A State-Based Peridynamic Formulation for Functionally Graded Kirchhoff Plates

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Abstract

Functionally graded materials is a potential alternative to traditional fiber reinforced composite materials since they have continuously varying material properties which does not cause stress concentrations. In this study, a state-based peridynamic model is presented for functionally graded Kirchhoff plates. Equations of motion of the new formulation is obtained by using Euler-Lagrange equation and Taylor’s expansion. The formulation is verified by considering several benchmark problems including a clamped plate subjected to transverse loading and simply supported plate subjected to transverse loading and inclined loading. The material properties are chosen such that Young’s Modulus is assumed to be varied linearly through the thickness direction and Poisson’s ratio is constant. Peridynamic results are compared against finite element analysis results and a very good agreement is obtained between the two approaches.

Keywords: Peridynamics; Kirchhoff Plate; Functionally Graded; State-Based; Euler-Lagrange equation; Non-local

1. Introduction

Functionally graded materials (FGMs) is a potential alternative to traditional fiber reinforced composite materials (FRCMs) since they have continuously varying material properties which does not cause stress concentrations. Such stress concentrations arise in FRCMs and cause delamination type failure. With the advancement of additive manufacturing technologies, it is expected that the usage of FGMs will increase in the near future.

There are many studies in the literature focusing on analysis of FGMs. Amongst these Vel and Batra [1] presented three dimensional exact solutions for free and forced vibrations of simply supported functionally graded plates using power series method and considering classical plate theory, first order shear deformation theory and third order shear deformation theory. Zenkour [2] studied the bending deformation of a simply supported functionally graded plate subjected to transverse uniform load by using generalised shear deformation theory without using shear correction factors. Shen [3] performed nonlinear bending analysis for simply supported functionally graded plates subjected to transverse loading by taking into account the effect of temperature and using Reddy’s higher-order shear deformation plate theory. Bian et. al. [4] utilised a plate theory using the concept of shape function of the transverse coordinate parameter to investigate functionally graded plates subjected to cylindrical bending. Carrera et al. [5] examined the effect of thickness stretching in functionally graded plates by utilising Carrera’s Unified Formulation. As an alternative formulation, a new continuum mechanics formulation, peridynamics, can be utilised.

Peridynamics (PD) was introduced by Silling [6] to overcome the limitations of Classical Continuum Mechanics (CCM). Governing equations of PD is in the form of integro-differential equations and do not contain spatial derivatives. Therefore, these equations are still valid even if the displacement field is discontinuous due to existence of cracks. Moreover, PD is a non-local continuum mechanics formulation and it has a length scale parameter, horizon, which defines range of non-local interactions between material points. PD has been used for analysing different types of materials including metals and composites [7-10] and is suitable for multiphysics analysis [11-15]. Moreover, several peridynamic formulations are available in the literature to analyse beams, plates and shells [16-21].

In this study, peridynamics is utilised to investigate the bending behaviour of functionally graded plates based on Kirchhoff Plate Theory for the first time in the literature. The governing equations are obtained by
using Euler-Lagrange equation and Taylor’s expansion. Several benchmark problems are considered to validate the newly developed formulation.

2. Functionally Graded Kirchhoff Plate

Kirchhoff developed a complete and adequate set of equation for linear theory of thin plates. According to the Kirchhoff plate theory, it is assumed that a transverse normal to the mid-surface in the undeformed state remains straight, normal to the mid-surface and it has no change in length during deformation. Moreover, the displacement field of any material point can be represented in terms of the displacement field of the material points at the mid-surface as

\[ u_1(x_1, x_2, z) = \bar{u}_1(x_1, x_2) + z \cdot \bar{u}_1(x_1, x_2) \quad x \\
\]
\[ u_2(x_1, x_2, z) = \bar{u}_2(x_1, x_2) + z \cdot \bar{u}_2(x_1, x_2) \quad x \\
\]
\[ w(x_1, x_2, z) = \bar{w}(x_1, x_2) \]

where \( \bar{u}_1, \bar{u}_2 \) and \( \bar{w} \) denote the displacement of a material point at the mid-surface in \( x_1 \), \( x_2 \) and \( z \) directions, respectively. Thus, the strain-displacement relationships can be written as

\[ \varepsilon_{11} = \frac{\partial \bar{u}_1}{\partial x_1} + z \frac{\partial \theta_1}{\partial x_1} \quad (2a) \]
\[ \varepsilon_{22} = \frac{\partial \bar{u}_2}{\partial x_2} + z \frac{\partial \theta_2}{\partial x_2} \quad (2b) \]
\[ \varepsilon_{33} = 0 \quad (2c) \]
\[ \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2} \left[ \frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1} + z \left( \frac{\partial \theta_1}{\partial x_2} + \frac{\partial \theta_2}{\partial x_1} \right) \right] \quad (2d) \]
\[ \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2} \left( \theta_1 + \frac{\partial \bar{w}}{\partial x_1} \right) = 0 \quad (2e) \]
\[ \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2} \left( \theta_2 + \frac{\partial \bar{w}}{\partial x_2} \right) = 0 \quad (2f) \]

in which \( \theta_1 = \frac{\partial \bar{u}_1}{\partial z} \) and \( \theta_2 = \frac{\partial \bar{u}_2}{\partial z} \) represent the rotation angles of a particular material point at the mid-surface which can be expressed in terms of \( \bar{w} \) from Eqs. (2e) and (2f) as

\[ \theta_1 = -\frac{\partial \bar{w}}{\partial x_1} \quad (3a) \]

and

\[ \theta_2 = -\frac{\partial \bar{w}}{\partial x_2} \quad (3b) \]

Therefore, eliminating \( \theta_1 \) and \( \theta_2 \), strain components in a thin plate can be easily represented in terms of only three mid-surface parameters, \( \bar{u}_1, \bar{u}_2 \) and \( \bar{w} \), as

\[ \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} \bar{u}_{1,11} - z \cdot \bar{w}_{11} & \frac{1}{2} (\bar{u}_{1,2} + \bar{u}_{2,1}) - z \cdot \bar{w}_{12} \\ \frac{1}{2} (\bar{u}_{1,2} + \bar{u}_{2,1}) - z \cdot \bar{w}_{12} & \bar{u}_{2,22} - z \cdot \bar{u}_{22} \end{bmatrix} \quad (4) \]

or
\[ \varepsilon_{ij} = \frac{1}{2} \left( \tilde{u}_{i,j} + \tilde{u}_{j,i} \right) - z \cdot \tilde{w},_{ij} \]  

(5)

where the indices \( i \) and \( j \) vary from 1 to 2. Note that from this stage, for the sake of conciseness, the notation “overbar”, \( \bar{v} \), will be removed from displacement components.

