Type-1 OWA Operators in Aggregating Multiple Sources of Uncertain Information : Properties and Real World Applications

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***Abstract*—The type-1 ordered weighted averaging (T1OWA) operator has demonstrated the capacity for directly aggregating multiple sources of linguistic information modelled by fuzzy sets rather than crisp values. Yager's OWA operators possess the properties of *idempotence*, *monotonicity*, *compensativeness*, and *commutativity*. This paper aims to address whether or not T1OWA operators possess these properties when the inputs and associated weights are fuzzy sets instead of crisp numbers. To this end, a partially ordered relation of fuzzy sets is defined based on the fuzzy maximum (*join*) and fuzzy minimum (*meet*) operators of fuzzy sets, and an *alpha-equivalently-ordered* relation of groups of fuzzy sets is proposed. Moreover, as the extension of *orness* and *andness* of an Yager's OWA operator, *joinness* and *meetness* of a T1OWA operator are formalised, respectively. Then, based on these concepts and the *Representation Theorem of T1OWA operators*, we prove that T1OWA operators hold the same properties as Yager's OWA operators possess, i.e.: *idempotence*, *monotonicity*, *compensativeness*, and *commutativity*. Various numerical examples and a case study of diabetes diagnosis are provided to validate the theoretical analyses of these properties in aggregating multiple sources of uncertain information and improving integrated diagnosis, respectively.**

***Index Terms*—OWA operator, type-1 OWA operator, aggregation, linguistic aggregation, fuzzy sets, soft decision making, diabetes, integrated diagnosis.**

I. Introduction

In domains where information fusion/integration or multi-factorial evaluation is needed, an aggregation process is necessary to combine multiple sources of information into a global result so that in the final decision, all the individual sources of information are taken into account [1]. For example, in medicine, diagnosis or measurement can rarely be decided based on an individual criterion. Particularly, in the age of big data, the use of information aggregation is rapidly increasing, both because data is easily collected by ubiquitous information technologies and because the availability of cost-effective computational power allows combining information from multiple sources to be readily feasible.

Yager's ordered weighted averaging (OWA) operators [2], [3] have become a popular tool to aggregate information from multiple sources due to their flexibility for modeling a wide variety of aggregation scenarios via the appropriate definition/selection of the OWA operator's weighting vector [3]. However, Yager's OWA operators exclusively aggregate crisp numbers, while in real-world decisions, one is often not certain about the exact value of a crisp attribute. For example, in medicine, patients often find it difficult to describe how they feel, and doctors/nurses often find it difficult to describe what they observed. Thus it is desirable to develop a technique that can aggregate multiple sources of uncertain information of attributes. T1OWA operators and the associated α-level T1OWA aggregations are such a technique [4], [5], in which uncertain information is modelled by fuzzy sets. In this way, with appropriately definitions of uncertain weights, T1OWA operators extend Yager's OWA operator [2], the *meet* operator of fuzzy sets and the *join* operators of fuzzy sets [7], [8].

Since their appearance, the T1OWA operators have received increasing attention in scientific applications [9]–[14]. To select optimal routes under uncertain environments, T1OWA operators have been designed to guide human decision-making in a fuzzy weighted graph [10]. In addition to the α-level approach to fast implementation of T1OWA operators, another new method of calculating T1OWA was proposed via an opposite direction searching [11]. A T1OWA unbalanced fuzzy linguistic aggregation method has been applied to credit risk evaluation [12]. In group decision making, T1OWA operators can be used to combine multiple granular linguistic information and improve consensus reaching processes [13]. In type-2 fuzzy logic system modelling, the type-reduction of general type-2 fuzzy sets can be

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efficiently implemented via T1OWA operators [14].

Despite the above mentioned advances on the development and applications of T1OWA operators, one issue remains unclear regarding aggregation mechanism properties. Yager's OWA operators are *idempotent*, *monotonic*, *compensative*, and *commutative* [2]. The question to be answered in our case is whether or not T1OWA operators hold these same properties when the inputs and associated weights become uncertain, being expressed as fuzzy sets instead of crisp numbers in soft decision making. This question is not trivial at all, because the mechanisms of operations on a group of (fuzzy) sets are completely different from those on crisp values, with more advanced computing techniques to be required. In this paper, we aim to answer this important question.

To this end, based on the α-cuts of fuzzy sets, we suggest a new relation of fuzzy sets, named the *alpha-equivalently-ordered* relation of a group of fuzzy sets, and address the *join* and the *meet* based *partial order relation* of fuzzy sets. Then we prove that the T1OWA operation is *commutative*, *idempotent*, *monotonic*, and *compensative* with respect to the fuzzy set partial order relation.

The rest of this paper proceeds as follows. In Section II, we briefly review two definitions of the T1OWA operator: one based on the *Extension Principle*, the other based on the α-cuts of fuzzy sets. Section III defines a fuzzy set partial order relation based on the meet and join operators of fuzzy sets. As the extension of the *andness* and *orness* of Yager's OWA operators. Section IV defines the *meetness* and *joinness* of T1OWA operators. The properties of a T1OWA operator are then analysed and proved in Section V. Section VI provides a case study of diabetes diagnosis and further validation of computing efficiency of α-level T1OWA aggregation. The paper concludes with Section VII.

II. Preliminaries

Although the T1OWA operators can be defined either via Zadeh's Extension Principle or via the α-cuts of fuzzy sets [4], [5], their final aggregation results coincide.

Let $F(X)$ be the power set of fuzzy subsets on the domain of discourse X. One can define the T1OWA operator via the Extension Principle [4] as follows:

Definition 1. *"Given n linguistic weights $\left\{\widetilde{W}^i\right\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0,1]$, a T1OWA operator is a mapping Φ,*

$$\begin{array}{rcl}\Phi\colon F(X)\times\cdots\times F(X) & \longrightarrow & F(X)\\ (\widetilde{A}^1,\cdots,\widetilde{A}^n) & \mapsto & \widetilde{Y}\end{array} \quad (1)$$

The membership function of outcome fuzzy set $\widetilde{Y}$ (aggregation result) is

$$\mu_{\widetilde{Y}}(y) = \sup_{\substack{\sum_{i=1}^n \bar{w}_i a_{\sigma(i)} = y\\ w_i\in U,\, a_i\in X}} \begin{pmatrix}\mu_{\widetilde{W}^1}(\omega_1)\wedge\cdots\wedge\mu_{\widetilde{W}^n}(\omega_n)\\ \wedge\mu_{\widetilde{A}^1}(a_1)\wedge\cdots\wedge\mu_{\widetilde{A}^n}(a_n)\end{pmatrix} \quad (2)$$

where $\bar{\omega}_i = \frac{\omega_i}{\sum_{i=1}^n \omega_i}$, and $\sigma\colon \{1,\cdots,n\}\longrightarrow\{1,\cdots,n\}$ is a permutation function such that $a_{\sigma(i)} \ge a_{\sigma(i+1)}$, $\forall i = 1,\cdots,n-1$, i.e., $a_{\sigma(i)}$ is the ith largest element in the set $\{a_1,\cdots,a_n\}$."

Definition (2) can lead to a procedure for implementing T1OWA operations, called the *Direct Approach* [4]. Alternatively, one can define T1OWA operators using the α-cuts of a fuzzy set [5] as follows:

Definition 2. *"Let $\left\{\widetilde{W}^i\right\}_{i=1}^n$ be a set of linguistic weights characterised by fuzzy sets on the domain of discourse $U = [0,1]$, and $\alpha\in[0,1]$. The α-level type-1 OWA operator with α-cuts $\left\{\widetilde{W}_\alpha^i\right\}_{i=1}^n$ is the operator that aggregates the α-cuts of the fuzzy sets $\{\widetilde{A}^1,\cdots,\widetilde{A}^n\}$ as follows:*

$$\Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right) = \left\{\frac{\sum_{i=1}^n \omega_i a_{\sigma(i)}}{\sum_{i=1}^n \omega_i}\middle|\, \omega_i\in\widetilde{W}_\alpha^i, a_i\in\widetilde{A}_\alpha^i, i = 1,\cdots,n\right\} \quad (3)$$

where σ is a permutation function such that $a_{\sigma(i)} \ge a_{\sigma(i+1)}, \forall i = 1,\cdots,n-1$, $\widetilde{W}_\alpha^i = \{\omega|\mu_{\widetilde{W}^i}(\omega)\ge\alpha\}$, and $\widetilde{A}_\alpha^i = \{x|\mu_{\widetilde{A}^i}(x)\ge\alpha\}$."

In fact, one can use the α-level sets $\Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right)$ to create a fuzzy set as follows:

$$\widetilde{G} = \bigcup_{0\le\alpha\le1} \alpha\Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right) \quad (4)$$

where the membership function is

$$\mu_{\widetilde{G}}(x) = \bigvee_{\alpha: x\in\Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right)} \alpha \quad (5)$$

The above two methods of aggregating fuzzy sets via Yager's OWA mechanism are equivalent [5] as stated below.

Theorem 1 (**Representation Theorem of T1OWA Operators**). *"Given a set of linguistic weights $\left\{\widetilde{W}^i\right\}_{i=1}^n$ in the form of fuzzy sets on U. For any fuzzy sets $\widetilde{A}^1,\cdots,\widetilde{A}^n$ on $F(X)$, let Y be the outcome aggregation result defined in (2) and $\widetilde{G}$ be the result defined in (4), then $\widetilde{Y} = \widetilde{G}$."*

According to this *Representation Theorem*, one can implement the T1OWA aggregation through a series of α-level T1OWA operators. This provides a new way of theoretically analysing the properties of T1OWA operators. The procedure for implementing the T1OWA aggregation through a series of α-level T1OWA operators is called the *Alpha-Level Approach* [5].

III. Join and Meet Based Partial Order Relation of Fuzzy Sets

Zadeh defined the *meet* and the *join* of fuzzy sets to aggregate linguistic variables $\widetilde{A}$ and $\widetilde{B}$ for the statements "*$\widetilde{A}$ and $\widetilde{B}$*" and "*$\widetilde{A}$ or $\widetilde{B}$*" respectively [6]. The *meet* and the *join* of fuzzy sets now become fundamental operators in developing type-2 fuzzy systems [7], [8].

