A novel hybrid Neumann expansion method for stochastic analysis of mistuned bladed discs

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Abstract

The paper presents a novel hybrid method to enhance the computational efficiency of matrix inversions during the stochastic analysis of mistuned bladed disc systems. The method is based on the use of stochastic Neumann expansion in the frequency domain, coupled with a matrix factorization in the neighbourhood of the resonant frequencies. The number of the expansion terms is used as an indicator to select the matrix inversion technique to be used, without introducing any additional computational cost. The proposed method is validated using two case studies, where the dynamics an aero-engine bladed disc is modelled first using a lumped parameter approach and then with high-fidelity finite element analysis. The frequency responses of the blades are evaluated according to different mistuning patterns via stiffness or mass perturbations under the excitation provided by the engine orders. Results from standard matrix factorization methods are used to benchmark the responses obtained from the proposed hybrid method. Unlike classic Neumann expansion methods, the new technique can effectively update the inversion of an uncertain matrix with no convergence problems during Monte Carlo simulations. The novel hybrid method is more computationally efficient than standard techniques, with no accuracy loss.

Introduction

A bladed disk typically consists in a set of disk/blade sectors designed to be identical. However, the identity between sectors is purely theoretical, because uncertainties associated to manufacturing, wear and tear create uneven distributions of the mechanical and geometric properties, both in individual blades and in the whole disk assembly. Dimensional and material uncertainties lead to departures of the natural frequencies of the blades from their nominal design values, creating the so-called “blade mistuning” phenomenon. During free-free vibrations, mistuning separates repeated eigenvalues associated with circumferential modes and distorts the corresponding mode shapes [1-5]. At the same time, the circumferential mode shapes increase the harmonic content of the nodal diameters, leading to the coupling with engine-induced vibrations. In the worst-case scenario, mistuning also causes mode localization phenomena, for which the vibrational energy is transferred and confined to only a few blades in the rotor. This may result in dynamic deformations that are significantly larger than those estimated at the design stage [6, 7]. Mistuning compromises the high-cycle fatigue resistance of bladed disks, and reduces the endurance and the reliability of the whole engine. It is therefore important to predict accurately the effects of geometric and material uncertainties on the maximum dynamic response in any of the blades of the disk assembly using stochastic analysis (e.g. Monte Carlo simulations, perturbation methods, spectral approach and stochastic reduced basis techniques) [8-10].

Analytical and numerical models of mistuning are generally more cost-effective than direct experimental characterizations [11]. High-fidelity finite element models of bladed disks are therefore commonly used to predict both the maximum dynamic excitation at representative design points and the associated behaviour due to pre-defined mistuning patterns [12, 13]. The uncertainty is commonly introduced by using added masses, spring elements or scatter of the material properties of the blades. However, a complete FE model of a bladed disk typically comprises millions of degrees of freedom (DOFs), making stochastic analysis too expensive even using state-of-the-art high-performance computers [14]. The problem is exacerbated by the fact that simplified FE models of single sectors cannot be used, since mistuning breaks the cyclic symmetry of bladed-disk systems. Many reduced order techniques have been proposed to increase the computational efficiency of the stochastic analysis of mistuning. Modal reduction based sub-structuring methods have been extensively developed in the last two decades, with typical examples represented by Component Mode Synthesis (CMS), Fundamental Mistuning Model (FMM), Component Mode Mistuning (CMM), and Integral Mode Mistuning (IMM) [15]. These techniques are able to improve significantly the efficiency of the forced frequency response analysis. Moreover, efficient stochastic simulation techniques have been developed to reduce the number of samples.
required for the Monte Carlo stochastic analysis, such as importance sampling, accelerated Monte Carlo simulations and Subset Simulation techniques. The latter have proven to yield the best performance so far [16].

