

Multipreconditioning with application to two-phase incompressible Navier–Stokes flow

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Abstract We consider the use of multipreconditioning to solve linear systems when more than one preconditioner is available but the optimal choice is not known. In particular, we consider a selective multipreconditioned GMRES algorithm where we incorporate a weighting that allows us to prefer one preconditioner over another. Our target application lies in the simulation of incompressible two-phase flow. Since it is not always known if a preconditioner will perform well within all regimes found in a simulation, we also consider robustness of the multipreconditioning to a poorly performing preconditioner. Overall, we obtain promising results with the approach.

1 Introduction

In challenging fluid flow simulations used to model hydraulic processes it is often not clear what the best choice of preconditioner might be for solving a given linear system $\mathcal{A}\mathbf{x} = \mathbf{b}$. Further, disparate flow regimes can be encountered in a simulation and the optimal preconditioner may change throughout. One can imagine trying to adaptively change the preconditioner based on tracking the current flow regime. However, this requires knowing *a priori* which preconditioner is likely best in any given regime as well as a suitable evaluation of the current flow, which may well

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vary within the domain. The required sophistication and good prior knowledge of the preconditioners' performance makes such an adaptive approach less appealing.

Instead we consider using multiple preconditioners simultaneously, aiming to get the best of each. If we can combine the preconditioners then we would like to know whether we can achieve performance similar to the (unknown) best preconditioner and, further, if together they provide an improvement over any individual approach. Another key question to ask would be that of robustness: whether inclusion of a poorly performing preconditioner significantly affects the overall performance.

These ideas are encompassed within multipreconditioning strategies, where either the iterative method or preconditioning incorporates more than one preconditioner. There are several ways in which multipreconditioning can be employed but it is salient to consider the computational cost incurred weighed against the performance improvements that might be gained. Note, however, that such a strategy might not simply be aiming to give the optimal performance for solving a given system but to provide an overall robustness during a simulation spanning differing regimes.

A simple way to incorporate multiple preconditioners into an iterative method is to change the preconditioner at each iteration, in which case a flexible solver such as FGMRES [9] is required. This is exemplified in *cycling*, where the preconditioner choice changes in a prescribed cyclic order [8]. However, results show convergence never better than the best choice of preconditioner on its own; though such a choice is unknown in advance. While only observed empirically, it stands to reason that this is unlikely to provide improvement over the best preconditioner for any given linear system, though it may help provide robustness over a sequence of problems.

Another strategy is to form a single preconditioner from the options available. This is employed in *combination* preconditioning, in which the action of the inverse of the preconditioner is a linear combination of other preconditioner inverses. The term was introduced in [10] and pursued further in [7], however, their main focus is on maintaining symmetry or positive definiteness (in some nonstandard inner product) so more efficient iterative methods can be used. Nonetheless, combination preconditioning could equally be applied to nonsymmetric cases with less restriction on requiring certain parameter choices or need for a nonstandard inner product.

A similar idea, using linear combinations of preconditioned operators, is found in the earlier *multi-splitting* method [6]. The idea is to utilise multiple different splitting methods to solve the linear system. The approach can be thought of as a stationary iteration with each splitting providing a preconditioner. Yet, as with combination preconditioning, fixed weights for the contributions must be chosen in advance.

Except for cycling, these approaches allow for parallelism in the application of multiple preconditioners. However, the performance of the underlying iterative method will depend on the overall effectiveness of the preconditioners and how they are combined. Instead, we consider a multipreconditioned GMRES method [5] that retains the parallelisable application of preconditioners but computes weights as part of the algorithm which are, in some sense, optimal. It considers not just one new search direction at each iteration but several, given by each preconditioner. We note that the idea was first applied to the conjugate gradient method for symmetric positive definite systems in [3]. However, with multiple preconditioners the search

space grows exponentially fast as we continue to iterate. Thus, a selective variant of the algorithm which restricts this growth to be linear is typically necessary.

2 Multipreconditioned GMRES (MPGMRES)

In the standard preconditioned GMRES (or FGMRES) method, at each iteration a new search direction, based on the preconditioned operator, is added to the search space and then a least-squares problem is solved to find a solution with minimum residual norm. The key idea behind *multipreconditioned GMRES* (MPGMRES) [5] is to add *multiple* new search directions at each iteration coming from the different preconditioners available. In fact, the method adds all new search directions from combinations of the preconditioned operators applied to vectors in the current search space, making the search space very rich. An Arnoldi-type block procedure is then used to obtain an orthonormal basis of the search space. MPGMRES then computes the optimal new iterate from this space in the minimum residual least-squares sense. Hence, note that the weights defining the contributions from each preconditioned operator are computed as part of the procedure, unlike in other approaches.