Definition 3. *Given two fuzzy sets $\widetilde{S}$ and $\widetilde{T}$, the join ($\cup$) and meet ($\cap$) operators are defined as*

$$\mu_{\widetilde{S}\cup\widetilde{T}}(v) = \sup_{\substack{s \vee t = v \\ s,t \in X}} \left(\mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(t)\right) \tag{6}$$

$$\mu_{\widetilde{S}\cap\widetilde{T}}(v) = \sup_{\substack{s \wedge t = v \\ s,t \in X}} \left(\mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(t)\right) \tag{7}$$

where sup is a t-conorm, $\wedge$ is the minimum operator and $\vee$ is the maximum operator.

It should be noted that the *join* (6) and the *meet* (7) operators can aggregate a set of criteria based on an imperative, such as "*one of the criteria should be satisfied*" and "*all the criteria should be satisfied*" respectively [6].

A. Join and Meet are T1OWA Operators

By appropriately choosing linguistic weights in a T1OWA operator, the *join* operator (6) of fuzzy sets is, in fact, a special T1OWA operator.

Theorem 2. *Let a T1OWA operator, J, be defined by the first linguistic weight being the singleton weight $\tilde{1}$: $\widetilde{W_1} = \tilde{1}$, all other weights being the singleton weight $\tilde{0}$: $\widetilde{W_i} = \tilde{0}$ ($i \neq 1$), where,*

$$\mu_{\tilde{1}}(\omega) = \begin{cases} 1 & for\ \omega = 1 \\ 0 & for\ \omega \neq 1 \end{cases} \tag{8}$$

$$\mu_{\tilde{0}}(\omega) = \begin{cases} 1 & for\ \omega = 0 \\ 0 & for\ \omega \neq 0 \end{cases} \tag{9}$$

For any groups of fuzzy sets $\left\{\widetilde{A^i}\right\}_{i=1}^{n}$,

$$J\left(\widetilde{A}^1, \widetilde{A}^2, \cdots, \widetilde{A}^n\right) = \bigcup_{i=1}^{n} \widetilde{A^i} \tag{10}$$

Proof. Omitted □

Example 2 in the *Supplemental Material* shows the *join* operation as a T1OWA operator in nature.

Interestingly, in T1OWA aggregation, if the first linguistic weight moves towards $\tilde{1}$, all the others towards $\tilde{0}$ (see *Example 3* in the *Supplemental Material*), then this operator demonstrates a *join-like* behavior. This type of operator is called a *join-like* T1OWA operator.

Similarly, the meet operation (7) of fuzzy sets is also a special T1OWA operator.

Theorem 3. *Let a T1OWA operator, M, be defined by the last linguistic weight being the singleton weight $\tilde{1}$: $\widetilde{W_n} = \tilde{1}$, all the others being the singleton weight $\tilde{0}$: $\widetilde{W_i} = \tilde{0}$ ($i \neq n$). For any groups of fuzzy sets $\left\{\widetilde{A^i}\right\}_{i=1}^{n}$,*

$$M\left(\widetilde{A}^1, \widetilde{A}^2, \cdots, \widetilde{A}^n\right) = \bigcap_{i=1}^{n} \widetilde{A^i} \tag{11}$$

Proof. Omitted □

Example 5 in the *Supplemental Material* shows the results of the *Meet* operator to aggregate three fuzzy aggregated objects.

Correspondingly, in T1OWA aggregation, if the last linguistic weight moves towards $\tilde{1}$, all the other weights towards $\tilde{0}$, then this operator demonstrates *meet-like* type behavior (see *Example 6* in the *Supplemental Material*). We call it a *meet-like* T1OWA operator.

B. Partial Order Relation of Fuzzy Sets

The set of real numbers $\mathbb{R}$ is linearly ordered, and the $(\mathbb{R}, \wedge, \vee)$ forms a lattice. Then, for any $a, b \in \mathbb{R}$, a partially ordered relation " $\geq$ "(" $\leq$ ") can be defined as

$$\begin{aligned} s \geq t \quad &\Longleftrightarrow s \vee t = s \\ &\Longleftrightarrow s \wedge t = t \end{aligned} \tag{12}$$

As a matter of fact, according to Zadeh' Extension Principle, the *meet* ($\cap$) and *join* ($\cup$) operators are just fuzzification of the *min* ($\wedge$) and *max* ($\vee$) operators of crisp numbers, respectively. In this way, $\widetilde{S} \cap \widetilde{T}$ and $\widetilde{S} \cup \widetilde{T}$ are no other than the fuzzified minimum, $\widetilde{S}$, and fuzzfied maximum, $\widetilde{T}$, of the fuzzy sets. It can be proved that $(F(\mathbb{R}), \cap, \cup)$ is a distributive lattice [15], with partial order relation defined as follows:

Definition 4. *Given two fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$, a partially ordered relation " $\succcurlyeq$ " is defined as*

$$\begin{aligned} \widetilde{S} \succcurlyeq \widetilde{T} \quad &\Longleftrightarrow \quad \widetilde{S} \cup \widetilde{T} = \widetilde{S} \\ &\Longleftrightarrow \quad \widetilde{S} \cap \widetilde{T} = \widetilde{T} \end{aligned} \tag{13}$$

We have the following theorem:

Theorem 4. *Let $\widetilde{S}$ and $\widetilde{T} \in F(R)$ be fuzzy numbers with core centres v_1 and v_2 respectively, and $v_1 \geq v_2$, then based on the t-conorm and t-norm,*
$\widetilde{S} \succcurlyeq \widetilde{T} \Longleftrightarrow \mu_{\widetilde{S}}(s) \leq \mu_{\widetilde{T}}(s)$ *for $s \leq v_2$ and $\mu_{\widetilde{S}}(s) \geq \mu_{\widetilde{T}}(s)$ for $s \geq v_1$.*

Proof. 1) First, if $\widetilde{S} \succcurlyeq \widetilde{T}$, then according to (13), for any $s \leq v_2$, we have $\mu_{\widetilde{T}}(v_2) = 1$ and

$$\begin{aligned} \mu_{\widetilde{T}}(s) &= \mu_{\widetilde{S}\cap\widetilde{T}}(s) \\ &= \sup_{\substack{s_1 \wedge s_2 = s \\ s_1, s_2 \in X}} \left(\mu_{\widetilde{S}}(s_1) \wedge \mu_{\widetilde{T}}(s_2)\right) \end{aligned}$$

Because $x \wedge v_2 = s$,

$$\begin{aligned} \mu_{\widetilde{T}}(s) &\geq \mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(v_2) \\ &= \mu_{\widetilde{S}}(s) \wedge 1 \\ &= \mu_{\widetilde{S}}(s) \end{aligned}$$

For any $s \geq v_1$, we have $\mu_{\widetilde{S}}(v_1) = 1$ and

$$\begin{aligned} \mu_{\widetilde{S}}(s) &= \mu_{\widetilde{S}\cup\widetilde{T}}(s) \\ &= \sup_{\substack{s_1 \vee s_2 = s \\ s_1, s_2 \in X}} \left(\mu_{\widetilde{S}}(s_1) \wedge \mu_{\widetilde{T}}(s_2)\right) \end{aligned}$$

Because $v_1 \vee s = s$,

$$\begin{aligned} \mu_{\widetilde{S}}(s) &\geq \mu_{\widetilde{S}}(v_1) \wedge \mu_{\widetilde{T}}(s) \\ &= 1 \wedge \mu_{\widetilde{T}}(s) \\ &= \mu_{\widetilde{T}}(s) \end{aligned}$$

2) If $\mu_{\widetilde{S}}(s) \leq \mu_{\widetilde{T}}(s)$ for any $x \leq v_2$ and $\mu_{\widetilde{S}}(s) \geq \mu_{\widetilde{T}}(s)$ for any $s \geq v_1$, we prove $\widetilde{S} \cup \widetilde{T} = \widetilde{S}$ in the following.

Let us denote $\widetilde{C} \equiv \widetilde{S} \cup \widetilde{T}$. For any s, s_1 and $s_2 \in X$ with $s_1 \wedge s_2 = s$, then $s = s_1$ or $s = s_2$. Hence, the membership function of fuzzy set $\widetilde{C}$ can be decomposed as follows:

$$\mu_{\widetilde{C}}(s) = u_1(s) \vee u_2(s)$$

where

$$u_1(s) = \bigvee_{s_1 : s_1 \leq s} \left(\mu_{\widetilde{S}}(s_1) \wedge \mu_{\widetilde{T}}(s) \right) = \mu_{\widetilde{T}}(s) \wedge \left(\bigvee_{s_1 : s_1 \leq s} \mu_{\widetilde{S}}(s_1) \right)$$

$$u_2(s) = \bigvee_{s_1 : s_1 \leq s} \left(\mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(s_1) \right) = \mu_{\widetilde{S}}(s) \wedge \left(\bigvee_{s_1 : s_1 \leq s} \mu_{\widetilde{T}}(s_1) \right)$$

Then if $s \leq v_2$, the $\mu_{\widetilde{S}}(\cdot)$ and $\mu_{\widetilde{T}}(\cdot)$ are both non-decreasing functions. So we have $u_1(s) = \mu_{\widetilde{T}}(s) \wedge \mu_{\widetilde{S}}(s)$, and $u_2(s) = \mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(s)$, which lead to

$$\mu_{\widetilde{C}}(s) = \mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(s) = \mu_{\widetilde{S}}(s)$$

If $v_2 \leq s \leq v_1$, $\mu_{\widetilde{T}}(\cdot)$ is non-increasing, and $\mu_{\widetilde{S}}(\cdot)$ is non-decreasing. Then we have $u_1(s) = \mu_{\widetilde{T}}(s) \wedge \mu_{\widetilde{S}}(s)$, and $u_2(s) = \mu_{\widetilde{S}}(s) \wedge 1 = \mu_{\widetilde{S}}(s)$, which lead to

$$\mu_{\widetilde{C}}(s) = \mu_{\widetilde{S}}(s) \vee \left(\mu_{\widetilde{T}}(s) \wedge \mu_{\widetilde{S}}(s) \right) = \mu_{\widetilde{S}}(s)$$

If $v_1 \leq x$, $\mu_{\widetilde{T}}(\cdot)$ is non-increasing, and $\mu_{\widetilde{S}}(\cdot)$ is non-increasing. So we have $\sup_{s_1 : s_1 \leq s} \mu_{\widetilde{S}}(s) = 1$, and $\sup_{s_1 : s_1 \leq x} \mu_{\widetilde{T}}(s) = 1$. Then,

$$\mu_{\widetilde{C}}(s) = \mu_{\widetilde{S}}(s) \vee \mu_{\widetilde{T}}(s) = \mu_{\widetilde{S}}(s)$$

Hence $\widetilde{S} \cup \widetilde{T} = \widetilde{S}$.