For simulations based on stochastic FEMs, the high computational costs of mistuning analyses are mainly due to the repeated inversions of the dynamic stiffness matrix, especially when the scale of the finite element (FE) model is large. However, any direct or reduced method will require the inversion of a dynamic stiffness matrix to obtain the frequency responses of the mistuned systems. Efficient inversion techniques are therefore highly sought after, because the size of the dynamic matrix of bladed disc systems is large, as the scale of the FE model usually involves millions of DOFs. Even reduced order models require considering a large number of modes and interface DOFs. Moreover, thousands of simulations are required in stochastic methods to obtain reliable statistical estimates. Matrix factorization methods, such as the LU decomposition, are commonly used to reduce the computational costs associated with matrix inversions. The main idea behind these techniques is to express the dynamic stiffness as a product between two banded triangular matrices, consisting in permutations of lower and upper triangular matrices. This way, the dynamic response can be efficiently computed by inverting two banded sub-matrices. The repeated inversions in stochastic analysis can also be potentially avoided by using either the Sherman-Morrison-Woodbury (SMW) formula or the Neumann expansion method (NEM). The SMW formula gives an explicit and efficient expression for the inverse of a matrix perturbed by adding a rank-one update based on the knowledge of the unperturbed inverse matrix (this also called the ‘Matrix Inversion Lemma’). The main advantage of using the matrix identity formula is to eliminate expensive repeated inversions in the stochastic analysis, therefore obtaining a significant reduction of the associated computational costs. For this reason, the SMW formula has been applied to the dynamic response of a simple linear system in [17], and also successfully used in the non-linear analysis of systems with frictional dampers [18]. Moreover, this method in principle allows the mistuning analysis of bladed discs via the same FE models that are employed in the analysis of tuned bladed discs via a cyclic symmetry approach [19]. The case study from reference [20] shows that the SMW algorithm can be efficiently used to update the inversion of an uncertain matrix without the need of multiple separated inversions. However, the computational time associated with this modified approach is only 20% smaller when compared to a full LU factorization. The reduction in computational cost depends on the ratio between the numbers of mistuned blades introduced versus the size of the representative model. However, the technique becomes inefficient and even more computationally expensive than LU factorization when more than two blades are perturbed. In this sense, NEM provides an alternative approach to circumvent the direct inversion of an uncertain dynamic stiffness matrix. NEM is based on an iterative process involving the solution of hierarchical linear system of algebraic equations. The NEM technique represents the uncertainty matrix as the sum of deterministic and stochastic matrices. Neumann series are then used to expand the coefficients of the perturbation matrix, and the solution of the inversion problem is therefore also represented as a series. The random fluctuation range allowed by the NEM is generally larger than the one treatable by other perturbation techniques, because it is easier to incorporate high order terms [21]. However, NEM breaks down when the spectral radius of the iterative matrix is larger than one [22, 23]. A scalar-modified NEM approach has been proposed to effectively guarantee convergence with large perturbations [24]. Yuan et.al [20] have investigated the feasibility of using NEMs in stochastic analysis of mistuned bladed disc systems. Results show that NEM can decrease the computational costs by 200% compared with direct matrix factorization approaches, with a maximum relative error smaller than 2%. The convergence of the NEM is however still not guaranteed when the excitation frequency approaches resonance, especially when the system has a low damping and/or a high perturbation level. To some extent, the scalar-modified NEM can reduce the size of the divergence region near resonance, or even eliminate the divergence at low perturbation levels. However, its convergence cannot be completely guaranteed within the practical range of damping and stiffness perturbations routinely used in mistuning analysis.