To understand how this *complete* MPGMRES algorithm works, suppose we have $\ell \geq 2$ preconditioners \mathcal{P}_i , $i = 1, \dots, \ell$. We start with an initial residual vector $\mathbf{r}^{(0)}$, which we normalise to give the first basis vector $V^{(1)} = \beta^{-1}\mathbf{r}^{(0)}$, with $\beta = \|\mathbf{r}^{(0)}\|_2$, and collect together the preconditioned (normalised) residuals

$$Z^{(1)} = \beta^{-1} \left[\mathcal{P}_1^{-1}\mathbf{r}^{(0)}, \dots, \mathcal{P}_\ell^{-1}\mathbf{r}^{(0)} \right] \in \mathbb{R}^{n \times \ell}. \quad (1)$$

Using an Arnoldi-type block procedure we orthogonalise columns of $W = \mathcal{A}Z^{(1)}$ with respect to our current basis $V^{(1)}$ and amongst themselves by using a reduced QR factorisation. Normalising then provides new basis vectors $V^{(2)} \in \mathbb{R}^{n \times \ell}$.

At each iteration, k , we increase the MPGMRES search space by applying each of the preconditioners to our newest basis vectors $V^{(k)}$, computing

$$Z^{(k)} = \left[\mathcal{P}_1^{-1}V^{(k)}, \dots, \mathcal{P}_\ell^{-1}V^{(k)} \right] \in \mathbb{R}^{n \times \ell^k}. \quad (2)$$

The Arnoldi-type block procedure is then used to orthogonalise $W = \mathcal{A}Z^{(k)}$ with respect to the current basis $\tilde{V}_k = [V^{(1)} \dots V^{(k)}]$ and within itself. This yields new basis vectors $V^{(k+1)} \in \mathbb{R}^{n \times \ell^k}$ and, by storing the coefficients from the Arnoldi-type step in an upper Hessenberg matrix \tilde{H}_k , we obtain an Arnoldi-type decomposition

$$\mathcal{A}\tilde{Z}_k = \tilde{V}_{k+1}\tilde{H}_k, \quad (3)$$

where $\tilde{Z}_k = [Z^{(1)} \dots Z^{(k)}]$. Note that any linear dependency in columns of \tilde{Z}_k , due to redundancy in the user-provided preconditioners, can be avoided using deflation; see [5, §3]. Now that we have a search space then, similarly to FGMRES, we solve

a linear least-squares problem for the minimum residual solution to

$$\min_{\mathbf{x} \in \mathbf{x}^{(0)} + \text{range}(\tilde{Z}_k)} \|\mathbf{b} - \mathcal{A}\mathbf{x}\|_2 = \min_{\mathbf{y}} \left\| \|\mathbf{r}^{(0)}\|_2 \mathbf{e}_1 - \tilde{H}_k \mathbf{y}, \right\|_2, \quad (4)$$

where $\mathbf{x} = \mathbf{x}^{(0)} + \tilde{Z}_k \mathbf{y}$. Note that there is a natural generalisation of the standard GMRES polynomial minimisation property, as detailed in [5].

While the search space for complete MPGMRES is very rich, we note that it grows exponentially at each iteration, and thus becomes prohibitive in practice. As such, a variant which selects only some of the potential search directions, ideally ensuring only linear growth, is natural to consider as a more practical alternative.

3 Selective MPGMRES (sMPGMRES)

To balance the benefits gained by adding multiple search directions with the storage and compute costs, we might wish to fix the number of preconditioner applications and matrix–vector products independent of the iteration, allowing for parallelisation of these operations via use of a fixed number of processors. To do so, we consider limiting the growth of the search space to be linear with respect to the iteration number k by using a *selective MPGMRES* (sMPGMRES) algorithm outlined in [5].