Plane stress constitutive relation is held in the thin plate of functionally graded material as

\[ \sigma = CE \]  

(6a)

and

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{pmatrix} = \frac{E(z)}{1-v(z)^2} \begin{bmatrix}
1 & \nu(z) & 0 \\
\nu(z) & 1 & 0 \\
0 & 0 & \frac{1-v(z)}{2}
\end{bmatrix} \begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2 \varepsilon_{12}
\end{pmatrix}
\]

(6b)

where \( E(z) \) and \( v(z) \) represent elastic modulus and Poisson’s ratio, respectively, and both of them vary in the thickness direction \( z \). Eq. (6b) can also be written in indicial notation as

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]  

(6c)

where \( C_{ijkl} \) is the components of the material property tensor and it is defined as

\[ 
C_{ijkl} = \frac{E(z)}{1-v(z)} \left[ \frac{1-v(z)}{2} \left( \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} \right) + v(z) \delta_{ij} \delta_{kl} \right] 
\]

(7)

The linear elastic strain energy density of the plate can be expressed as

\[ W = \frac{1}{2} \sigma_{ij} \varepsilon_{ji} = \frac{1}{2} C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} \]  

(8)

Substituting Eqs. (6) and (7) into Eq. (8) and rearranging the terms results in

\[
W = \frac{E(z)}{2(1-v(z)^2)} \left\{ \frac{1-v(z)}{2} \left( u_{i,j}u_{i,j} + u_{i,j}u_{j,i} + u_{i,i}u_{j,j} \right) + \frac{3v(z)-1}{2} u_{i,i}u_{j,j} + z^2 \left( w_{i,j}w_{j,i} + (1-v(z))(w_{i,j}w_{j,i} - w_{i,i}w_{j,j}) \right) \right. \\
\left. -2z \left[ u_{i,i}w_{j,j} + (1-v(z)) \left( \frac{1}{2} (u_{i,j} + u_{j,i})w_{i,j} - u_{i,i}w_{j,j} \right) \right] \right\}
\]

(9)

Since the strain energy density changes in the thickness direction, the average strain energy density can be obtained by integrating Eq. (9) over the thickness and dividing by thickness which yields

\[
W = \frac{1}{h} \int_{-h/2}^{h/2} \frac{E(z)}{2} \left\{ \frac{1-v(z)}{2} \left( u_{i,j}u_{i,j} + u_{i,j}u_{j,i} + u_{i,i}u_{j,j} \right) + \frac{3v(z)-1}{2} u_{i,i}u_{j,j} + z^2 \left( w_{i,j}w_{j,i} + (1-v(z))(w_{i,j}w_{j,i} - w_{i,i}w_{j,j}) \right) \right. \\
\left. -2z \left[ u_{i,i}w_{j,j} + (1-v(z)) \left( \frac{1}{2} (u_{i,j} + u_{j,i})w_{i,j} - u_{i,i}w_{j,j} \right) \right] \right\} dz
\]

(10)

Note that Eq. (10) is composed of three independent parts. The first and second rows represent the strain energy densities that occur due to in-plane and flexural (bending) deformation, respectively, and the last row arises due to coupling of in-plane and flexural deformations. If the material properties are constant over the thickness, the last row of Eq. (10) will be cancelled out after integration and Eq. (10) will become uncoupled. Thus, the strain energy density of the plate can be written concisely as

\[ W = W_I + W_{II} + W_{III} \]  

(11)
where $W_I$, $W_{II}$ and $W_{III}$ denote the strain energy densities due to in-plane deformation, flexural deformation and coupled deformation, respectively, which can be written as

$$W_I = \frac{1}{h} \int \frac{E(z)}{2(1-v(z)^2)} \left[ \frac{1-v(z)}{2} (u_{i,j}u_{i,j} + u_{i,i}u_{j,j} + u_{i,j}u_{j,j}) + \frac{3v(z)-1}{2} u_{i,i}u_{j,j} \right] dz$$ (12a)

$$W_{II} = \frac{1}{h} \int \frac{E(z)x^2}{2(1-v(z)^2)} \left[ w_{ii}w_{jj} + (1 - v(z))(w_{ij}w_{ij} - w_{ii}w_{jj}) \right] dz$$ (12b)

$$W_{III} = -\frac{1}{h} \int \frac{E(z)x}{2(1-v(z)^2)} \left\{ u_{i,i}w_{jj} + (1 - v(z)) \left( \frac{1}{2}(u_{ij} + u_{ji})w_{ij} - u_{ii}w_{jj} \right) \right\} dz$$ (12c)

### 3. Peridynamic Formulation for Functionally Graded Kirchhoff Plate

PD is a non-local theory, which is different than the classical local continuum theory, since state of each material point is not only influenced by the material points located in its immediate vicinity but also influenced by material points which are located within a region of a finite radius named as “horizon”, $H$. The equation of motion in PD can be written as

$$\rho(x)\ddot{u}(x, t) = \int_{H} f(u' - u, x' - x) dV' + b(x, t)$$ (13)

where $\rho$ and $t$ represent density and time, respectively, $u$ and $\dot{u}$ denote displacement and $b$ is the body load vector. In Eq. (13), $f$ represents the interaction (bond) force vector between material points located at $x$ and $x'$. Closed-form solution to Eq. (13) is generally not available. Numerical approaches such as meshless approach are widely used to solve Eq. (13). Therefore, the PD equations of motion for a particular material point $k$ can be expressed in summation form as

$$\rho(k)\ddot{u}_{k} = \sum_{j=1}^{N} f_{(k)(j)}V_{(j)} + b_{(k)}$$ (14)

where $N$ indicates the total number of material points inside the horizon of the material point, $k$. The interaction force vector, $f_{(k)(j)}$, between material points $k$ and $j$ is defined as

$$f_{(k)(j)} = t_{(k)(j)} - t_{(j)(k)}$$ (15)

As shown in Fig. 1, the PD force density vector $t_{(k)(j)}$ represents the force acting on the main material point $k$ by its family member material point $j$, and, on the contrary, $t_{(j)(k)}$ represents the force acting on material point $j$ by its family member material point, $k$.

![Peridynamic interaction force between material points](22)

Unlike the classical elasticity theory, according to PD theory, the strain energy density function of a particular material point $k$ depends on the relative displacement between $k$ and all other material points in its horizon which can be expressed as
\[ W_{(k)} = W_{(k)}(u_{(1k)} - u_{(k)}, u_{(2k)} - u_{(k)}, u_{(3k)} - u_{(k)}, \ldots) \]  

where \( u_{(k)} \) is the displacement vector of point \( k \) and \( u_{(i)} \) \( (i = 1, 2, 3, \ldots) \) is the displacement vector of the \( i \)th material point within the horizon of the material point \( k \).

The total potential energy stored in the body can be obtained by summing potential energies of all material points including strain energy and energy due to external loads as

\[ U = \sum_k W_{(k)}(u_{(1k)} - u_{(k)}, u_{(2k)} - u_{(k)}, u_{(3k)} - u_{(k)}, \ldots) V_{(k)} - \sum_k b_{(k)} u_{(k)} V_{(k)} \]  

where \( b_{(k)} \) is the body load vector of the material point \( k \) with a unit of “force per unit volume”. Similarly, the total kinetic energy of the body can be obtained by summing kinetic energies of all material points as

\[ T = \frac{1}{2} \sum_k \rho_{(k)} \dot{u}_{(k)} \cdot \dot{u}_{(k)} V_{(k)} \]  

Thus, the Lagrangian of the body can be expressed as

\[ L = T - U = \frac{1}{2} \sum_k \rho_{(k)} \dot{u}_{(k)} \cdot \dot{u}_{(k)} V_{(k)} - \sum_k W_{(k)}(u_{(1k)} - u_{(k)}, u_{(2k)} - u_{(k)}, u_{(3k)} - u_{(k)}, \ldots) V_{(k)} + \sum_k b_{(k)} u_{(k)} V_{(k)} \]  