□

Theorem 4 provides a more strict finding than that investigated by Ramik and Rimanex [15] in the context of fuzzification of the min and max operators, which states that $\widetilde{S} \succcurlyeq \widetilde{T} \iff$ there must be v_1, u_* and v_2 with $v_1 \geq u_* \geq v_2$, $\mu_{\widetilde{S}}(v_1) = \mu_{\widetilde{T}}(v_2) = 1$, $\mu_{\widetilde{S}}(s) \leq \mu_{\widetilde{T}}(s)$ for any $s < u_*$ and $\mu_{\widetilde{S}}(s) \geq \mu_{\widetilde{T}}(s)$ for any $s > u_*$.

Based on the α-cuts of fuzzy sets, the following order relation has been defined [15]:

Definition 5. *"For any fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$, an ordering relation $\widetilde{\geq}$ is defined as*

$$\widetilde{S} \widetilde{\geq} \widetilde{T} \iff \widetilde{S}_{\alpha+} \geq \widetilde{T}_{\alpha+} \text{ and } \widetilde{S}_{\alpha-} \geq \widetilde{T}_{\alpha-} \forall \alpha \in [0,\ 1]$$

where $\widetilde{S}_\alpha = \left[\widetilde{S}_{\alpha-}, \widetilde{S}_{\alpha+}\right]$ and $T_\alpha = \left[\widetilde{T}_{\alpha-}, \widetilde{T}_{\alpha+}\right]$ are the α-cuts of $\widetilde{S}$ and $\widetilde{T}$, respectively. "

The following example shows an ordering relation between two fuzzy sets.

Example 1 (Ordering Relation)**.** *Figure 1 illustrates two fuzzy sets such that $\widetilde{A} \widetilde{\geq} \widetilde{B}$.*

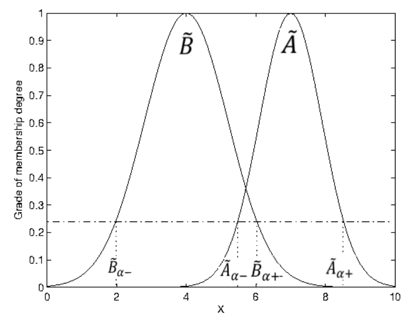

Fig. 1: Two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ having ordering relation.

The relation $\widetilde{\geq}$ is a partially ordered relation on $F(\mathbb{R})$, known as the *fuzzy max* order [15]. Interestingly, the two apparently different order relations , $\succcurlyeq$ and $\widetilde{\geq}$, are equivalent on $F(\mathbb{R})$ as it was proved in [15]:

Lemma 1. *The following three relations are equivalent for any fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$:*
i) $\widetilde{S} \widetilde{\geq} \widetilde{T}$; ii) $\widetilde{S} \cup \widetilde{T} = \widetilde{S}$; iii) $\widetilde{S} \cap \widetilde{T} = \widetilde{T}$

IV. Joinness and meetness of a T1OWA operator

A popular way to evaluate the behaviour of an OWA operator is to use the measure of *orness* and its dual *andness* proposed by Yager [2], [3]. The two measures aim to assess the similarity of an OWA operator with the maximum and minimum operators, respectively based on the associated weighting vector.

Similarly, in T1OWA aggregation, the following definitions of *joinness* and *meetness* associated with the linguistic weights evaluate how the T1OWA aggregation behaves like the operations of *join* and *meet*, respectively.

Definition 6. *For a T1OWA operator with fuzzy set weights $\left\{\widetilde{W}_i\right\}_{i=1}^n$ on $U \subseteq [0,\ 1]$, its joinness is:*

$$\mu_{joinness}(u) = \sup_{j_{\omega_1,\cdots,\omega_n} = u} \mu_{\widetilde{W}_1}(\omega_1) * \cdots * \mu_{\widetilde{W}_n}(\omega_n) \quad (14)$$

where $*$ is a t-norm operator, and

$$j_{\omega_1,\cdots,\omega_n} = \frac{1}{(n-1)\sum_{i=1}^{n} \omega_i} \sum_{i=1}^{n} (n-i)\omega_i \quad (15)$$

The corresponding meetness of the T1OWA is:

$$\mu_{meetness}(u) = \sup_{m_{\omega_1,\cdots,\omega_n} = u} \mu_{\widetilde{W}_1}(\omega_1) * \cdots * \mu_{\widetilde{W}_n}(\omega_n) \quad (16)$$

where

$$m_{\omega_1,\cdots,\omega_n} = 1 - \frac{1}{(n-1)\sum_{i=1}^{n} \omega_i} \sum_{i=1}^{n} (n-i)\omega_i \quad (17)$$

Clearly, the defined *joinness* and *meetness* of a T1OWA are fuzzy sets describing the linguistic expressions of aggregations behaving like the *join* and *meet*, respectively.

It is not difficult to calculate that the *joinness* and *meetness* of the *join* operator as a particular T1OWA operator (see the Theorem 2), are $joinness\left(\left\{\widetilde{W}_i\right\}_{i=1}^n\right) = \tilde{1}$ and $meetness\left(\left\{\widetilde{W}_i\right\}_{i=1}^n\right) = \tilde{0}$, which further confirms that this particular T1OWA operator is the *join* operator of fuzzy sets. Correspondingly, the *joinness* and *meetness* of the *meet* operator as a particular T1OWA operator (see the Theorem 3), are $joinness\left(\left\{\widetilde{W}_i\right\}_{i=1}^n\right) = \tilde{0}$ and $meetness\left(\left\{\widetilde{W}_i\right\}_{i=1}^n\right) = \tilde{1}$, confirming that this particular T1OWA operator is the *meet* operator of fuzzy sets.

Moreover, *Example 4* in the *Supplemental Material* depicts the *joinness* of the T1OWA operator shown in *Example 3* in the *Supplemental Material*.

V. Properties of T1OWA Operators

Yager's OWA operators possess the properties of *idempotence*, *monotonicity*, *compensativeness*, and *commutativity* [2]. In this section, we investigate the conditions for these properties to be verified by T1OWA operators.

Firstly, because Yager's OWA operators and the *sup* operators in the (6) and (7) are commutative, the T1OWA operator is *commutative* as well according to its definition in (2).

Theorem 5. *For any T1OWA operator* Φ *and* $\widetilde{A}^1, \cdots, \widetilde{A}^n \in F(\mathbb{R})$,

$$\Phi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right) = \Phi\left(\widetilde{A}^{p_1}, \cdots, \widetilde{A}^{p_n}\right)$$

where the sequence $\{p_1, \cdots, p_n\}$ *is a permutation of the sequence* $\{1, \cdots, n\}$.

The T1OWA operators with linguistic weights also verify the property of *idempotence* as addressed by the following Theorem.

Theorem 6. *For any fuzzy number* $\widetilde{A}$, *the T1OWA operators* Φ *with fuzzy number weights verify*

$$\Phi\left(\widetilde{A}, \cdots, \widetilde{A}\right) = \widetilde{A}$$

Proof. Let $y \in \mathbb{R}$ and $w_1, \cdots, w_n, a_1, \cdots, a_n \in \mathbb{R}$ such that $y = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$ with $\bar{w}_i = w_i \Big/ \sum_{i=1}^n w_i$. Convexity of $\widetilde{A} \in F(\mathbb{R})$ implies that

$$\begin{aligned}
&\mu_{\widetilde{A}}(y) = \mu_{\widetilde{A}}\left(\sum_{i=1}^n \bar{w}_i a_{\sigma(i)}\right) \\
&\geq \mu_{\widetilde{A}}(a_{\sigma(1)}) \wedge \cdots \wedge \mu_{\widetilde{A}}(a_{\sigma(n)}) \\
&= \mu_{\widetilde{A}}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(a_n) \\
&\geq \mu_{\widetilde{W}^1}(w_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(w_n) \wedge \mu_{\widetilde{A}}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(a_n)
\end{aligned}$$

The above inequality is true for any possible set of values $w_1, \cdots, w_n$, $a_1, \cdots, a_n \in \mathbb{R}$ such that $y = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$ and therefore it is true that

$$\begin{aligned}
\mu_{\widetilde{A}}(y) \geq & \sup_{\substack{\sum_{k=1}^n \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U, a_i \in \mathbb{R}}} \begin{pmatrix} \mu_{\widetilde{W}^1}(w_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(w_n) \\ * \mu_{\widetilde{A}^1}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}^n}(a_n) \end{pmatrix} \\
= & \mu_{\widetilde{Y}}(y)
\end{aligned}$$

where $\widetilde{Y} = \Phi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right)$.