The search directions in MPGMRES are given by a collection of column vectors Z . To limit the growth of the search space we limit the size of Z , in particular to be proportional to the number of preconditioners, independent of k . To do this we select only certain search directions from the span of the columns of Z , giving a selective MPGMRES algorithm. There are many strategies to choose these directions, for instance, instead of applying the preconditioners to all columns of $V^{(k)}$, as in (2), we might apply them to just a single vector from $V^{(k)}$, selecting this vector differently for each preconditioner. This selection choice need not be the same at each iteration and could incorporate randomness if desired. The corresponding $Z^{(k)}$ is then

$$Z^{(k)} = \left[\mathcal{P}_1^{-1} V_{:,s_1}^{(k)}, \dots, \mathcal{P}_\ell^{-1} V_{:,s_\ell}^{(k)} \right], \quad (5)$$

where $V_{:,s_i}^{(k)}$ is the s_i th column of $V^{(k)}$ and s_i might change with k .

An alternative to applying each preconditioner to just one vector from $V^{(k)}$ is to apply them all to a linear combination of these vectors, namely to $V^{(k)} \boldsymbol{\alpha}^{(k)}$ for some vector $\boldsymbol{\alpha}^{(k)}$ of appropriate size detailing the contribution from each column of $V^{(k)}$. The corresponding $Z^{(k)}$ is then

$$Z^{(k)} = \left[\mathcal{P}_1^{-1} V^{(k)} \boldsymbol{\alpha}^{(k)}, \dots, \mathcal{P}_\ell^{-1} V^{(k)} \boldsymbol{\alpha}^{(k)} \right] \in \mathbb{R}^{n \times \ell}. \quad (6)$$

Note that a natural choice for $\boldsymbol{\alpha}^{(k)}$ is the vector $\mathbf{1}$, of all ones. All of these selection methods result in choosing a lower dimensional subspace of the full space and then

minimising over this subspace. With these selection strategies, where we limit $Z^{(k)}$ to ℓ new directions each iteration, \tilde{V}_{k+1} has $k\ell + 1$ basis vectors while the number of columns of \tilde{Z}_k is $k\ell$. Hence, the storage is proportional to k , as in FGMRES, as opposed to exponential in k , like complete MPMGRES.

Now suppose we have reason to favour one preconditioner over another and, for simplicity, that there are just two candidate preconditioners \mathcal{P}_1 and \mathcal{P}_2 . We would like our selective approach to incorporate knowledge of which preconditioner to favour. As such, we might choose an $\alpha^{(k)} = \alpha$ to weight more the contributions coming from one of the preconditioners. Consider the initial steps in sMPGMRES: we start with new search directions $Z^{(1)}$ and orthogonalise them to be $V^{(2)}$

$$Z^{(1)} = \beta^{-1} \left[\mathcal{P}_1^{-1} \mathbf{r}^{(0)}, \mathcal{P}_2^{-1} \mathbf{r}^{(0)} \right] \xrightarrow{\text{orthog.}} V^{(2)}, \quad (7)$$

then add search directions $Z^{(2)}$ which are orthogonalised to be $V^{(3)}$

$$Z^{(2)} = \left[\mathcal{P}_1^{-1} V^{(2)} \alpha, \mathcal{P}_2^{-1} V^{(2)} \alpha \right] \xrightarrow{\text{orthog.}} V^{(3)}. \quad (8)$$

So $\alpha = (\alpha_1, \alpha_2)^T$ weighs the contributions from each of the two preconditioners as $V^{(2)} \alpha = \alpha_1 V_{:,1}^{(2)} + \alpha_2 V_{:,2}^{(2)}$ and the two columns of $V^{(2)}$ come from the two different preconditioned residuals. If we let $\alpha = (\alpha, 1 - \alpha)^T$, for some $\alpha \in (0, 1)$, then the parameter α states how much we favour the first preconditioner, with $\alpha = \frac{1}{2}$ giving equal weighting and being equivalent to using the vectors of all ones ($\alpha = \mathbf{1}$), as suggested above. Similar strategies could be used to weight contributions from more than two preconditioners.

In this weighted version of sMPGMRES the ordering of the preconditioners $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ is important as we weight them differently. However, even with equal weighting (that is, $\alpha = \mathbf{1}$) ordering is important. This more nuanced asymmetry within sMPGMRES is an aspect not mentioned in [5]. The asymmetry comes about from the need to orthogonalise the new search directions in $Z^{(k)}$ within themselves. The contribution from the first preconditioner is allowed to be in any new direction but this direction is taken out of the contribution from subsequent preconditioners, and so on as we orthogonalise *in order* the contributions from all preconditioners. This means that if the direction from the last preconditioner is mostly within the span of the preceding directions it may well contribute very little of value, despite coming from a good preconditioner when applied by itself. As a general rule then, we might value less these final search directions as the useful components may have already been taken out. This suggests taking a weighting α which decreases in the components, instead of being equal, might be preferred. Nonetheless, in practice with a small number of good preconditioners, $\alpha = \mathbf{1}$ might suffice to be as good. We will see that when we favour a preconditioner the ordering will matter, even if we are weighting the preconditioners in the same way. Further, ordering can still have a significant impact even when just two preconditioners are used and they are weighted equally, especially when one of the preconditioners is poorer.