Utilising the Lagrangian term given in Eq. (19), the equations of motion can be obtained from Euler-Lagrange equation as

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{u}_{(k)}} - \frac{\partial L}{\partial u_{(k)}} = 0 \]  

which yields

\[ \rho_{(k)} \ddot{u}_{(k)} = \left[ \sum_k \frac{\partial W_{(k)}}{\partial (u_{(1k)} - u_{(k)})} \frac{\partial (u_{(1k)} - u_{(k)})}{\partial u_{(k)}} + \sum_k \sum_k \frac{\partial W_{(k)}}{\partial (u_{(ik)} - u_{(k)})} \frac{\partial (u_{(ik)} - u_{(k)})}{\partial u_{(k)}} + b_{(k)} \right] = 0 \]  

where point \( i^k \) is a family member of the material point \( k \) and similarly point \( i^k \) is a family member of the material point \( i^k \). Note that the following relationship holds

\[ \frac{\partial (u_{(ik)} - u_{(k)})}{\partial u_{(k)}} = \delta_{ik} \]  

Substituting Eq. (22) into Eq.(21) and rearranging the summation index leads to

\[ \rho_{(k)} \ddot{u}_{(k)} = \sum_j \frac{\partial W_{(k)}}{\partial (u_{(ij)} - u_{(k)})} - \sum_j \frac{\partial W_{(ij)}}{\partial (u_{(k)} - u_{(ij)})} + b_{(k)} \]  

Comparing Eq. (23) with Eqs. (14) and (15) and pairing the corresponding terms, one can define as

\[ t_{(k)(j)} = \frac{1}{V_{(j)}} \frac{\partial W_{(k)}}{\partial (u_{(ij)} - u_{(k)})} \]  

and

\[ t_{(j)(k)} = \frac{1}{V_{(j)}} \frac{\partial W_{(ij)}}{\partial (u_{(k)} - u_{(ij)})} \]
which can also be explicitly written as

\[ \mathbf{t}_{(k)(j)} = \left\{ \begin{array}{c} \mathbf{t}_{1}^{(k)(j)} \\ \mathbf{t}_{2}^{(k)(j)} \\ \mathbf{t}_{z}^{(k)(j)} \end{array} \right\} = \frac{1}{V(j)} \left( \begin{array}{c} \frac{\partial W(k)}{\partial (u_{1}^{(k)} - u_{1}^{(j)})} \\ \frac{\partial W(k)}{\partial (u_{2}^{(k)} - u_{2}^{(j)})} \\ \frac{\partial W(k)}{\partial (w_{(k)} - w_{(j)})} \end{array} \right) \]  

(25a)

and

\[ \mathbf{t}_{(j)(k)} = \left\{ \begin{array}{c} \mathbf{t}_{1}^{(j)(k)} \\ \mathbf{t}_{2}^{(j)(k)} \\ \mathbf{t}_{z}^{(j)(k)} \end{array} \right\} = \frac{1}{V(j)} \left( \begin{array}{c} \frac{\partial W(j)}{\partial (u_{1}^{(j)} - u_{1}^{(k)})} \\ \frac{\partial W(j)}{\partial (u_{2}^{(j)} - u_{2}^{(k)})} \\ \frac{\partial W(j)}{\partial (w_{(j)} - w_{(k)})} \end{array} \right) \]  

(25b)

where \( u_1, u_2 \) and \( w \) are the components of the displacement vector.

The strain energy density function given Eq. (12) can be written in PD form for the material point \( k \). This can be achieved by transforming all differential terms into the equivalent form of integration. As derived in Appendix A, the partial differential terms in Eq. (12) for the material point \( k \) can be expressed in PD form as:

\[ u_{i,i}^{(k)} + u_{i,j}^{(k)} + u_{i,l}^{(k)} = \frac{12}{\pi \delta^2 h} \sum_{i,k=1}^{N} \left( \frac{u_{i}^{(k)} - u_{i}^{(j)}}{\xi_{(i)k}} \right) n_t^{(k)} n_l^{(k)} V_{(k)} \]  

(26a)

\[ u_{i,i}^{(k)} = \left( \frac{2}{\pi \delta^2 h} \right)^2 \sum_{i,k=1}^{N} u_{i}^{(k)} - u_{i}^{(j)} n_t^{(k)} V_{(k)} \sum_{j=1}^{N} \left( \frac{u_{i}^{(k)} - u_{i}^{(j)}}{\xi_{(i)k}} \right) n_l^{(k)} V_{(k)} \]  

(26b)

\[ w_{l,l}^{(k)} = \left( \frac{4}{\pi \delta^2 h} \right)^2 \left( \sum_{i,k=1}^{N} \frac{w_{ij}^{(k)} - w_{ij}^{(j)}}{\xi_{(i)k}^2} V_{(i)k} \right)^2 \]  

(26c)

\[ w_{l,l}^{(k)} - w_{l,l}^{(k)} = \]  

\[ 2 \left( \frac{2}{\pi \delta^2 h} \right)^2 \left( \sum_{i,k=1}^{N} \frac{w_{ij}^{(k)} - w_{ij}^{(j)}}{\xi_{(i)k}^2} V_{(i)k} \right)^2 \]  

(26d)

\[ u_{i,l}^{(k)} w_{l,l}^{(k)} = \left( \frac{2}{\pi \delta^2 h} \right)^2 \sum_{i,k=1}^{N} \left( \frac{u_{i}^{(k)} - u_{i}^{(j)}}{\xi_{(i)k}} \right) n_t^{(k)} V_{(i)k} \sum_{i,k=1}^{N} \frac{w_{ij}^{(k)} - w_{ij}^{(j)}}{\xi_{(i)k}^2} V_{(i)k} \]  

(26e)

\[ \left( \frac{1}{2} u_{i,j}^{(k)} + u_{i,j}^{(j)} \right) w_{i,j}^{(k)} - u_{i,l}^{(k)} w_{l,l}^{(k)} = \]  

\[ \left( \frac{2}{\pi \delta^2 h} \right)^2 \left( 2 \sum_{i,k=1}^{N} \frac{u_{i}^{(k)} - u_{i}^{(j)}}{\xi_{(i)k}} n_l^{(k)} V_{(i)k} \sum_{i,k=1}^{N} \frac{w_{ij}^{(k)} - w_{ij}^{(j)}}{\xi_{(i)k}^2} n_t^{(k)} n_l^{(k)} V_{(i)k} \right) + \]
where \( n_1 = \cos \varphi, n_2 = \sin \varphi \) with \( \varphi \) being the bond angle with respect to \( x \)-axis. Substituting the relationships given in Eqs. (26a-f) in Eq. (12), the strain energy density components for the material point \( k \) can be written in PD form as

\[
W_{II}(k) = \frac{2}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z)}{(1-v(z))^2} \left[ (1-v(z)) \left( \frac{1}{\pi \delta^2} \sum_{i=1}^{N} \left( \frac{u^{k}(i,k)-u^{k}(i,k)}{\xi^{k}(i,k)} \right) n^{k}(i,k) V_{i(k)} \right) + v(z) - \frac{1}{3} \sum_{i=1}^{N} \left( \frac{u^{k}(i,k)-u^{k}(i,k)}{\xi^{k}(i,k)} \right) n^{k}(i,k) V_{i(k)} \right] dz
\]