This work presents an adaptive hybrid expansion approach to overcome the convergence issues associated with NEM. The hybrid approach proposed is based on using the Neumann expansion in the whole frequency domain, with the exception of frequency bands that contain resonances. An adaptive indicator is proposed to detect the level of divergence of the Neumann expansion and therefore select a proper inversion technique. The paper is organized as follows: the formulation of the mistuned bladed disc system is firstly presented, and then followed by the detailed presentation of the hybrid expansion approach, with particular emphasis on the development of the adaptive indicator and the simulation strategy. Two case studies are then presented to validate the effectiveness of the proposed method, using a low order and a high-order model, respectively (where the order refers to the number of degrees of freedom, i.e. DOF). The results of the hybrid simulation are benchmarked by those obtained from a standard LU matrix factorization method.
The mistuned bladed disc system

The Equations of Motion (EoM) for forced harmonic vibration of a linear bladed disc system can be expressed in the frequency domain as [19]:

\[
[-\omega^2 M + K + iD]q = F
\]  

(1)

In Eq. (1), \(M, K, D\) are the global mass, stiffness and damping matrices, while \(q\) and \(F\) represents the vectors of the dynamic displacements and external generalized forces, respectively. If structural damping is considered in the system (i.e. hysteretic damping), the matrix \(D\) is proportional to the stiffness matrix and expressed as \(\eta K\), where \(\eta\) is the damping coefficient.

The external force applied here is a harmonic excitation that differs only in phase from one blade to the next. This is expressed as [8]:

\[
F = F_0[1 \ e^{j\phi_2} \ e^{j\phi_3} \ e^{j\phi_4} \ ...]
\]  

(2)

where:

\[
\phi_i = \frac{2\pi C(i - 1)}{N_b}
\]  

(3)

In Eq. (3), \(C\) is the engine order, \(\phi_i\) is the phase angle of the \(i^{th}\) blade and \(N_b\) is the total number of the blades. The structural uncertainties are generally introduced into the system by using either stiffness or mass perturbations. The modified EoM of the system can be written by introducing the nominal mass and stiffness matrices \(M_0, K_0\) and their respective perturbations \(\Delta M, \Delta K\):

\[
[-\omega^2(M_0 + \Delta M) + (K_0 + \Delta K) + iD]q = F
\]  

(4)

The dynamic stiffness in Eq. (4) can be expressed as a sum of two matrices: a dynamic stiffness corresponding to the tuned system \(Z_0\), and a mistuning matrix \(\Delta Z\), which leads to the perturbation of the response with respect to the tuned system; hence, we have

\[
(Z_0 + \Delta Z)q = Zq = F
\]  

(5)

where \(Z_0\) and \(\Delta Z\) are expressed as:

\[
Z_0 = -\omega^2 M_0 + K_0 + iD
\]  

(6)

\[
\Delta Z = -\omega^2 \Delta M + \Delta K
\]  

(7)

The response of the mistuned bladed disc is therefore given by:

\[
q = (Z_0 + \Delta Z)^{-1}F = Z^{-1}F
\]  

(8)

Novel hybrid expansion approach

Matrix inversions in general require a large amount of CPU time, especially in the case of large-scale models and when stochastic response properties are of interest. Furthermore, although the dynamic stiffness matrix \(Z\) is usually narrowly banded, \(Z^{-1}\) is not. The multiplication on the right hand side of Eq. (8) cannot be performed efficiently when the number of DOFs is large. The work by Yuan et al [20] shows that, taking advantage of a NEM approach, the computational costs in the range of frequency where the Neumann expansions is convergent can be reduced by 200% with respect to direct matrix factorization approaches. The maximum relative error was found to be lower than 2%. Divergence occurs only in a very small portion of the whole frequency range. Based on this fact, the basic idea behind the proposed hybrid approach is to employ the matrix factorization technique within the frequency range close to resonance and the NEM for the remaining frequency band of interest. An adaptive indicator is introduced to select automatically the most appropriate inversion technique at a particular excitation frequency.