In order to prove $\mu_{\widetilde{Y}}(y) = \mu_{\widetilde{A}}(y)$, we only need to find a specific combination of $\hat{w}_1, \cdots, \hat{w}_n$, $\hat{a}_1, \cdots, \hat{a}_n \in \mathbb{R}$ such that $\mu_{\widetilde{W}^1}(\hat{w}_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(\hat{w}_n) \wedge \mu_{\widetilde{A}}(\hat{a}_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(\hat{a}_n)$ reaches $\mu_{\widetilde{A}}(y)$. For $\widetilde{W}^i$ $(\forall i)$ being a fuzzy number, there exists at least one value $\hat{w}_i$ such that $\mu_{\widetilde{W}^i}(\hat{w}_i) = 1$ $(\forall i)$. Taking $\hat{a}_i = y$ $(\forall i)$, we have

$$\begin{aligned}
&\mu_{\widetilde{W}^1}(\hat{w}_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(\hat{w}_n) \wedge \mu_A(\hat{a}_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(\hat{a}_n) \\
&= \mu_{\widetilde{A}}(y) \wedge \cdots \wedge \mu_{\widetilde{A}}(y) \\
&= \mu_{\widetilde{A}}(y)
\end{aligned}$$

Consequently $\mu_{\widetilde{G}}(y) = \mu_{\widetilde{A}}(y)$. □

In what follows, we investigate how monotonicity is verified by T1OWA operators. Firstly, we propose the *alpha-equivalently-ordered* relation between two sets of fuzzy numbers:

Definition 7. *Let* $\left\{\widetilde{A}^i\right\}_{i=1}^n$ *and* $\left\{\widetilde{B}^i\right\}_{i=1}^n$ *be two sets of fuzzy numbers. The* σ *and* η *represent permutations of* $\{1, \cdots, n\}$ *defined by* $\left\{\widetilde{A}^i_{\alpha+}\right\}_{i=1}^n$ *and* $\left\{\widetilde{A}^i_{\alpha-}\right\}_{i=1}^n$, *respectively. If for any* $\alpha \in [0,1]$

$$\begin{aligned}
&\widetilde{A}^{\sigma(1)}_{\alpha+} \geq \widetilde{A}^{\sigma(2)}_{\alpha+} \geq \cdots \geq \widetilde{A}^{\sigma(n)}_{\alpha+} \Longrightarrow \\
&\widetilde{B}^{\sigma(1)}_{\alpha+} \geq \widetilde{B}^{\sigma(2)}_{\alpha+} \geq \cdots \geq \widetilde{B}^{\sigma(n)}_{\alpha+}
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{A}^{\eta(1)}_{\alpha-} \geq \widetilde{A}^{\eta(2)}_{\alpha-} \geq \cdots \geq \widetilde{A}^{\eta(n)}_{\alpha-} \Longrightarrow \\
&\widetilde{B}^{\eta(1)}_{\alpha-} \geq \widetilde{B}^{\eta(2)}_{\alpha-} \geq \cdots \geq \widetilde{B}^{\eta(n)}_{\alpha-}
\end{aligned}$$

then the fuzzy sets $\left\{\widetilde{B}^i\right\}_{i=1}^n$ *are said to be alpha-equivalently-ordered with the sets* $\left\{\widetilde{A}^i\right\}_{i=1}^n$.

The following example illustrates the *alpha-equivalently-ordered* relation, while *Example 7* in the *Supplemental Material* gives a counterexample of the *alpha-equivalently-ordered* relation.

Example 2 (Alpha-equivalently-ordered Relation)**.** *Figure 2 illustrates a group of three fuzzy numbers* $\{\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3\}$ *being alpha-equivalently-ordered with the group of three fuzzy numbers* $\{\widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3\}$.

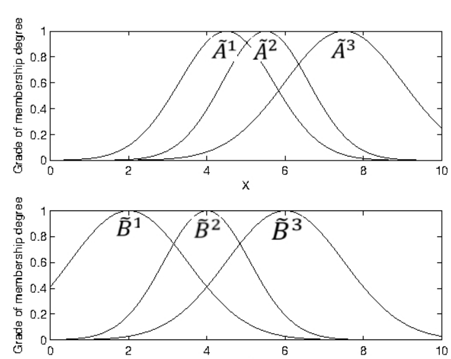

Fig. 2: *Alpha-equivalently-ordered* fuzzy numbers $\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3$ (bottom) with $\widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3$ (up)

The following Theorem states the conditions under which T1OWA operators are *monotonic* in the sense of partial order relation of fuzzy sets.

Theorem 7. *Let Φ be a T1OWA operator. Supposing the two sets of fuzzy numbers $\left\{\widetilde{A^i}\right\}_{i=1}^{n}$ and $\left\{\widetilde{B^i}\right\}_{i=1}^{n}$ be alpha-equivalently-ordered. If $\forall i$, $\widetilde{A^i} \succcurlyeq \widetilde{B^i}$, then*

$$\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right) \succcurlyeq \Phi\left(\widetilde{B}^1,\cdots,\widetilde{B}^n\right)$$

Proof. As defined in (3), for each $\alpha \in [0,1]$, the α-level aggregation of $\left\{\widetilde{A^i}\right\}_{i=1}^{n}$ by Φ is

$$\Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right) = \left\{ \frac{\sum_{i=1}^{n} w_i a_{\sigma(i)}}{\sum_{i=1}^{n} w_i} \middle| w_i \in \widetilde{W}_\alpha^i, a_i \in \widetilde{A}_\alpha^i, i = 1,\cdots,n \right\}$$

We know from Theorem 1 that $\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right)_\alpha = \Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right)$, therefore

$$\begin{aligned}\left(\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right)\right)_{\alpha+} &= \Phi_\alpha\left(\widetilde{A}_\alpha^1,\cdots,\widetilde{A}_\alpha^n\right)_+ \\ &= \max_{\substack{\widetilde{W}_{\alpha-}^i \le w_i \le \widetilde{W}_{\alpha+}^i \\ \widetilde{A}_{\alpha-}^i \le a_i \le \widetilde{A}_{\alpha+}^i}} \frac{\sum_{i=1}^{n} w_i a_{\sigma(i)}}{\sum_{i=1}^{n} w_i} \\ &= \max_{\widetilde{W}_{\alpha-}^i \le w_i \le \widetilde{W}_{\alpha+}^i} \frac{\sum_{i=1}^{n} w_i \widetilde{A}_{\alpha+}^{\sigma(i)}}{\sum_{i=1}^{n} w_i}\end{aligned}$$

Because $\widetilde{A^i} \succcurlyeq \widetilde{B^i}$, and $\left\{\widetilde{B^i}\right\}_{i=1}^{n}$ is *alpha-equivalently-ordered* with $\left\{\widetilde{A^i}\right\}_{i=1}^{n}$, then we have that $\widetilde{A}_{\alpha+}^{\sigma(1)} \ge \widetilde{A}_{\alpha+}^{\sigma(2)} \ge \cdots \ge \widetilde{A}_{\alpha+}^{\sigma(n)}$ implies $\widetilde{B}_{\alpha+}^{\sigma(1)} \ge B_{\alpha+}^{\sigma(2)} \ge \cdots \ge \widetilde{B}_{\alpha+}^{\sigma(n)}$. Thus,

$$\begin{aligned}\left(\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right)\right)_{\alpha+} &\ge \max_{\widetilde{W}_{\alpha-}^i \le w_i \le \widetilde{W}_{\alpha+}^i} \frac{\sum_{i=1}^{n} w_i \widetilde{B}_{\alpha+}^{\sigma(i)}}{\sum_{i=1}^{n} w_i} \\ &= \Phi_\alpha\left(\widetilde{B}_\alpha^1,\cdots,\widetilde{B}_\alpha^n\right)_+\end{aligned}$$

Because $\Phi\left(\widetilde{B}^1,\cdots,\widetilde{B}^n\right)_\alpha = \Phi_\alpha\left(\widetilde{B}_\alpha^1,\cdots,\widetilde{B}_\alpha^n\right)$, we conclude that $\left(\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right)\right)_{\alpha+} \ge \left(\Phi\left(\widetilde{B}^1,\cdots,\widetilde{B}^n\right)\right)_{\alpha+}$. A similar reasoning leads to $\left(\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right)\right)_{\alpha-} \ge \left(\Phi\left(\widetilde{B}^1,\cdots,\widetilde{B}^n\right)\right)_{\alpha-}$. Hence

$$\Phi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right) \succcurlyeq \Phi\left(\widetilde{B}^1,\cdots,\widetilde{B}^n\right)$$

□

The following example illustrates how the monotonic relation of aggregation in terms of $\succcurlyeq$ can be maintained for the aggregated objects which are *alpha-equivalently-ordered*.

Example 3 (Monotonic Relation)**.** *The fuzzy sets $\left\{\widetilde{B^i}\right\}_{i=1}^{3}$ depicted in Figure 2 are alpha-equivalently-ordered with $\left\{\widetilde{A^i}\right\}_{i=1}^{3}$, and $\widetilde{A^i} \succcurlyeq \widetilde{B^i}$ $(i = 1,2,3)$. Figure 4 illustrates the results of aggregating the fuzzy numbers in Figure 2 by a T1OWA operator Φ with the linguistic weights defined in Figure 3 respectively: $\bar{G} = \Phi\left(\widetilde{A}^1,\widetilde{A}^2,\widetilde{A}^3\right), \hat{G} = \Phi\left(\widetilde{B}^1,\widetilde{B}^2,\widetilde{B}^3\right)$. It is clear that for each $\alpha \in [0,1], \bar{G}_{\alpha-} \ge \hat{G}_{\alpha-}$, and $\bar{G}_{\alpha+} \ge \hat{G}_{\alpha+}$, i.e. $\bar{G} \succcurlyeq \hat{G}$.*

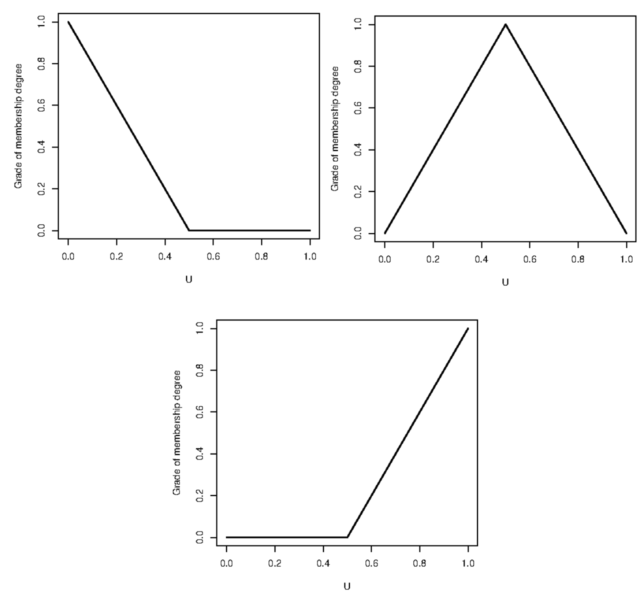

Fig. 3: Linguistic weights of a T1OWA operator: $\widetilde{W}^1$ (top-left), $\widetilde{W}^2$ (top-right), and $\widetilde{W}_3$ (bottom)

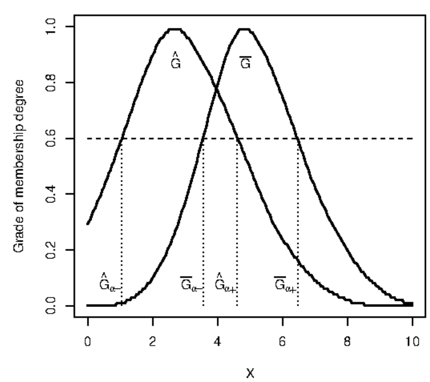

Fig. 4: Monotonic relation preserved in the results of aggregating the fuzzy numbers in Fig. 2 by a T1OWA operator defined by the linguistic weights in Fig.3

The next Theorem states that the *meet* and *join* operators are the lower bound and upper bound of T1OWA aggregation in the sense of *partial order relation*.