4 Numerical results for sMPGMRES

Here we apply sMPGMRES within a two-phase incompressible Navier–Stokes flow problem. That is, to solve linear systems associated with discretisation of

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \left(\mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right) + \nabla p = \rho \mathbf{f}, \quad (9a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (9b)$$

for velocity \mathbf{u} and pressure p where density ρ and dynamic viscosity μ are piecewise constant, representing the two phases. An important dimensionless quantity that appears is the dominating Reynolds number Re over the two phases, a parameter which quantifies the ratio of inertial to viscous forces within a fluid. Our results will also exhibit how performance depends on Re . An auxiliary equation to describe how ρ and μ vary in time with the flow is required, such as a level set equation; for the full model see [2]. We consider seeking the \mathcal{Q}_2 – \mathcal{Q}_1 finite element solution using Newton iteration to treat the nonlinearity. We utilise block preconditioners, in particular those introduced in [2]. These are two-phase versions of the *pressure convection–diffusion* (PCD) and *least-squares commutator* (LSC) approaches [4]. To answer questions of robustness we further use a SIMPLE-type preconditioner, also discussed in [2]. We restrict our results to focus on the two preconditioner case ($\ell = 2$) using (6) with $\alpha^{(k)} = (\alpha, 1 - \alpha)^T$ for some $\alpha \in (0, 1)$. We follow exactly the simplified problem of a lid-driven cavity used in [2] along with the same implementations, as such we omit the details for brevity. The only difference is we now use sMPGMRES to solve the Newton systems via the MATLAB implementation¹ which accompanies [5].

We focus on iteration counts, as opposed to timings, since our implementation runs in serial and so does not take advantage of the inherent parallelism. Note that, when we tabulate our results using sMPGMRES, the iteration counts given in bold emphasise the best choice of weighting parameter α which provides the minimum number of iterations for a given pair of preconditioners. The preconditioner given on the left of a set of results is used as the first preconditioner in sMPGMRES.

Table 1 displays results for combining two-phase PCD and LSC. We see that the best iteration counts are seen towards the centre of the table, that is with a weighting parameter α closer to $\frac{1}{2}$, though we see some bias towards larger α for both orderings as the asymmetry of ordering might suggest. In this example most choices of α will provide some improvement over either of PCD or LSC individually while the best choice can allow convergence using up to 32% fewer iterations. Note that the choice $\alpha = \frac{1}{2}$ typically gives iterations counts close to optimum. Given that it is not clear that we necessarily should do any better than the best preconditioner by itself, these results are quite promising and show that sMPGMRES can improve performance in terms of the number of iterations required. Furthermore, we see in this case that the performance is not particularly sensitive to α . To examine robustness, we now include the SIMPLE-type preconditioner, a method which performs poorly here.

¹ www.mathworks.com/matlabcentral/fileexchange/34562-multi-preconditioned-gmres

Table 1 Average preconditioned sMPGMRES iterations upon Newton linearisation using weighted combinations of PCD and LSC with density ratio 1.2×10^{-3} , viscosity ratio 1.8×10^{-2} (values for air-water flow), $h = 1/64$, and varying Reynolds number Re and time-step Δt .

Δt	Re	PCD	α in PCD-LSC					LSC	α in LSC-PCD					PCD
			0.9	0.7	0.5	0.3	0.1		0.9	0.7	0.5	0.3	0.1	
10^{-1}	10	16	14	14	14	16	18	18	15	14	15	15	21	16
	$10^{1.5}$	16	14	13	13	14	16	17	14	13	14	15	18	16
	100	15	14	13	14	16	17	26	15	14	13	14	16	15
	$10^{2.5}$	16	14	12	12	12	14	14	12	11	12	14	24	16
	1000	19	15	13	14	15	15	19	13	13	13	32	38	19
1	10	19	18	17	18	21	23	24	19	18	18	18	21	19
	$10^{1.5}$	21	19	18	18	20	22	23	19	18	18	19	22	21
	100	25	21	18	20	21	23	32	21	19	18	21	26	25
	$10^{2.5}$	27	24	19	18	18	20	22	18	17	19	25	37	27
	1000	31	27	24	24	24	26	34	25	23	26	37	63	31
10	10	20	19	18	19	22	23	25	20	19	19	19	22	20
	$10^{1.5}$	24	21	20	21	23	25	26	22	20	21	22	27	24
	100	30	26	23	26	27	28	42	27	25	23	26	30	30
	$10^{2.5}$	35	31	29	27	30	35	38	32	29	28	32	41	35
	1000	44	47	40	44	41	49	58	43	38	39	46	85	44

