO(4)$, it was possible to find a closed form solution to the fully assessable and controllable 6 dimensional problem. $\mathfrak{p}$-curves are interesting because, with just 3 input controls, any configuration in 6 dimensions is attainable. Mathematica was used for the manipulations and the final solution was checked against the original problem statement. The relationships between the symmetry, Casimir invariant functions and Lax operators, Lax pairs and the isomorphism between $SO(4)$ and $SU(2) \times SU(2)$ were investigated. It is only by exploiting all those relationships that the simplifications necessary to find the analytical solution can be found. These relationships are applicable to other control problems.

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