Theorem 8. *Any T1OWA operator Ψ is between the join, J, and the meet, M:*

$$J\left(\widetilde{A}^1,\cdots\widetilde{A}^n\right) \succcurlyeq \Psi\left(\widetilde{A}^1,\cdots,\widetilde{A}^n\right) \succcurlyeq M\left(\widetilde{A}^1,\cdots\widetilde{A}^n\right)$$

Proof. According to (3), for each $\alpha \in [0,1]$, the α-level aggregation of $\left\{\widetilde{A^i}\right\}_{i=1}^n$ by the T1OWA operator, J, is

$$J_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right) = \left\{ \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \middle| w_1 \in \tilde{1}_\alpha,\ w_i \in \tilde{0}_\alpha\ (i \neq 1),\ a_i \in \widetilde{A}_\alpha^i (\forall i) \right\}$$

We have that $\tilde{1}_\alpha = \{1\}$, $\tilde{0}_\alpha = \{0\}$. Thus,

$$\begin{aligned} J_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right) &= \left\{a_{\sigma(1)} | a_i \in \widetilde{A}_\alpha^i,\ i = 1, \cdots n\right\} \\ &= \left\{\max\{a_1, \cdots, a_n\} | a_i \in \widetilde{A}_\alpha^i (\forall i)\right\} \end{aligned}$$

As a result, the end points of the α-cut intervals are

$$J_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right)_+ = \max\left(\widetilde{A}_{\alpha+}^1, \cdots, \widetilde{A}_{\alpha+}^n\right);$$
$$J_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right)_- = \max\left(\widetilde{A}_{\alpha-}^1, \cdots, \widetilde{A}_{\alpha-}^n\right)$$

The α-level aggregation of $\left\{\widetilde{A^i}\right\}_{i=1}^n$ by a general T1OWA operator Ψ is,

$$\Psi_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right) = \left\{ \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \middle| w_i \in \widetilde{W}_\alpha^i, a_i \in \widetilde{A}_\alpha^i (\forall i) \right\}$$

Furthermore,

$$\begin{aligned} &\left(\Psi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right)\right)_{\alpha+} \\ &= \Psi_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right)_+ \\ &= \max_{\substack{\widetilde{W}_{\alpha-}^i \le w_i \le \widetilde{W}_{\alpha+}^i \\ \widetilde{A}_{\alpha-}^i \le a_i \le \widetilde{A}_{\alpha+}^i}} \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \\ &= \max_{\widetilde{W}_{\alpha-}^i \le w_i \le \widetilde{W}_{\alpha+}^i} \frac{\sum_{i=1}^n w_i \widetilde{A}_{\alpha+}^{\sigma(i)}}{\sum_{i=1}^n w_i} \\ &\le \max_{\widetilde{W}_{\alpha-}^i \le w_i \le \widetilde{W}_{\alpha+}^i} \frac{\sum_{i=1}^n w_i \max\left(\widetilde{A}_{\alpha+}^1, \cdots, \widetilde{A}_{\alpha+}^n\right)}{\sum_{i=1}^n w_i} \\ &= \max\left(\widetilde{A}_{\alpha+}^1, \cdots, \widetilde{A}_{\alpha+}^n\right) \end{aligned}$$

We have proved that $\left(\Psi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right)\right)_{\alpha+} \le J_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right)_+$. Similarly, we have $\left(\Psi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right)\right)_{\alpha-} \le J_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right)_-$. So we prove that

$$J\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right) \succcurlyeq \Psi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right)$$

We omit the proof of the other inequality: $\Psi\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right) \succcurlyeq M\left(\widetilde{A}^1, \cdots, \widetilde{A}^n\right)$, because it follows the same above line of reasoning. □

According to Theorem 8, the *join* and *meet* operators are two extreme cases of T1OWA operators. T1OWA aggregation is located between the *meet* and the *join* of all the individual operands, i.e., T1OWA operators are *compensative*: low aggregation in the sense of approaching the *meet* operation is compensated by high aggregation in the sense of approaching the *join* operation.

Example 8 in the *Supplemental Material* illustrates the validation of how a T1OWA aggregation maintains the compensative property in terms of *partial roder relations* of fuzzy numbers.

VI. A Case Study and Experimental Results

A. Diabetes Diagnosis by T1OWA Based Fuzzy Inference System

T1OWA operators have gained many real-world applications in different domains [10], [12], [13]. In this subsection, we further provide a practical application of T1OWA operators to 'Pima Indian Diabetes' for integrated patient diagnosis.

The 'Pima Indian Diabetes' dataset [18] describes the clinical conditions of 768 females who develop Type-2 diabetes. All patients in this dataset were women ($\geq$ 21 years old): 500 (65.1%) healthy and 268 (34.9%) with diabetes. It can be seen that this is an imbalanced dataset. Eight attributes describe the patients: age (years), plasma glucose concentration (*plaGlu*), number of times pregnant, triceps skin fold thickness (mm), diastolic blood pressure (mmHg), body mass index (*BMI*) ((weight in kg)/(height in m^2)), 2-hour serum insulin ($mmol/L$), and diabetes pedigree function. The outcome is a class variable (0 or 1): *1=diabetic*, *0=non-diabetetic*.

In contrast to other studies using the 'Pima Indian Diabetes' dataset, we take into account two further underlying issues with the data. One issue is that clearly this is an imbalanced dataset: hence, the widely used assessment metric, *classification rate* (CR) (also known as *accuracy*), is not appropriate and not reliable to assess such a real clinical scenario. For imbalanced datasets, which are very common in clinical studies, the F_1-score and balanced CR (BCR) are a preferred metric, as they makes more sense than others: $F_1 - score = (2 \cdot recall \cdot precision)/(recall + precision)$; $BCR = (sensitivity + specificity)/2$.

The second issue is that the majority of the existing studies using this dataset did not consider its underlying missing value problem. Indeed, there are no specifically labelled missing values in the dataset. But this cannot be the case, because so many zeros are used to represent the status of attributes where they are biologically impossible, such as the attributes of glucose concentration (5 records of zeros), triceps skin fold thickness (227 records of zeros), blood pressure (35 records of zeros), insulin (374 records of zeros), and body mass index (11 records of zeros). It is highly plausible that these zero values were actually originally used to encode missing values in these fields. In our study, we considered these zeros as missing values and used the nearest-neighbor method to impute them. Then we apply different T1OWA aggregations to non-stationary fuzzy sets [16], [17] to find the optimal diagnoses of diabetes.

Given a standard fuzzy system as a baseline system, the T1OWA based non-stationary fuzzy system

(T1OWANFS) (see Figure 8 in the *Supplemental Material*) proceeds as follows. First, in each run, the crisp numbers of each input variable are fuzzified by a fuzzifier function, such as singleton or non-singleton function. These fuzzified input sets then feed into the inference engine with the given rulebase to conduct operations of union and intersection on these fuzzy sets, and perform composition of the relations. Such a process is repeated n runs, so n fuzzy set outputs are produced. Then a T1OWA aggregation operation is applied to these sets to achieve an overall solution. Finally, a crisp output is generated via defuzzification of this overall output fuzzy set.

The rulebase in this study consists of the following four rules based on two attributes of *plaGlu* and *BMI*:

1) *Rule 1: if (plaGlu is high) then Diabetic;*
2) *Rule 2: if (plaGlu is medium) and (BMI is high) then Diabetic;*
3) *Rule 3: if (plaGlu is low) then Non-Diabetic;*
4) *Rule 4: if (plaGlu is medium) and (BMI is low) then Non-Diabetic.*

where the variables *plaGlu*, *BMI* and *outcome* are described by baseline fuzzy sets (see Figure 9 of the *Supplemental Material*). Their corresponding non-stationary fuzzy sets were generated based on these baseline sets [16], [17]. In our study, the non-stationary fuzzy system ran ten times to generate the diagnoses for each patient (Figure 10 in the *Supplemental Material* shows an example of ten fuzzy decision outputs for a patient). The system performance is evaluted in terms of F_1-score and BCR.

We use five different types of T1OWA operators in this case study to aggregate the fuzzy diagnosis from non-stationary fuzzy inference engine (see Figure 8 in the *Supplemental Material*):

1) the standard *join* operator: denoted as *join_NFS*;
2) the standard *meet* operator: denoted as *meet_NFS*;
3) join-like T1OWA operators with the linguistic weight $\widetilde{W}_1$ as in Figure 3a and others $\widetilde{W}_i (i \neq 1)$ as in Figure 3b in the *Supplemental Material* to aggregate the 10 output sets for diabetes diagnosis: denoted as *JLT1OWA_NFS*;
4) meet-like T1OWA operators with the last linguistic weight $\widetilde{W}_{10}$ as in Figure 3a and others $\widetilde{W}_i (i \neq 10)$ as in Figure 3b in the *Supplemental Material*: denoted as *MLT1OWA_NFS*;
5) a T1OWA operator with linguistic weights implementing the fuzzy majority represented by the type-2 quantifier *'most'* [4]: denoted as *T2MT1OWA_NFS*.