\[
W_{III}(k) = \left( \frac{2}{\pi \delta^2 h} \right)^2 \sum_{i=1}^{N} \left( \frac{u^{k}(i,k)-u^{k}(i,k)}{\xi^{k}(i,k)} \right) n^{k}(i,k) V_{i(k)} \sum_{i=1}^{N} \left( \frac{w^{k}(i,k)-w^{k}(i,k)}{\xi^{k}(i,k)} \right) n^{k}(i,k) V_{i(k)}
\]

\[
\begin{align*}
W_{II}(j) &= \frac{3}{h^2} \int_{-\frac{3h}{2}}^{\frac{3h}{2}} \frac{E(z)}{(1-v(z))^2} \left[ (1-v(z)) \left( \frac{1}{\pi \delta^2} \sum_{j=1}^{N} \left( \frac{u^{j}(i,j)-u^{j}(i,j)}{\xi^{j}(i,j)} \right) n^{j}(i,j) V_{i(j)} \right) + v(z) - \frac{1}{3} \sum_{i=1}^{N} \left( \frac{u^{j}(i,j)-u^{j}(i,j)}{\xi^{j}(i,j)} \right) n^{j}(i,j) V_{i(j)} \right] dz
\end{align*}
\]
After substituting Eqs. (27) and (28) into Eqs. (11) and (23) and performing some algebraic manipulations, the PD equation of motion can be written in terms of displacements only as

\[
\rho \dddot{\mathbf{u}}_{m}^{(k)} = \frac{12}{\pi \delta h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(x)}{1+\nu(x)} \frac{h}{\xi_{(k),(j)}} \left\{ \frac{1}{2} \sum_{i=1}^{N} \left[ \sum_{i=1}^{N} \frac{w_{(i),(k)}^{(j)} - w_{(i),(j)}^{(k)}}{\xi_{(k),(k)}} \mathbf{V}_{(i)} \right] - \left(1 - \frac{1}{2} \nu(x) \right) \sum_{i=1}^{N} \left[ \sum_{i=1}^{N} \frac{w_{(i),(k)}^{(j)} - w_{(i),(j)}^{(k)}}{\xi_{(k),(k)}} \mathbf{n}_{(i)} \right] \right\} \frac{dz}{2} 
\]

with \( m = 1,2 \)

and

\[
\rho \dot{w}_{(k)} = 4 \frac{2}{\pi \delta h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(x)}{1+\nu(x)} \frac{h}{\xi_{(i),(j)}} \left\{ \frac{1}{2} \sum_{i=1}^{N} \left[ \sum_{i=1}^{N} \frac{w_{(i),(k)}^{(j)} - w_{(i),(j)}^{(k)}}{\xi_{(k),(k)}} \mathbf{V}_{(i)} \right] - \left(1 - \frac{1}{2} \nu(x) \right) \sum_{i=1}^{N} \left[ \sum_{i=1}^{N} \frac{w_{(i),(k)}^{(j)} - w_{(i),(j)}^{(k)}}{\xi_{(k),(k)}} \mathbf{n}_{(i)} \right] \right\} \frac{dz}{2} 
\]
\[
\left(\frac{2}{\pi \delta h}\right)^2 \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{1}{2(1-v(z))^2} \frac{1}{(1-v(z))^2} \sum_{i=1}^{N} \frac{1}{\xi_{(i,k)(k)}} \left(2 \sum_{j=1}^{N} \frac{u_{i,j}^{(k)} - u_{i,j}^{(k)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} - \sum_{j=1}^{N} \frac{u_{i,j}^{(i,j)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} \right) + \\
(1 - v(z)) \left[ 4 n_{i}^{(i,j)(j)} \left( \sum_{k=1}^{N} \frac{u_{i,j}^{(k)} - u_{i,j}^{(k)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} - \sum_{j=1}^{N} \frac{u_{i,j}^{(i,j)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} \right) - \\
3 \left( \sum_{k=1}^{N} \frac{u_{i,j}^{(k)} - u_{i,j}^{(k)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} - \sum_{j=1}^{N} \frac{u_{i,j}^{(i,j)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} \right) \right] V(j) + b_{z}^{(k)}
\]

(30)

In particular case of our study, when the plate is subjected to a constraint condition on all edges, i.e. no free edges, the strain energy density function of a particular material point of the plate can be considerably simplified as explained in Appendix B, and takes the following form as:

\[
W = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \left\{ \frac{1 - v(z)}{2} \left( u_{i,j} u_{i,j} + u_{i,j} u_{i,j} + u_{i,j} u_{i,j} + \frac{3v(z)-1}{2} u_{i,j} u_{i,j} \right) + z^2 \cdot w_{i} \cdot w_{j} - 2z \cdot u_{i} \cdot w_{j} \right\} \, dz
\]

(31)

with

\[
W_{I} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \left( \frac{1 - v(z)}{2} \left( u_{i,j} u_{i,j} + u_{i,j} u_{i,j} + u_{i,j} u_{i,j} + \frac{3v(z)-1}{2} u_{i,j} u_{i,j} \right) \right) \, dz
\]

(32a)

\[
W_{II} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{z^2}{2(1-v(z))^2} \, dz \cdot w_{i} \cdot w_{j}
\]

(32b)

\[
W_{III} = -\frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{z}{2(1-v(z))^2} \, dz \cdot u_{i} \cdot w_{j}
\]

(32c)

Corresponding PD form for the material point \( k \) can be written as:

\[
W_{I}^{(k)} = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{1}{2(1-v(z))^2} \left(1 - v(z)\right) \left( \frac{1}{\pi \delta h} \sum_{i=1}^{N} \frac{u_{i,j}^{(k)} - u_{i,j}^{(k)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} \right) + \left( v(z) - \frac{1}{3} \right) \sum_{i=1}^{N} \frac{u_{i,j}^{(k)} - u_{i,j}^{(k)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} \right) \, dz
\]

(33a)

\[
W_{II}^{(k)} = 2 \left( \frac{2}{\pi \delta h} \right)^2 \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{z^2}{2(1-v(z))^2} \, dz \left( \sum_{i=1}^{N} \frac{w_{i}^{(k)} - w_{i}^{(k)}}{\xi_{(i,k)(k)}} V_{(i,k)} \right)^2
\]

(33b)

\[
W_{III}^{(k)} = -2 \left( \frac{2}{\pi \delta h} \right)^2 \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{z}{2(1-v(z))^2} \, dz \sum_{i=1}^{N} \frac{u_{i,j}^{(k)} - u_{i,j}^{(k)}}{\xi_{(i,j)(j)}} n_{i}^{(i,j)(j)} V_{(i,j)} \sum_{i=1}^{N} \frac{w_{i}^{(k)} - w_{i}^{(k)}}{\xi_{(i,k)(k)}} V_{(i,k)} \right)
\]