Neumann expansion techniques
When the external frequency of excitation $\omega$ is far from the resonance region, the original Neumann expansion technique is used to avoid the repeated inversions of the mistuned dynamic stiffness $Z$. By using the expansion coefficients for the inversion of the perturbation part $\Delta Z$, the inverted dynamic stiffness in Eq. (8) can be then expressed as [24]:

$$
(Z_0 + \Delta Z)^{-1} = (I - E + E^2 - E^3 + E^4 - E^5 + \cdots)Z_0^{-1},
$$

where

$$
E = Z_0^{-1}\Delta Z.
$$

By substituting the Eq. (9) into Eq. (8), the solution vector $q$ is represented by the following series:

$$
q = q_0 - E^1q_0 + E^2q_0 - E^3q_0 + \cdots
$$

where $q_0$ is the frequency response of the tuned system. Assuming that $q_i = E^iq_0$, this series is equivalent to the solution of the following set of recursive equations:

$$
q_i = Eq_{i-1}
$$

The expansion in Eq. (9) is truncated to the $i$-th term if the series has converged according to the following criterion:

$$
\frac{\|q_i\|_2}{\|\sum_{n=0}^i(-1)^nq_n\|_2} \leq r_{err}
$$

where $r_{err}$ is a predefined tolerance (0.01 is commonly used [24] and the same value is adopted here) and $\|q_i\|_2$ is the norm of the response vector defined as:

$$
\|q_i\|_2 = \sqrt{q_i^Tq_i}
$$

**Matrix factorization method**

For an external excitation with frequency sufficiently close to resonance, factorization is used to reduce the computational cost of the matrix inversion. As the diagonal elements in the matrix are no longer guaranteed to be real and positive due to the complex damping matrix, a Cholesky decomposition technique would not be effective, and an LU matrix factorization approach is therefore employed for the inversions [25]. The dynamic stiffness is expressed as a product of two essentially triangular matrices (lower and upper with permutation) as follows:

$$
LU = Z
$$

It is therefore possible to solve the following systems of equations with respect to the unknown vectors $X$ and $q$ separately:

$$
LX = F
$$

$$
Uq = X
$$

The $X$ and $q$ vectors can be efficiently obtained from Eqs. (16)-(17), because the lower triangle matrices $L$, $U$ possess the same bandwidth as $Z$.

**Adaptive indicator for the selection of the simulation techniques**

An indicator able to account for the effectiveness of the NEM approach is essential to switch to the more computationally efficient technique at a given frequency. NEM does not converge when the maximum eigenvalue ($\lambda_{max}$) of $E$ is larger than one [24]. However, the use of the maximum eigenvalue as the indicator is not efficient from the computational standpoint, because of the high CPU costs involved in solving eigenvalue problems. Also, even if $\lambda_{max}$ is less than 1 at a particular frequency, it is not completely guaranteed that using Neumann expansion would be computationally cheaper than matrix factorization. This is because, for frequency
values close to resonance, NEM may require calculating more than 50 terms to converge. Clearly, in cases as the latter, matrix factorization methods must be preferred to the NEM. This point will be further discussed later in the case studies. The number of the expansion terms (NoET) in the NEM is here used as an adaptive indicator for the hybrid method. The NoET is defined as:

\[
\text{NoET} = i, \text{ Once when } \frac{\|q_i\|_2}{\|q_{n=0}^{(-1)^n} q_n\|_2} \leq r_{\text{err}},
\]

(18)

Where the inequality is the same as in Eq. (9). With the increase of the number of expansion terms \(i\), the ratio on the left hand side will be gradually approaching the tolerance \(r_{\text{err}}\). Once the condition is satisfied, the term, \(i\) would be recorded as the value of the indicator at that particular frequency point. The threshold of the NoET to activate the switch between the methods generally depends on number of DOF and it must be chosen before the simulations. However, the NoET can be simply identified via a comparison of the computational costs respectively associated with the factorization method and the NEM. The methodology used to perform the adaptive hybrid Neumann expansion is the following:

1. The simulations start at a set frequency \(f_1\) by using the NEM approach with an increasing frequency step \(\Delta f\). The number of expansion terms in the NEM is used as the switch indicator.
2. Once the lower frequency boundary point \(f_{b1}\) is identified, simulations are restarted at a shifted frequency point \(f_2\) in between the first and second resonance frequencies. The simulation is performed again using NEM, and it starts at the shifted frequency point \(f_2\), but the frequency is decreased by a suitable \(\Delta f\). The number of expansion terms is still used as the switch indicator.
3. Once the upper boundary point \(f_{b2}\) is also identified, the inversion technique will be changed to the LU factorization to compute the forced frequency response in the frequency range between the \(f_{b1}\) and \(f_{b2}\).
4. The process is iterated by starting the simulations at the frequency \(f_2\) in order to calculate the forced response close to the second resonance.