Figure 5 depicts an example of corresponding results of aggregating ten fuzzy decisions from non-stationary fuzzy inference engine for one patient (See Figure 10 in the *Supplemental Material*) by the above five T1OWA operators. These aggregated fuzzy outputs are then defuzzified to generate crisp values as final outputs.

The above examples demonstrate the advantages of these T1OWA operators to aggregate the uncertain information modelled by fuzzy sets. The *T2MT1OWA_NFS* operator implements the *'soft'* majority in aggregating a group of uncertain decisions (perhaps expressed linguistically as *"most of the decisions should be satisfied"*), which is much closer to the real human perception in decision making than traditional aggregation methods. Figure 11 in the *Supplemental Material* illustrates the *joinness* of the operator, *T2MT1OWA_NFS*, which clearly shows that the quantifier *'most'* guided operator approaches the meet operation (expressed linguistically as *"all decisions should be satisfied"*). As a matter of fact, such a linguistic quantifier based aggregation can be treated as a manifestation of a semantically guided aggregation [2], [4].

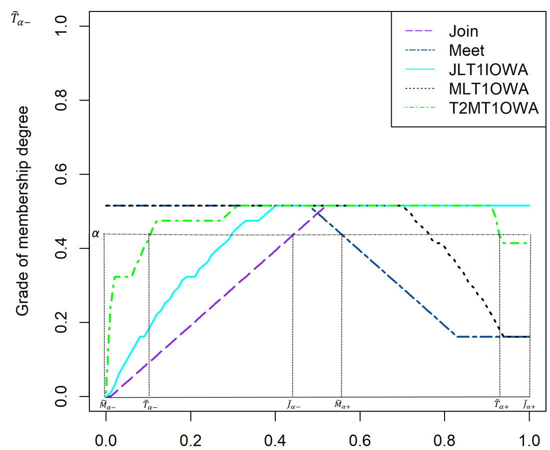

Fig. 5: Example of aggregation results of 10 fuzzy output decisions (from non-stationary fuzzy inference engine for a patient) by the five different T1OWA operators.

To validate the *compensative* property of these T1OWA operators, let us assume the aggregation results of the five operators shown in Figure 5 (*join_NFS*, *meet_NFS*, *JLT1OWA_NFS*, *MLT1OWA_NFS* and *T2MT1OWA_NFS*) be represented as $\widetilde{J}$, $\widetilde{M}$, $\widetilde{JL}$, $\widetilde{ML}$, and $\widetilde{T}$, respectively. Taking *T2MT1OWA_NFS* as an example, for any α level in Figure 5, it can be seen that

$$\widetilde{J}_{\alpha+} \geq \widetilde{T}_{\alpha+} \geq \widetilde{M}_{\alpha+} \text{ and } \widetilde{J}_{\alpha-} \geq \widetilde{T}_{\alpha-} \geq \widetilde{M}_{\alpha-} \forall \alpha \in [0, 1]$$

Therefore, according to Definition 5, and because the relations $\succcurlyeq$ and $\gtrapprox$ are equivalent, we get $\widetilde{J} \succcurlyeq \widetilde{T} \succcurlyeq \widetilde{M}$, i.e., the T2MT1OWA operator verifies *Theorem* 8, and the *compensative* property holds in this case study.

Furthermore, we show a comparison (in terms of F_1-score and BCR) with the following existing methods: (i) standard fuzzy weighted average (FWA) operators [19]; and (ii) a zero-order Takagi-Sugeno fuzzy system with two rules (TSFSTR) [20]. Table I summarises the performances of these approaches. It can be seen that the

TABLE I: Performances of different approaches to diabetes diagnosis

Approach	CR	Recall	Specificity	Precision	F_1-score	BCR
Meet_NFS	0.760	0.519	0.890	0.716	0.602	0.704
MLT1OWA_NFS	0.760	0.519	0.890	0.716	0.602	0.704
T2MT1OWA_NFS	0.759	0.586	0.852	0.680	0.629	0.719
JLT1OWA_NFS	0.746	0.683	0.780	0.625	**0.652**	**0.731**
Join_NFS	0.746	0.683	0.780	0.625	**0.652**	**0.731**
FWA	0.751	0.619	0.822	0.651	0.635	0.721
TSFSTR	0.716	0.455	0.856	0.629	0.528	0.656

TABLE II: Time-costs of Type-1 OWA Aggregations for Diabetes Diagnoses (in minutes)

Setting	$n_x = 100, n_u = 30, n = 10$		$n_x = 10, n_u = 5, n = 3$	
Method	**Alpha-Level Approach**	**Direct Approach**	**Alpha-Level Approach**	**Direct Approach**
join_NFS	2.927	Infeasible	0.295	8.786
JLT1OWA_NFS	2.861	Infeasible	0.294	123640.300
MLT1OWA_NFS	2.919	Infeasible	0.302	109579.800
meet_NFS	2.982	Infeasible	0.297	8.589

JLT1OWA_NFS and *join* achieved the best performance in terms of F_1-score and BCR. The JLT1OWA_NFS significantly improved the recall without sacrificing much precision, so that a better F_1-score was achieved.

B. Validation of Computing Efficiencies of Alpha-Level Approach to T1OWA Operations in Real World Applications

The *Direct Approach* [4] and *Alpha-Level Approach* [5] generate the same results of aggregating fuzzy sets as shown in the *Representation Theorem of Type-1 OWA Operators* (Theorem 1). However, the *Direct Approach* is an exponential-time algorithm that takes $O(K^n)$ operations [5], in which the constant K depends on $n_u \cdot n_x$, where n is the number of fuzzy sets to be aggregated, n_u is the number of sampling points on the domain [0, 1] of the T1OWA operator's linguistic weights, and n_x is the number of sampling points on the domain of the fuzzy sets to be aggregated. In comparison, the *Alpha-Level Approach* is a linear-time algorithm, taking $O(n)$ operations [5]. Therefore, the *Alpha-Level Approach* can be used to implement T1OWA aggregations in real-time applications.

In the subsection VI-A, the T1OWA based fuzzy decision making for diabetes was implemented by the *Alpha-Level Approach*. The domains of the linguistic weights and fuzzy sets have to be discretized. The default settings are: $n_u = 30$ and $n_x = 100$, while $n = 10$ (i.e. ten fuzzy decisions). However, such settings are unworkable for the *Direct Approach* in implementation due to the oversized vectors which need to be created by the computer. For comparison, therefore, simplified settings are used, such that $n_u = 5$ and $n_x = 10$, while $n = 3$. Even under such simplified settings, it is estimated that the *Direct Approach* still takes days to complete the diagnoses for all 768 patients using the *meet-like* or *join-like* T1OWA operators. Our solution to calculation of time-cost was: firstly, the time-cost, tc_1, of the *Direct Approach* to aggregating three fuzzy decisions for only one patient is calculated; then, the total time-cost is $768 \cdot tc_1$. Table II shows the time-costs of the two approaches to diagnosing diabetic patients by the T1OWA operators, *join_NFS*, *meet_NFS*, *JLT1OWA_NFS* and *MLT1OWA_NFS*. This validated that the *Alpha-Level Approach* can achieve much higher computing efficiency than the *Direct Approach* to aggregating fuzzy sets in the manner of OWA operation in real-world applications.

The experimental results were generated in *R* on a computer with Intel(R) Core i5-4440@3.10GHz and 16GB memory. The *R* codes for type-1 OWA aggregations are available upon request.

VII. Discussion and Conclusion

As a generalization of Yager's OWA operator, T1OWA operators provide an efficient tool to aggregate uncertain information modelled by fuzzy sets in soft decision making. By appropriately selecting fuzzy sets for the weights, various forms of T1OWA operators can be created to fulfill different tasks under multi-granular linguistic contexts. This has been demonstrated in the above case study of diabetic diagnosis, which is an imbalanced data problem. By appropriately selecting the uncertain weights to favour the rare class, such as the *join-like* T1OWA operator for the diabetic class, the T1OWA aggregation approach has the potential to enable standard classifiers to be a cost-sensitive approach, whereby the cost of misclassifying the rare class is higher than the cost of misclassifying the other class. This topic merits further research. In addition, to date, T1OWA operators only consider the t-norm (minimum) and t-conorm (maximum); but how T1OWA aggregations and properties vary by using different forms of t-norm and t-conorm is an interesting research problem.

In summary, this paper has proven that T1OWA operators verify the same properties which hold for Yager's OWA operator, namely: *idempotence*, *monotonicity*, *compensativeness*, and *commutativity*. Such theoretical analyses provide a solid foundation for T1OWA operators to be applied widely in different scenarios.