(33c)

which can also be written concise form as
\[
\rho \ddot{u}_m^{(k)} = \frac{12}{\pi \delta^2 h} \frac{1}{h^2} E(z) \frac{h}{2} + 1 + v(z) \frac{1}{\xi_j(k)} \int_0^h \frac{E(z) \frac{h}{2}}{(1-v(z)^2)^2} dz \sum_{j=1}^N \frac{u_j^{(k)} - u_j^{(k)}}{\xi_j(k)} n_j^{(k)} n_j^{(k)} V_j^{(j)} + \frac{1}{\xi_i(k)} \int_0^h \frac{E(z) \frac{h}{2}}{(1-v(z)^2)^2} dz \sum_{j=1}^N \frac{u_j^{(k)} - u_j^{(k)}}{\xi_j(k)} n_j^{(k)} n_j^{(k)} V_j^{(j)} \left[ V_j^{(j)} - \frac{1}{\xi_i(k)} \int_0^h \frac{E(z) \frac{h}{2}}{(1-v(z)^2)^2} dz \sum_{j=1}^N \frac{w_j^{(k)} - w_j^{(k)}}{\xi_j(k)} V_j^{(j)} - \sum_{i=1}^N \frac{w_i^{(j)} - w_i^{(j)}}{\xi_j(j)} V_i^{(i)} \right] \right] V_j^{(j)} + b_m^{(k)}
\]

\[
\begin{align*}
\rho \ddot{w}_k^{(k)} &= 4 \left( \frac{2}{\pi \delta^2 h} \right) \frac{1}{h} \int_0^h \frac{E(z) \frac{h}{2}}{(1-v(z)^2)^2} dz \sum_{j=1}^N \frac{1}{\xi_j(k)} \left[ \sum_{i=1}^N \frac{w_j^{(k)} - w_j^{(k)}}{\xi_j(k)} V_j^{(j)} - \sum_{i=1}^N \frac{w_i^{(j)} - w_i^{(j)}}{\xi_j(j)} V_i^{(i)} \right] V_j^{(j)} + b_m^{(k)} \\
\rho \ddot{w}_k^{(k)} &= 2 \left( \frac{2}{\pi \delta^2 h} \right) \frac{1}{h} \int_0^h \frac{E(z) \frac{h}{2}}{(1-v(z)^2)^2} dz \sum_{j=1}^N \frac{1}{\xi_j(k)} \left[ \sum_{i=1}^N \frac{w_j^{(k)} - w_j^{(k)}}{\xi_j(k)} V_j^{(j)} - \sum_{i=1}^N \frac{w_i^{(j)} - w_i^{(j)}}{\xi_j(j)} V_i^{(i)} \right] V_j^{(j)} - \frac{1}{\xi_i(k)} \int_0^h \frac{E(z) \frac{h}{2}}{(1-v(z)^2)^2} dz \sum_{j=1}^N \frac{w_j^{(k)} - w_j^{(k)}}{\xi_j(k)} V_j^{(j)} - \sum_{i=1}^N \frac{w_i^{(j)} - w_i^{(j)}}{\xi_j(j)} V_i^{(i)} \right] \right] V_j^{(j)} + b_m^{(k)} \\
\end{align*}
\]

\[
\begin{align*}
4. \text{Numerical Results} \end{align*}
\]

To demonstrate the validity of the presented PD formulation for functionally graded Kirchhoff plates, the PD solutions are compared with the corresponding finite element (FE) analysis results. Here, the material properties are chosen such that Young’s Modulus, \( E(z) \), is assumed to be varied linearly through the thickness direction and Poisson’s ratio, \( v(z) \), remains a constant as

\[
E(z) = (E_t - E_b) \frac{z}{h} + \frac{1}{2} (E_t + E_b) \quad \text{(GPa)}
\]

and

\[
v(z) = v
\]

where \( E_t \) and \( E_b \) denote the Young’s modulus of the top and bottom surfaces of the plate, and \( h \) represents the total thickness of the plate.
4.1. Clamped plate subjected to transverse loading

A clamped functionally graded plate with a length and width of \( L = W = 1 \, \text{m} \) and a thickness of \( h = 0.02 \, \text{m} \) is considered as shown in Fig. 2. The Poisson’s ratio of the plate is \( \nu = 1/3 \) and the Young’s modulus of the top and bottom surfaces are \( E_t = 200 \, \text{GPa} \) and \( E_b = 100 \, \text{GPa} \), respectively. The model is discretized into one single row of material points along the thickness direction and the distance between material points is \( \Delta x = 1/101 \, \text{m} \). A fictitious region is introduced outside the edges as the external boundaries with a width of \( 2\delta \). The plate is subjected to a distributed transverse load of \( p_z = 101 \, \text{N/m} \) through the \( y \)-centre line, respectively. The line load is converted to a body load of \( b = \frac{p_z W}{\Delta x^2} = 510050 \, \text{N/m}^3 \) and it is distributed to one column of material volumes through the central line.

![Figure 2. Clamped plate subjected to transverse loading.](image)

<table>
<thead>
<tr>
<th>Layer 50</th>
<th>( E_{50} = 199 , \text{GPa} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer i</td>
<td>( E_i = 100x(1+(i-1)/50) , \text{GPa} )</td>
</tr>
<tr>
<td>Layer 1</td>
<td>( E_1 = 101 , \text{GPa} )</td>
</tr>
</tbody>
</table>

![Figure 3. Variation of the Young’s modulus in thickness direction for the FE model.](image)

The FE model of the plate is created by using the SHELL181 element in ANSYS with dimensions of \( 1 \, \text{m} \times 1 \, \text{m} \times 0.02 \, \text{m} \). To model the functionally graded plate, the model is divided to 50 layers with varying homogeneous materials properties through the thickness. The Young’s modulus varies linearly over the thickness from the first layer \( E_1 = 101 \, \text{GPa} \) to the last layer \( E_{50} = 199 \, \text{GPa} \) as shown in Fig. 3. The model is meshed with 0.01 \, \text{m} element size.

\[ \text{Maximum difference between the PD and FE results is less than 0.5%.} \]

The PD and FE transverse displacement contours are compared in Fig. 4. They yield similar displacement variations. The maximum difference between the PD and FE results is less than 0.5%. Moreover, the transverse and in-plane displacement components along the central \( y \)-axis are compared in Fig. 5 and very good agreement is obtained between the two approaches. These results verify the accuracy of the current PD formulation for a functionally graded Kirchhoff plate theory under clamped boundary conditions.
Figure 4. Variation of transverse displacements (a) PD, (b) FEM

Figure 5. Variation of (a) transverse, $w$ and (b) in-plane, $u_1$, displacements along the central $y$-axis.

4.2 Simply supported plate subjected to transverse loading

A simply supported plate (see Fig. 6) has the same geometrical and material properties as in the clamped plate case. Again, it is discretized with a single row of material points along the thickness direction and the discretization size is $\Delta x = \frac{1}{101} \, m$. A fictitious region is created outside the region of boundaries and its width is equal to two times the size of the horizon, $\delta$. The plate is subjected to a distributed transverse line load of $p_z = 101 \, N/m$ through the $y$-central line. It is imposed to central column of material points with a body load of $b = \frac{p_z \delta W}{\delta^2} = 510050 \, N/m^3$. 

\[
\]
The transverse displacement components of FE and PD theory show very close variations, as shown in Fig. 7. The maximum difference between the PD and FE results is less than 0.5%. Furthermore, the transverse and in-plane displacement variations along the central $y$-axis are on top of each other for the FE and PD results, as shown in Fig. 8. This confirms the current PD formulation of functionally graded plate theory under simply supported boundary conditions.