In general, if the nominal resonance frequencies of the system are \([f_0^1, f_0^2, f_0^3, \ldots f_0^i, \ldots]\), the shifted frequencies considered in step (2) are taken as follows:

\[
f_{i+1} = \frac{1}{2} (f_0^i + f_0^{i+1})
\]

(19)

**Case study 1: Low order model**

Figure 1 shows a 2 DOFs per sector lumped parameter model, with one DOF representing the motion of the blade and the other the motion of the disk sector [8, 20, 26]. A stiffness \(k^b\) and mass \(m^b\) are associated to the blade DOF; similarly, the parameters \(k^d\) and \(m^d\) represent the stiffness and mass of the disc, respectively. The stiffness \(k^c\) is related to the coupling between blades through the disc. \(q_i^b, q_i^d\) are the generalized coordinates of blade and disc for the \(i^{th}\) sector, respectively. Engine order excitations are applied to the blade DOFs.

![Figure 1. Two DOFs per sector model](image)
In this model a material damping model, proportional to the stiffness matrix, is used. The equation of motion of the system can be expressed as:

\[
-\omega^2 M_0 + (1 + \eta_0)(K_0 + \Delta K)q = F
\]  

(19)

\[
\Delta K = \begin{bmatrix}
\Delta K^b \\
-\Delta K^b \\
\Delta K^b
\end{bmatrix}
\]  

(20)

\[
\Delta K^b = diag(\Delta k^b_1, \Delta k^b_2, \ldots, \Delta k^b_N), \quad \Delta k^b_i = \delta^b_i k^b
\]  

(21)

In Eq (20), \(\Delta K^b\) represents a perturbation matrix affecting only the stiffness of the blades; more in detail, \(\delta^b_i\) gives the perturbation of the stiffness for the \(i^{th}\) blade from its nominal value \(k^b\). Further information regarding the formulation of these perturbation matrices can be found in references [8, 26]. The present case study investigates structural uncertainties related only to the stiffness. The values of the perturbations are randomly generated according to a log-normal probability density function, in order to avoid negative blade stiffness values due to the tail of the distribution. Both small and large mistuning levels can be introduced into this model. Figure 5 shows three examples of mistuning patterns for the blade stiffness perturbations.

![Figure 2 Three mistuning patterns for the perturbations in the blade stiffness](image)

The parameters of the lumped model are identified from the response of the full-scale finite element model shown in Figure 9, which is based on the trends of the natural frequency curves in the veering region. It is worth noticing that this parametric identification is not aimed at capturing the specific modes from the full-scale finite element model, but aims to identify the variation of the natural frequencies only in the veering regions, which is the most significant for mistuning analysis purposes. This fitting process follows the methodology proposed in [27] to identify the main lumped stiffness and mass parameters \((k^b, k^c, m^d, m^b)\). A non-dimensional frequency \(\tilde{\xi} = \omega/\omega_0\) is also introduced, whereby \(\omega_0\) corresponds to the modal frequency of the lumped parameter blade-alone model, calculated from non-dimensional stiffness and mass parameters. Figure 5 shows the direct comparison of the variation of the NFs versus the NDs between the lumped parameter and the FE model. The two models are in good agreement, in particular (as expected) within the veering region at low NDs. It can be therefore concluded that the lumped parameter model is an adequate representation of the dynamic behaviour of the two dominating flapping modes of the bladed disc. The modal analysis has been performed using a Block Lanczos method on a PC with an Intel 3.2 GHz dual-core processor and 4 GB RAM.
Comparisons between the natural frequencies versus the nodal diameters between the lumped parameter and the finite element models