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Supplemental Material for the Manuscript *"Type-1 OWA Operators in Aggregating Multiple Sources of Uncertain Information : Properties and Real World Applications"*

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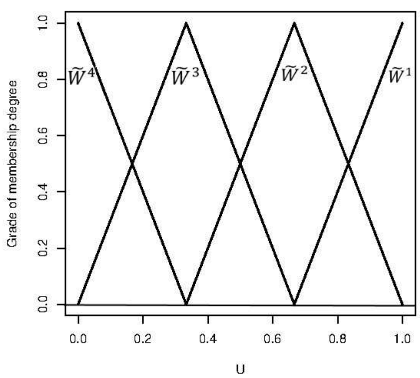
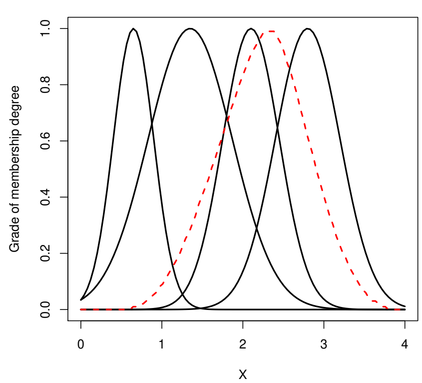
I. More examples of Type-1 OWA Aggregations

Example 1 (A T1OWA Operator). *Figure 1 depicts the results of a T1OWA operator to aggregate four fuzzy sets. Figure 1a shows the linguistic weights. Figure 1b shows the aggregated fuzzy set objects and aggregation result.*



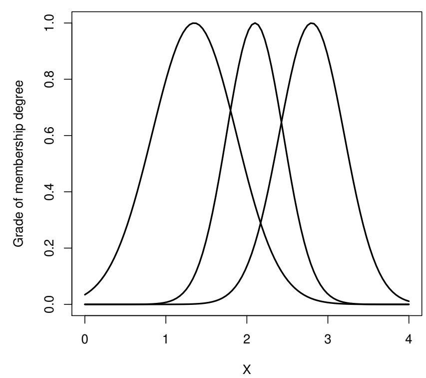
(a) Fuzzy sets as linguistic weights in T1OWA aggregation



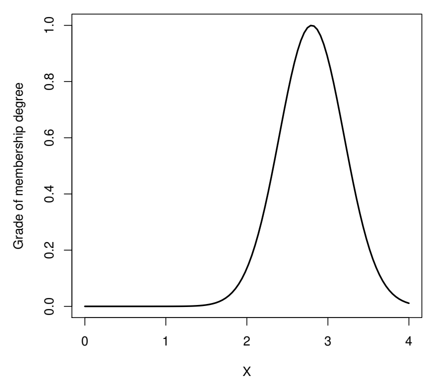
(b) Aggregation result in dashed lines; aggregated fuzzy set objects in solid lines

Fig. 1: A T1OWA operator

Example 2 (Join Operator). *Figure 2 illustrates the result of join operator to aggregate fuzzy sets as the extended maximum of fuzzy sets, a special T1OWA operator.*



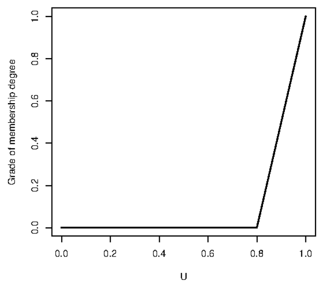
(a) Fuzzy sets to be aggregated



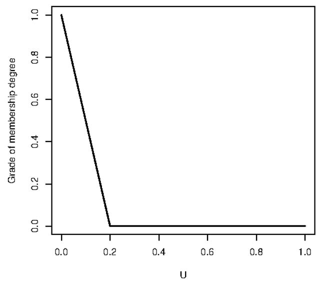
(b) Aggregating result

Fig. 2: Aggregation of T1OWA operator as join

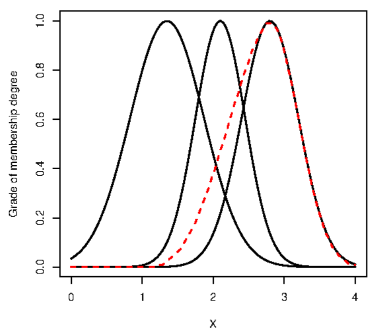
Example 3 (Join-Like T1OWA Operator). *The Join-like T1OWA operation requests that the first linguistic weight move towards the singleton fuzzy set, $\tilde{1}$, and all others towards the singleton fuzzy set $\tilde{0}$ (Figure 4a and Figure 4b). Using these weights to aggregate the fuzzy sets, Figure 3c illustrates how this T1OWA exhibits the join-like behaviour when aggregating three fuzzy sets. Clearly, the final aggregation fuzzy set is closer to the rightmost fuzzy set (i.e., the fuzzified maximum)*



(a) $\widetilde{W}_1$

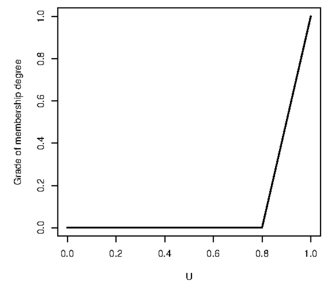


(b) $\widetilde{W}_i$ $(i \neq 1)$

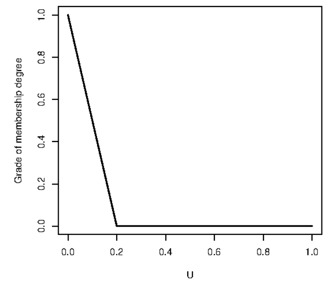


(c) Aggregation result

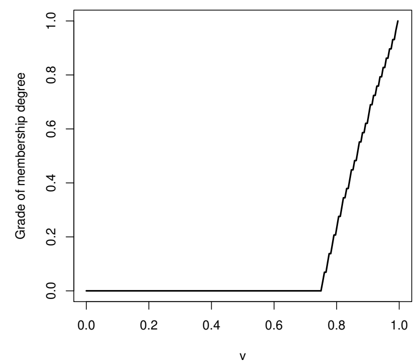
Fig. 3: A join-like T1OWA operator. (a)(b)-Linguistic weights;(c)-Aggregation result. Solid lines representing the aggregated objects; dashed line representing the aggregation result.



(a) W_1



(b) W_i $(i \neq 1)$



(c) Joinness

Fig. 4: *Joinness* of a join-like T1OWA operator

Example 4 (Joinness of Join-Like T1OWA Operator)**.** *The Figure 4c shows an example of the joinness of the join-like T1OWA operator defined by the fuzzy weights in Figure 4a and Figure 4b.*

Example 5 (Meet Operator)**.** *Figure 5 illustrates the result of meet operator to aggregate fuzzy sets as the extended minimum of fuzzy sets, a special T1OWA operator.*

Example 6 (Meet-Like T1OWA Operator)**.** *Meet-like T1OWA operation requests that the last linguistic weight to be close to $\tilde{1}$ (Figure 4a), and all other weights close to $\tilde{0}$ (Figure 4b). Figure 6 illustrates how this T1OWA exhibits the meet-like behaviour when aggregating three fuzzy sets. Clearly, the final aggregation fuzzy set is closer to the leftmost fuzzy set (i.e., the fuzzified minimum).*

Example 7 (Counterexample of α-Equivalently Ordered Relation)**.** *Figure 7 shows the groups of fuzzy numbers $\{\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3\}$ are not α-equivalently ordered with another group $\{\widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3\}$. Figure 7a shows that at the $\alpha = 0.2$ level: $\widetilde{A}^3_{0.2+} \geq \widetilde{A}^2_{0.2+} \geq \widetilde{A}^1_{0.2+}$, i.e., the permutation operator $\sigma = (3, 2, 1)$, but $\widetilde{B}^3_{0.2+} \geq \widetilde{B}^1_{0.2+} \geq \widetilde{B}^2_{0.2+}$; while Fig.7b shows that $\widetilde{A}^3_{0.2-} \geq \widetilde{A}^2_{0.2-} \geq \widetilde{A}^1_{0.2-}$, i.e., the permutation operator $\eta = (3, 2, 1)$, but $\widetilde{B}^3_{0.2-} \geq \widetilde{B}^1_{0.2-} \geq \widetilde{B}^2_{0.2-}$.*

Example 8. *Supposing the numerical domains $U = \{0.0, 0.5, 1.0\}$ and $X = \{0.0, 1.0, 2.0\}$. Let the given linguistic weights $\widetilde{W} = \begin{pmatrix} \omega_i \\ \mu_{\widetilde{W}}(\omega_i) \end{pmatrix}_{\omega_i \in U}$ on U be*

$$\widetilde{W}^1 = \begin{pmatrix} 0.0 & 0.5 & 1.0 \\ 1.0 & 0.5 & 0.0 \end{pmatrix}; \quad \widetilde{W}^2 = \begin{pmatrix} 0.0 & 0.5 & 1.0 \\ 0.0 & 1.0 & 0.0 \end{pmatrix};$$
$$\widetilde{W}^3 = \begin{pmatrix} 0.0 & 0.5 & 1.0 \\ 0.0 & 0.5 & 1.0 \end{pmatrix}$$

and the fuzzy sets to be aggregated be

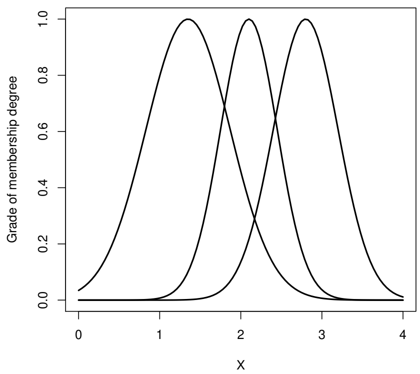
$$\widetilde{A}^1 = \begin{pmatrix} 0.0 & 1.0 & 2.0 \\ 0.0 & 0.5 & 1.0 \end{pmatrix}; \quad \widetilde{A}^2 = \begin{pmatrix} 0.0 & 1.0 & 2.0 \\ 1.0 & 0.5 & 0.0 \end{pmatrix};$$
$$\widetilde{A}^3 = \begin{pmatrix} 0.0 & 1.0 & 2.0 \\ 0.0 & 1.0 & 0.0 \end{pmatrix}$$

The aggregation result by the join operator J can be calculated as follows.

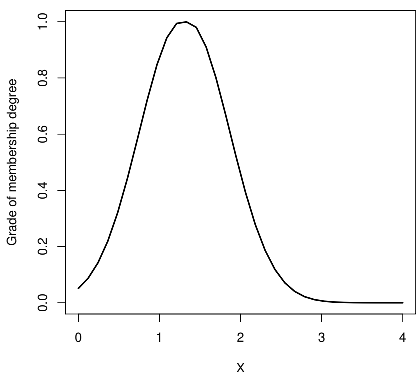
1) *For $\alpha = 0.0$,*

$$J_\alpha\left(\widetilde{A}^1_\alpha, \widetilde{A}^2_\alpha, \widetilde{A}^3_\alpha\right)_+ = \max\left(\widetilde{A}^1_{\alpha+}, \widetilde{A}^2_{\alpha+}, \widetilde{A}^3_{\alpha+}\right) = 2.0$$
$$J_\alpha\left(\widetilde{A}^1_\alpha, \widetilde{A}^2_\alpha, \widetilde{A}^3_\alpha\right)_- = \max\left(\widetilde{A}^1_{\alpha-}, \widetilde{A}^2_{\alpha-}, \widetilde{A}^3_{\alpha-}\right) = 0.0$$

So $G_{\alpha-} \leq J_{\alpha-}, G_{\alpha+} \leq J_{\alpha+}$.