Figure 6. Simply supported plate subjected to transverse loading.

Figure 7. Variation of transverse displacements (a) PD, (b) FEM
4.3 Simply supported plate subjected to inclined loading

This problem case is similar to the previous case except Poisson’s ratio of 0.2 and an inclined load of $p_x = p_z = 101 \, N/m$ through the y-central line are considered as shown in Fig. 9.

**Figure 8.** Variation of (a) transverse, $w$ and (b) in-plane, $u_1$, displacements along the central $y$-axis.

**Figure 9.** Simply supported plate subjected to inclined loading.
As depicted in Fig. 10, the transverse displacement components of FE and PD theory agree very well with each other. Moreover, the transverse and in-plane displacements variations along the central $y$-axis are also in very good agreement as depicted in Fig. 11.

5. Conclusions

In this study, a state-based peridynamic model was presented for functionally graded Kirchhoff plates. Equations of motion of the new formulation was obtained by using Euler-Lagrange equations and Taylor’s expansion. The formulation was verified by considering several benchmark problems including a clamped plate subjected to transverse loading and simply supported plate subjected to transverse loading and inclined loading. The material properties were chosen such that Young’s Modulus is assumed to be varied linearly through the thickness direction and Poisson’s ratio is constant. Peridynamic results were compared against finite element analysis results and a very good agreement is obtained between the two approaches.
References


Appendix A

The in-plane displacement function $u$ can be expanded as Taylor series up to first order as

$$u(x + \xi) - u(x) = J u(x) \xi n$$

where $\xi, \xi, n$ and $J$ are defined as

$$\xi = \{\xi_{n_1} \xi_{n_2}\}$$

$$\xi = |\xi|$$

$$n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$$

$$J = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix}$$

Eq. (A1) can be rewritten in indicial notation form as

$$u_{ij}(x + \xi) - u_{ij}(x) = \xi u(x)_{i,j} n_j$$

Eq. (A3a) can also be written with a different free index, $l$ as

$$u_{il}(x + \xi) - u_{il}(x) = \xi u(x)_{i,m} n_m$$

(A3b)

Multiplying Eq. (A3a) with Eq.(A3b) leads to

$$[u_{il}(x + \xi) - u_{il}(x)][u_{il}(x + \xi) - u_{il}(x)] = \xi^2 u(x)_{i,j} u(x)_{l,m} n_j n_m$$

which can be rewritten by multiplying both sides with $n_r n_s$ as

$$[u_{il}(x + \xi) - u_{il}(x)][u_{il}(x + \xi) - u_{il}(x)] n_r n_s = \xi^2 u(x)_{i,j} u(x)_{l,m} n_j n_m n_r n_s$$

(A5)

If we suppose that the point $x$ is kept as a fixed point, multiplying both sides of Eq.(A5) by $\frac{1}{\xi}$ and integrating over a circular horizon with central point of $x$ and radius of $\delta$ results in

$$\int_0^{2\pi} \int_0^{\delta} \frac{[u_{il}(x + \xi) - u_{il}(x)][u_{il}(x + \xi) - u_{il}(x)]}{\xi} n_r n_s \xi d\xi d\varphi = u(x)_{i,j} u(x)_{l,m} \int_0^{2\pi} \int_0^{\delta} n_j n_m n_r n_s \xi^2 d\xi d\varphi$$

(A6)
According to the orthogonality of the trigonometric functions, the integral at the right hand side of Eq.(A6), \( \int_0^{2\pi} n_j n_m n_r d\varphi \), can be expressed in following different forms as

\[
\int_0^{2\pi} (n_1)^2 \, d\varphi = \int_0^{2\pi} \cos^2 \varphi \, d\varphi = \int_0^{2\pi} \sin^2 \varphi \, d\varphi = \frac{\pi}{4}
\]

\[
\int_0^{2\pi} (n_1)^4 \, d\varphi = \int_0^{2\pi} (n_2)^4 \, d\varphi = \int_0^{2\pi} \cos^4 \varphi \, d\varphi = \int_0^{2\pi} \sin^4 \varphi \, d\varphi = \frac{3\pi}{4}
\]

\[
\int_0^{2\pi} (n_1)^3 n_2 \, d\varphi = \int_0^{2\pi} n_1 (n_2)^3 \, d\varphi = \int_0^{2\pi} \cos \varphi n_2 \, d\varphi = \int_0^{2\pi} \cos \varphi \sin \varphi \, d\varphi = 0
\]

Thus, Eq. (A6) can be evaluated as

\[
\int_0^{2\pi} \int_0^{\delta} \left[ u_i(x+\xi) - u_i(x) \right] \left[ u_i(x+\xi) - u_i(x) \right] \, d\xi d\varphi = \frac{\pi \delta^3}{12} u_i(x)_{i,j} u(x)_{l,m} \left( \delta_{jm} \delta_{rs} + \delta_{jr} \delta_{ms} + \delta_{js} \delta_{mr} \right) = \frac{\pi \delta^3}{12} \left[ u_i(x)_{i,j} u(x)_{i,j} + u(x)_{i,i} u(x)_{l,l} + u(x)_{l,l} u(x)_{i,i} \right] \quad (A8)
\]

Multiplying both sides of Eq.(A8) with \( \delta_{ri} \delta_{ls} \) results in

\[
\int_0^{2\pi} \int_0^{\delta} \left[ u_i(x+\xi) - u_i(x) \right] \left[ u_i(x+\xi) - u_i(x) \right] \, d\xi d\varphi = \frac{\pi \delta^3}{12} u_i(x)_{i,j} u(x)_{l,m} \left( \delta_{jm} \delta_{rs} + \delta_{jr} \delta_{ms} + \delta_{js} \delta_{mr} \right) = \frac{\pi \delta^3}{12} \left[ u_i(x)_{i,j} u(x)_{i,j} + u(x)_{i,i} u(x)_{l,l} + u(x)_{l,l} u(x)_{i,i} \right] \quad (A9)
\]

Discretizing Eq. (A9) for the material point \( k \) and performing some algebraic manipulations results in

\[
u_{i,j}^{(k)} + u_{i,j}^{(k)} + u_{i,j}^{(k)} = \frac{12 \pi \delta^3}{\lambda_n} \sum_{i,k=1}^N \left( u_i^{(k)} - u_i^{(k)} \right) \left( u_i^{(k)} - u_i^{(k)} \right) \frac{u_{i,j}^{(k)} n_{i,j}^{(k)}}{\xi_{i,j}^{(k)}} V_{i,j}^{(k)} \quad (A10)
\]

Next, expanding the in-plane displacement function \( u \) again and ignoring higher order terms yields

\[
u_i(x+\xi) - u_i(x) = \xi u_i(x)_{i,j} n_j + \frac{1}{2} \xi^2 u_i(x)_{i,j} n_j n_k \quad (A11)
\]

Multiplying each term in Eq. (A11) with unit orientation \( n_l \) and dividing by the norm of distance, \( \xi \) results in

\[
u_i(x+\xi) - u_i(x) \xi \frac{n_l}{\xi} = u_i(x)_{i,j} n_j + \frac{1}{2} \xi u_i(x)_{i,j} n_j n_k \quad (A12)
\]