**Adaptive indicator**

Figure 4 shows the relative fraction of the computing time when using the NEM approach, with different normalized NoET. The CPU time required by the matrix factorization method is also indicated as benchmark. The percentage of CPU time fractions linearly increases with the NoET used. In terms of the computational effort, the NEM becomes less efficient than the factorization techniques when more than 20 terms are required for relative error of 0.01. Figure 5 shows the map plots of the required NoET and $\lambda_{\text{max}}$ when using the Neumann expansion with a varying damping coefficient within the normalized frequency range [0.8 – 1.15]. The map plot in Figure 5(a) is truncated for a number of expansion terms equal to 20, while in Figure 5(b) the representation is truncated when the NEM does not converge. The white color region in shown in Figure 5(a) is wider than the one shown in Figure 5(b) for the same mistuning pattern, meaning that the size of the inefficient region for the use of NEM techniques is actually larger than that associated with NEM divergence. From these results, it may be argued that the continued use of NEM would increase the computational effort in this convergent but computationally expensive region. Moreover, these results also show that the use of the NoET as the adaptive indicator is more effective to reduce computational expense than the use of $\lambda_{\text{max}}$. Furthermore, no additional computational cost is introduced when the NEM is used, because the indicator can be extracted from the NEM itself. Figure 6 shows the truncated 2D map plot of the required NoET used in the NEM approach, within the normalized frequency range [0.8 – 1.15] and for 100 Monte Carlo simulations at the perturbation level of 1%. The adaptive indicator, i.e. the number of the expansion terms, can effectively identify the “zig-zag” boundaries occurring during the Monte Carlo simulations.
Figure 4 The percentage of computing time using NEM against the one required by the matrix factorization method at different numbers of expansion terms

(a) Figure 5 Surface responses related to (a) the number of expansion terms and (b) maximum eigenvalue by using the Neumann expansion technique (Structural damping 0.5%, mistuning level 1%)

Figure 6. 2D truncated surface response of the number of the required expansion terms to achieve a convergence in 100 MCS at perturbation level of 1% and structural damping of 0.5%

Accuracy and computational effort

Figure 7 shows the comparison between the forced frequency responses (amplitude factors) of the six blades when using the hybrid expansion and the LU factorization method. The threshold of the adaptive indicator is set at a value of 20, which is obtained from the observation of the relative computational costs shown in Figure 4. The results from the hybrid expansion method are almost identical to those from the benchmark LU factorization across the whole frequency range. The largest errors of the frequency response occur at normalized frequencies between 0.97 and 1.02, which are close to the switch points for which the NoET is approaching to 20. The enlarged view on the Figure presents the departure of the amplitude factors in this region. The average error close to the switch frequency points from 0.977 to 0.978 is however less than 0.1%. The results from the hybrid expansion method maintain a high accuracy across the overall frequency range when compared to the benchmark results.
Figure 7 Comparisons of the frequency response in six blades from the hybrid expansion method with exact solution from the baseline.

Figure 8 shows the 2D map plots of the ratios of the CPU time required by the 100 Monte Carlo simulations when using the hybrid expansion method and the LU factorization approach. Different damping coefficients and perturbation levels are considered. The hybrid expansion method outperforms the LU factorization when the ratio of CPU time is less than 1. Figure 8(a) shows the map of these ratios without using the proposed simulation strategy. The area where ratios are less than 1 corresponds to cases with extremely low perturbation levels. The associated CPU times are less than 50% compared to the factorization method case. However, the computational costs are significantly increased at higher perturbation levels, being 1.2 to 2 times higher than the benchmark LU cases when the perturbation level rises from 2.5% to 5%. This behaviour is mainly attributed to the additional CPU efforts required by the NEM in the inefficient region for computing the indicator, with double computing time been spent compared to the benchmark factorization. Figure 8(b) shows the map of the CPU times when the proposed hybrid simulation strategy is used. Compared to Figure 8(a), the region corresponding to ratios lower than 1.0 is significantly expanded and it is almost covering the whole range of the damping and mistuning levels considered in this example. The computational costs are reduced considerably when the perturbation level varies from 1.5% to 5%. The overall computational efficiency can be effectively improved on average by 50% when using the hybrid expansion within the proposed simulation strategy.