Table 2 Average preconditioned sMPGMRES iterations upon Newton linearisation using weighted combinations of LSC and SIMPLE with density ratio 1.2×10^{-3} , viscosity ratio 1.8×10^{-2} (values for air-water flow), $h = 1/64$, and varying Reynolds number Re and time-step Δt .

Δt	Re	LSC	α in LSC-SIMPLE					SIMPLE	α in SIMPLE-LSC					LSC
			0.9	0.7	0.5	0.3	0.1		0.9	0.7	0.5	0.3	0.1	
10^{-1}	10	18	18	19	24	48	97	164	48	24	19	18	19	18
	$10^{1.5}$	17	16	17	21	32	85	154	41	21	17	16	18	17
	100	26	23	22	27	35	51	131	38	26	23	22	24	26
	$10^{2.5}$	14	14	15	18	27	35	116	33	19	15	15	15	14
	1000	19	19	20	22	24	39	109	31	25	21	21	21	19
1	10	24	22	24	32	60	93	177	40	27	23	22	24	24
	$10^{1.5}$	23	21	24	31	60	95	185	40	27	23	22	25	23
	100	32	30	31	40	70	103	188	60	36	31	30	31	32
	$10^{2.5}$	22	20	21	26	50	109	190	46	25	21	20	22	22
	1000	34	31	33	43	51	78	190	62	38	35	32	32	34
10	10	25	22	25	33	60	96	179	40	28	23	23	24	25
	$10^{1.5}$	26	24	26	34	62	98	192	41	29	24	25	27	26
	100	42	38	39	47	69	104	207	64	42	37	36	39	42
	$10^{2.5}$	38	33	35	42	82	125	233	54	38	33	34	37	38
	1000	58	49	51	64	96	150	294	85	57	51	49	52	58

Table 2 combines the LSC and SIMPLE-type preconditioners. We see that, when LSC is used as the first preconditioner, primarily there is relatively little gained from including the SIMPLE-type approach with the best choice either being to simply use LSC or else a large α favouring LSC, though the best reduction in iteration counts does reach to 15%. However, if we change the ordering to have the SIMPLE-type approach first, the picture looks slightly different. While the best iteration counts are very similar, this time any $\alpha \leq \frac{1}{2}$ gives results comparable to LSC. This suggests

that, while we do not gain much in the way of improved performance, the algorithm is still fairly robust to varying α so long as we do not favour the poorly performing preconditioner too strongly. This example also provides a case where, with equal weighting ($\alpha = \frac{1}{2}$), the ordering of the preconditioners can substantially matter, with one choice giving iteration counts that are similar or better than LSC and the other giving results that are somewhat worse than LSC. Furthermore, it is by putting the worst preconditioner first (which by the asymmetry is subtly favoured) that we obtain the better results. While at first this may sound counter-intuitive, we can make sense of this observation by considering what the selection in sMPGMRES is doing. If the good preconditioner is used first then we take this contribution away from that of the second preconditioner, likely making it even worse, then by equally weighting these we are allowing a large component of this much worse contribution to prevail. On the other hand, if the worse preconditioner is first, we remove this component from the contribution of the better preconditioner, which is unlikely to make this contribution worse and may possibly make it even better. Thus we see this latter combination is more favourable than the former, though we may not expect it to provide significantly better results than the best preconditioner by itself. We note that, in results not shown, a somewhat similar scenario occurs when combining PCD and the SIMPLE-type approach; see also [1] for further numerical results.

Our study show promise that sMPGMRES can combine multiple preconditioners to reduce overall iteration counts and, additionally, provide robustness in situations when one preconditioner is performing poorly. Further, weights can be incorporated to favour preconditioners and results are not particularly sensitive to any sensible choice of weights, though ordering can be important. It remains to confirm how much speed-up can be gained from sMPGMRES but initial results in [5] are positive.

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