Considering the material point \( x \) as fixed and integrating each term over a circular horizon results in

\[
\int_0^{2\pi} \int_0^{\delta} \left[ u_i(x+\xi) - u_i(x) \right] \frac{n_l}{\xi} \, d\xi d\varphi = \frac{\pi \delta^2}{2} u_i(x)_{i,j} \delta_{ij} \quad (A13)
\]

Note that, the following orthogonality property of the trigonometric functions has been used in Eq. (A13) as

\[
\int_0^{2\pi} (n_1)^2 \, d\varphi = \int_0^{2\pi} (n_2)^2 \, d\varphi = \int_0^{2\pi} \cos^2 \varphi \, d\varphi = \int_0^{2\pi} \sin^2 \varphi \, d\varphi = \pi
\]

\[
\int_0^{2\pi} n_1 n_2 \, d\varphi = \int_0^{2\pi} n_1^2 n_2 \, d\varphi = \int_0^{2\pi} n_1 (n_2)^2 \, d\varphi = \int_0^{2\pi} \cos \varphi \sin \varphi \, d\varphi = \int_0^{2\pi} \cos^2 \varphi \sin^2 \varphi \, d\varphi = 0
\]

Discretizing the integration in Eq. (A13) for the material point \( k \) and performing algebraic operations result in
Multiplying both sides of Eq. (A15) with $\delta_{ii}$ gives

$$u_{ii}^{(k)} = \frac{2}{\pi \delta^2 h} \sum_{i=1}^{N} \frac{u_{ii}^{(k)} - u_{ii}^{(k)}}{\xi^{(k)}_{(k)}} n_i V_{(i)}^{(k)}$$

(A16)

Following an analogical approach, the transverse displacement function, $w$, can be expanded in the form of Taylor series and can be written after ignoring higher order terms as

$$w(x + \xi) - w(x) = \xi w(x)_i n_i + \frac{1}{2} \xi^2 w(x)_{ij} n_i n_j + \frac{1}{3!} \xi^3 w(x)_{ijk} n_i n_j n_k$$

(A17)

Multiplying each term in Eq. (A17) by $\frac{n_i n_j}{\xi^2}$ and integrating over a circular horizon with central point of $x$ and radius of $\delta$ results in

$$\int_0^{2\pi} \int_0^{\delta [w(x)_{ij} n_i n_j + \frac{1}{2} \xi^2 w(x)_{ij} n_i n_j + \frac{1}{3!} \xi^3 w(x)_{ijk} n_i n_j n_k]} n_i n_j \xi d\xi d\varphi = \pi \delta^2 \frac{1}{16} \left( w(x)_{ij} \delta_{rs} + \delta_{ir} \delta_{js} + \delta_{is} \delta_{rj} \right) = \frac{\pi \delta^2}{16} \left( w(x)_{ii} \delta_{rs} + 2w(x)_{ir} \right)$$

(A18a)

Again, the orthogonality property which has been stated earlier was utilized to simplify the expression given in Eq. (A18a). Similar expression can be obtained by using different free and dummy indices as

$$\int_0^{2\pi} \int_0^{\delta [w(x)_{ij} n_i n_j + \frac{1}{2} \xi^2 w(x)_{ij} n_i n_j + \frac{1}{3!} \xi^3 w(x)_{ijk} n_i n_j n_k]} n_i n_j \xi d\xi d\varphi = \pi \delta^2 \frac{1}{16} \left( w(x)_{ij} \delta_{pq} + 2w(x)_{ij} \right)$$

(A18b)

Multiplying Eq. (A18a) by Eq. (A18b) gives

$$\int_0^{2\pi} \int_0^{\delta [w(x)_{ij} n_i n_j + \frac{1}{2} \xi^2 w(x)_{ij} n_i n_j + \frac{1}{3!} \xi^3 w(x)_{ijk} n_i n_j n_k]} n_i n_j \xi d\xi d\varphi = \pi \delta^2 \frac{1}{16} \left( w(x)_{ij} \delta_{pq} + 2w(x)_{ij} \right)$$

(A19)

Multiplying both sides of Eq. (A19) with $\delta_{rs} \delta_{pq}$ and performing algebraic manipulations result in

$$w(x)_{ij} w(x)_{jj} = \left( \frac{4}{\pi \delta^2} \right)^2 \left( \int_0^{2\pi} \int_0^{\delta [w(x)_{ij} n_i n_j + \frac{1}{2} \xi^2 w(x)_{ij} n_i n_j + \frac{1}{3!} \xi^3 w(x)_{ijk} n_i n_j n_k]} n_i n_j \xi d\xi d\varphi \right)^2$$

(A20)

which can be expressed in discrete form as

$$w_{ij}^{(k)} w_{jj}^{(k)} = \left( \frac{4}{(\pi \delta^2 h)} \right)^2 \left( \sum_{i=1}^{N} \frac{w_{ij}^{(k)} - w_{ij}^{(k)}}{\xi^{(k)}_{(k)}} V_{(i)}^{(k)} \right)^2$$

(A21)

Multiplying both sides of Eq. (A19) with $\delta_{rq} \delta_{sp}$ and performing algebraic manipulations result in

$$\int_0^{2\pi} \int_0^{\delta [w(x)_{ij} n_i n_j + \frac{1}{2} \xi^2 w(x)_{ij} n_i n_j + \frac{1}{3!} \xi^3 w(x)_{ijk} n_i n_j n_k]} n_i n_j \xi d\xi d\varphi = \pi \delta^2 \frac{1}{16} \left( w(x)_{ij} \delta_{pq} + 2w(x)_{ij} \right)$$

(A22)

Thus,
\[ w(x, y)_{p,q} - w(x, y)_{i,i}w(x, y)_{i,j} = \frac{1}{4} \left( \frac{16}{\pi \delta^2} \right)^2 \int_0^{2\pi} \int_0^\delta \frac{\delta w(x+\xi) - w(x)}{\xi^2} n_p n_q \xi d\xi d\phi \int_0^{2\pi} \int_0^\delta \frac{\delta w(x+\xi) - w(x)}{\xi^2} n_i n_j \xi d\xi d\phi - \frac{5}{2} w(x, y)_{i,i}w(x, y)_{i,j} \] (A23)

Substituting Eq. (A21) into the right hand side of Eq. (A23) and rearranging the free indices result in

\[ w(x, y)_{i,j}w(x, y)_{i,j} - w(x, y)_{i,i}w(x, y)_{i,j} = \left( \frac{8}{\pi \delta^2} \right)^2 \int_0^{2\pi} \int_0^\delta \frac{\delta w(x+\xi) - w(x)}{\xi^2} n_i n_j \xi d\xi d\phi \int_0^{2\pi} \int_0^\delta \frac{\delta w(x+\xi) - w(x)}{\xi^2} n_i n_j \xi d\xi d\phi - \frac{5}{2} \left( \frac{4}{\pi \delta^2} \right)^2 \int_0^{2\pi} \int_0^\delta \frac{\delta w(x+\xi) - w(x)}{\xi^2} \xi d\xi d\phi \] (A24)

and its corresponding discretized form can be written as

\[ w^{(k)}_{i,j}w^{(k)}_{i,j} - w^{(k)}_{i,i}w^{(k)}_{i,j} = \frac{1}{4} \left( \frac{8}{\pi \delta^2} \right)^2 \sum_{i=1}^{N} \frac{w^{(k)}_{i,i} - w^{(k)}_{j,j}}{\xi^2} \left( n^i (i)(k) n^j (j)(k) V(i)(k) \sum_{j=1}^{N} \frac{w^{(k)}_{i,j} - w^{(k)}_{j,i}}{\xi^2} n^i (i)(k) n^j (j)(k) V(i)(k) - \right) \frac{5}{2} \left( \frac{4}{\pi \delta^2} \right)^2 \int_0^{2\pi} \int_0^\delta \frac{\delta w(x+\xi) - w(x)}{\xi^2} \xi d\xi d\phi \] (A25)