(a) Fuzzy sets to be aggregated



(b) Aggregating result

Fig. 5: Aggregation of T1OWA operator as meet

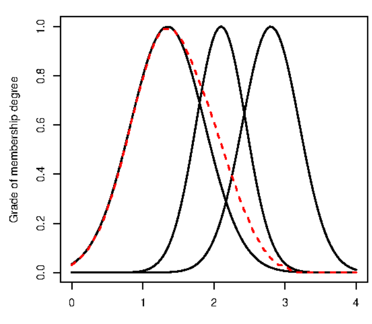
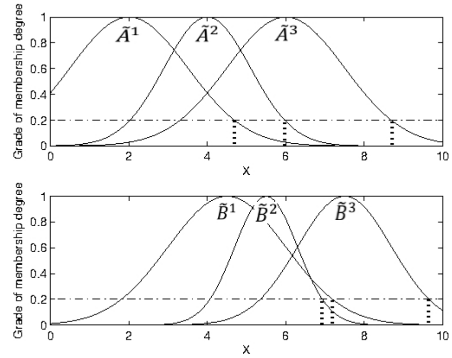
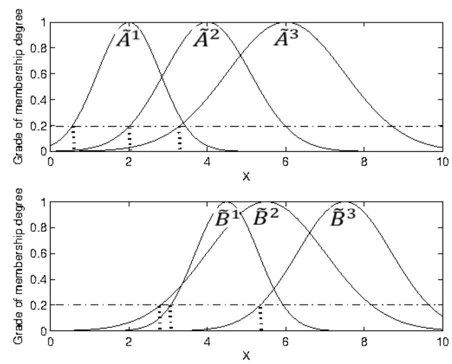


Fig. 6: Aggregation result by a meet-like T1OWA operator. Solid lines representing the aggregated objects; dashed line representing the aggregation result.



(a) $\widetilde{A}^3_{0.2+} \geq \widetilde{A}^2_{0.2+} \geq \widetilde{A}^1_{0.2+}$ and $\widetilde{B}^3_{0.2+} \geq \widetilde{B}^1_{0.2+} \geq \widetilde{B}^2_{0.2+}$



(b) $\widetilde{A}^3_{0.2-} \geq \widetilde{A}^2_{0.2-} \geq \widetilde{A}^1_{0.2-}$ and $\widetilde{B}^3_{0.2-} \geq \widetilde{B}^1_{0.2-} \geq \widetilde{B}^2_{0.2-}$

Fig. 7: Fuzzy numbers $\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3$ (bottom) not *α-equivalently ordered* with $\widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3$ (up) separately

2) *For $\alpha = 0.5$,*

$$\begin{aligned} J_\alpha\left(\widetilde{A}^1_\alpha, \widetilde{A}^2_\alpha, \widetilde{A}^3_\alpha\right)_+ &= \max\left(\widetilde{A}^1_{\alpha+}, \widetilde{A}^2_{\alpha+}, \widetilde{A}^3_{\alpha+}\right) \\ &= \max\{2.0, 1.0, 1.0\} \\ &= 2.0 \\ J_\alpha\left(\widetilde{A}^1_\alpha, \widetilde{A}^2_\alpha, \widetilde{A}^3_\alpha\right)_- &= \max\left(\widetilde{A}^1_{\alpha-}, \widetilde{A}^2_{\alpha-}, \widetilde{A}^3_{\alpha-}\right) \\ &= \max\{1.0, 0.0, 1.0\} \\ &= 1.0 \end{aligned}$$

So $G_{\alpha-} \leq J_{\alpha-}, G_{\alpha+} \leq J_{\alpha+}$.

3) *For $\alpha = 1.0$*

$$\begin{aligned} J_\alpha\left(\widetilde{A}^1_\alpha, \widetilde{A}^2_\alpha, \widetilde{A}^3_\alpha\right)_+ &= \max\left(\widetilde{A}^1_{\alpha+}, \widetilde{A}^2_{\alpha+}, \widetilde{A}^3_{\alpha+}\right) \\ &= \max\{2.0, 0.0, 1.0\} \\ &= 2.0 \\ J_\alpha\left(\widetilde{A}^1_\alpha, \widetilde{A}^2_\alpha, \widetilde{A}^3_\alpha\right)_- &= \max\left(\widetilde{A}^1_{\alpha-}, \widetilde{A}^2_{\alpha-}, \widetilde{A}^3_{\alpha-}\right) \\ &= \max\{2.0, 0.0, 1.0\} \\ &= 2.0 \end{aligned}$$

So $G_{\alpha-} \leq J_{\alpha-}, G_{\alpha+} \leq J_{\alpha+}$.

Then according to the Definition about partial order relation of fuzzy sets, we have $J \succcurlyeq G$. Similarly, $G \succcurlyeq M$, the meet operator.

II. Case Study

A. The T1OWA based non-stationary fuzzy system

Figure 8 illustrates the structure of T1OWA based non-stationary fuzzy inference system, the T1OWA operator is used to aggregate the multiple fuzzy decisions from non-stationary fuzzy inference engine.

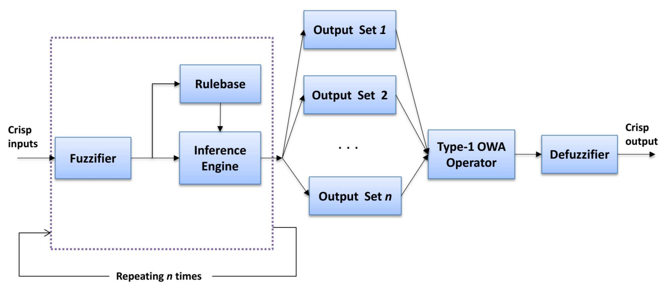


Fig. 8: T1OWA based non-stationary fuzzy inference system

B. Fuzzy sets of the variables - "plaGlu", "BMI" and "Outcome"

joinnessOfT2QuantMostPrecision0d0035

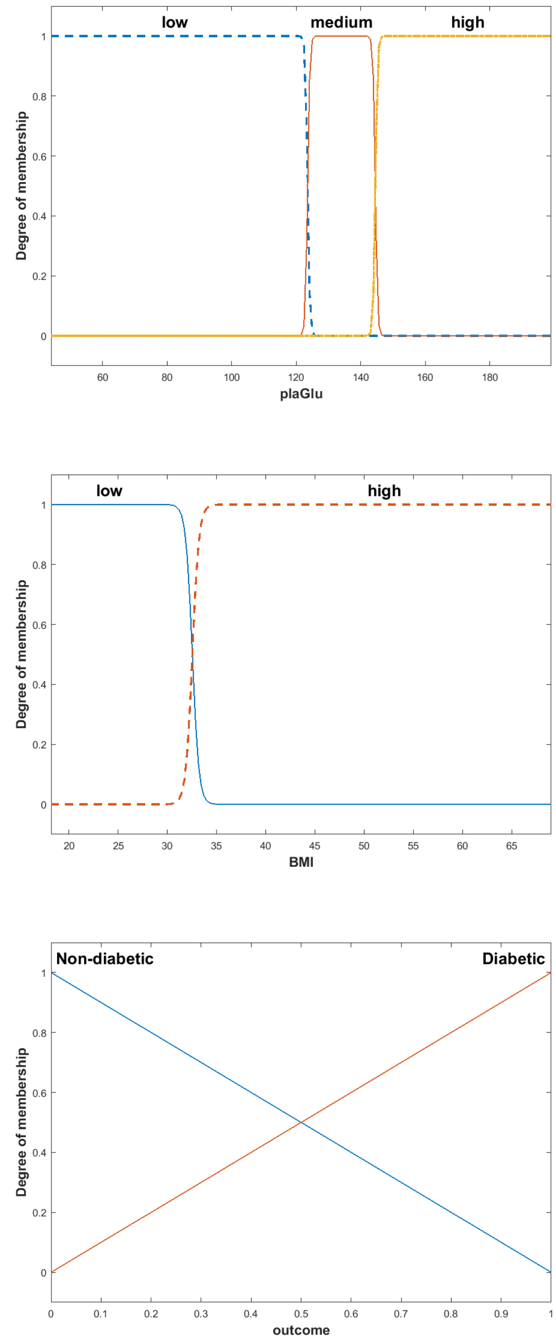
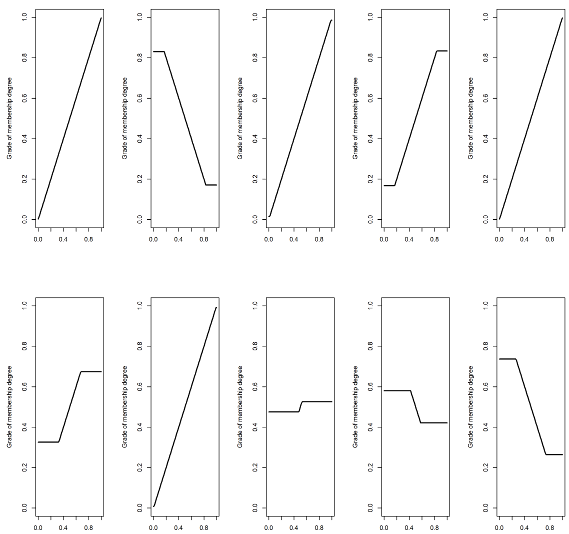


Fig. 9: Membership functions of fuzzy sets for the attributes of *plaGlu, BMI and outcome*



(a) 10 fuzzy output decisions

Fig. 10: Example of 10 fuzzy output decisions by the non-stationary fuzzy system for a patient

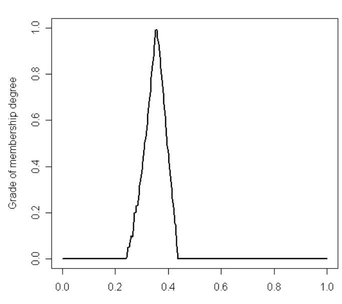


Fig. 11: Joinness of the type-2 quantifier "most" guided T1OWA operator