**Appendix B**

As mentioned earlier, the total strain energy stored in the plate can be expressed as:

\[ U = \iiint_V \left( W_i + W_{II} + W_{III} \right) dV = U_i + U_{II} + U_{III} \] (B1)

where

\[ U_i = \frac{1}{2} \iiint_V \frac{E(z)}{1-v(z)^2} \left[ (u_x + v_y)^2 - 2[1 - v(z)]u_x v_y + \frac{1-v(z)}{2} (u_y + v_x)^2 \right] dV \] (B2a)

\[ U_{II} = \frac{1}{2} \iiint_V \frac{E(z)z^2}{1-v(z)^2} \left[ (\nabla^2 w)^2 + 2[1 - v(z)](w_{xy}^2 - w_{xx}w_{yy}) \right] dV \] (B2b)

\[ U_{III} = - \iiint_V \frac{E(z)z}{1-v(z)^2} \left[ (u_x + v_y) \nabla^2 w + [1 - v(z)] \left[ (u_y + v_x) w_{xy} - (u_x w_{yy} + v_y w_{xx}) \right] \right] dV \] (B2c)

Utilizing the definition of differentiation of a compound function, the term, \( w_{xx}w_{yy} \), in Eq. (B2b) can be expressed as

\[ w_{xx}w_{yy} = (w_{yy}w_x)_x - (w_{xy}w_x)_y + w_{xy}^2 \] (B3a)

and

\[ w_{xx}w_{yy} = (w_{xx}w_y)_y - (w_{xy}w_y)_x + w_{xy}^2 \] (B3b)

Calculating the average of Eqs. (B3a) and (B3b) yields

\[ w_{xx}w_{yy} - w_{xy}^2 = \frac{1}{2} \left[ (w_{yy}w_x - w_{xy}w_y)_x + (w_{xx}w_y - w_{xy}w_x)_y \right] \] (B4)

Integrating the above expression over the mid-surface results in
\[
\iint_A \left[ (w_{yy}w_x - w_{xy}w_y) \right]_x + (w_{xx}w_y - w_{xy}w_x)_y \right] dA = \oint_{\partial A} \left[ (w_{yy}w_x - w_{xy}w_y) \cos \alpha - (w_{xy}w_x - w_{xx}w_y) \right] \sin \alpha \right] ds
\]  
(B5)

where the notation of the integral region \( A \) stands for the mid-surface of the plate. Green’s theorem was used for the above derivation and the closed integral path \( \partial A \) implies the boundary of the mid-surface of the plate. When a simply connected plate is subjected to clamped boundary conditions, i.e., \( w_x = w_y = 0 \) at the boundary, thus, the above integral equals to zero.

For a simply connected plate subjected to simply supported boundary conditions, if a local coordinate system \((x^*, o^*, y^*)\) is set at each edge, for instance where \(x^*\) and \(y^*\) axis are set as the normal and tangent orientation to the boundary, \(w_{yy^*y^*} = 0\) is satisfied on every edge.

The relationship between local and global coordinate systems can be expressed as
\[
\frac{\partial}{\partial x^*} = \cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y} \quad \text{(B6a)}
\]
\[
\frac{\partial}{\partial y^*} = -\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y} \quad \text{(B6b)}
\]

and
\[
w_{x'^*x^*} = \cos^2 \alpha w_{xx} + \sin^2 \alpha w_{yy} + 2 \sin \alpha \cos \alpha w_{xy} \quad \text{(B7a)}
\]
\[
w_{y'^*y^*} = \sin^2 \alpha w_{xx} + \cos^2 \alpha w_{yy} - 2 \sin \alpha \cos \alpha w_{xy} \quad \text{(B7b)}
\]

Utilizing Eq. (B6b), one can easily obtain that \(w_x \sin \alpha = w_y \cos \alpha\). Thus, the integrand of Eq.(B5) becomes
\[
\left( w_{yy}w_x - w_{xy}w_y \right) \cos \alpha - \left( w_{xy}w_x - w_{xx}w_y \right) \sin \alpha = \frac{w_y}{\sin \alpha} \left( \cos^2 \alpha w_{yy} + \sin^2 \alpha w_{xx} - 2 \sin \alpha \cos \alpha w_{xy} \right) = 0
\]  
(B8)

Substituting Eqs. (B8) and (B7b) into (B5), the bending strain energy, \(U_{II}\), can be simplified as
\[
U_{II} = \frac{1}{2} \iiint_v \frac{E(z) x^2}{1-v(z)^2} (\nabla^2 w)^2 \, dv
\]  
(B9)

and the corresponding strain energy density is
\[
W_{II} = \frac{1}{2} \left( w_{xx} + w_{yy} \right) \frac{E(z) x^2}{1-v(z)^2}
\]  
(B10)

The simplification of \(U_{III}\) can be derived in a similar way. According to the differentiation definition for a compound function, it is clear that
\[
u_{xy}w_{xy} = (u \cdot w_{xy})_y - (u \cdot w_{yy})_x + u_x w_{yy} \quad \text{(B11a)}
\]
\[
u_{xy}w_{xy} = (v \cdot w_{xy})_x - (v \cdot w_{xx})_y + v_y w_{xx} \quad \text{(B11b)}
\]

Thus,
\[
(u_y + v_x)w_{xy} - (u_x w_{yy} + v_y w_{xx}) = (v \cdot w_{xy} - u \cdot w_{yy})_x + (u \cdot w_{xy} - v \cdot w_{xx})_y
\]  
(B12)
\[ \iint_A \left[ (v \cdot w_{xy} - u \cdot w_{yy})_x + (u \cdot w_{xy} - v \cdot w_{xx})_y \right] dA = \oint_{\partial A} \left[ (v \cdot w_{xy} - u \cdot w_{yy}) \cos \alpha - (u \cdot w_{xy} - v \cdot w_{xx}) \sin \alpha \right] ds \]  

(B12)

For any shape of plate with boundaries are constrained as fixed, the in-plane displacements \( u \) and \( v \) are equal to zero. Therefore,

\[ \iint_A \left[ (u, y + v, x)w_{xy} - (u, xw_{yy} + v, yw_{xx}) \right] dA = 0 \]  

(B12)

Substituting Eq. (B12) into (B2c) results in a simpler form of strain energy as

\[ U_{III} = -\iiint_V (u, x + v, y)(w_{xx} + w_{yy}) \frac{E(z)x}{1-v(z)^2} dx dy dz \]  

(B13)

and the corresponding strain energy density is

\[ W_{III} = -(u, x + v, y)(w_{xx} + w_{yy}) \frac{E(z)x}{1-v(z)^2} \]  

(B14)