Figure 8 The ratios between CPU times required by the hybrid method and LU factorization for 100 Monte Carlo simulations, with varying damping and mistuning levels. Results without (a) and with (b) the hybrid simulation strategy.
Case study 2: Higher order model

The second case study consists in a full-scale finite element model of the bladed disc built by using the commercial software ANSYS Rel. 15.0 (Ansys Inc, 2013) (Figure 9). The model consists of 4224 3D-SOLID 95 8-node elements with 2 linear elements per unit thickness. Each node has three translational DOFs, generating a total of 160218 DOFs. The rotor is entirely made of titanium, considered as a homogenous and isotropic material (Young’s modulus of 115 GPa, density of 4800 kg/m³ and Poisson’s ratio of 0.32). The blade has a high slenderness ratio (10), and a uniform rectangular cross section. The model is clamped at the inner diameter of the disc by zeroing all DOFs at that particular location. A modal analysis has been performed using a Block Lanczos method on using a PC with an Intel 2.4GHz dual-core processor and 24 GB RAM.

Figure 9 Finite element model of (a) a bladed disc and (b) a single blade sector

Figure 10 shows the variations of the natural frequencies (NFs) of the first modes against the nodal diameters (NDs) of the bladed disc. The modal frequencies are normalized with respect to those associated with the first bending mode at the 12th ND. With increasing nodal diameters, the disc dynamic stiffness increases. Hence, the disc NFs also increasing, approaching those corresponding to isolated blades. The horizontal lines at high NDs represent the blade-dominated modes, while the slope-varying lines at low NDs indicate disk-dominated ones. The veering regions indicate some significant disk-blade interactions, which have an important effect on the forced frequency response of mistuned systems [15]. It is possible to observe that the 2nd and 3rd out-of-plane flapping modes exhibit significant NF changes within some specific intervals of the NDs. This suggests the existence of inter-blade coupling in these modes that facilitates the energy transfer between the blades through the disc. This mechanism can generate mode localization and extremely large forced frequency responses[15].

Figure 10 Natural frequencies versus nodal diameters from the full scale Finite Element model of the
bladed disc. FE

Perturbation method

The lumped mass approach is used to represent the uncertainty due to geometrical mistuning [28]. The lumped masses are usually placed at points corresponding to the highest modal displacements, so that the uncertainty can be successfully propagated in a specific frequency bandwidth that includes the modes considered. Three-dimensional mass elements (MASS21 in ANSYS) are lumped at the tip of the blades to provide the desired perturbation level for the first bending mode. In this case, a maximum of 2% perturbation level is assumed for the first bending eigenvector. Figure 11 shows the sensitivity of the natural frequencies corresponding to the first five modes from their nominal value when lumped masses are added to the disc. The mass values to achieve the predefined perturbation level for the first bending mode are about 1% of the bladed disc sector mass. It is worth noting that a negative mass can be used in the finite element to obtain a negative natural frequency change. Similarly to the low order model case, the log normal distribution is still used as the perturbation function to generate the mistuning patterns.

![Diagram](https://via.placeholder.com/150)

Figure 11 The sensitivity of the NF deviations of the first five modes to the perturbed masses

Figure 12 shows the normalized forced frequency response of the six blades for both the tuned and the mistuned systems when using the 0th engine order excitation. In the tuned system, all the blades have the same frequency responses. The external force can only excite the mode at a particular nodal diameter corresponding to the applied engine order. In contrast, the forced frequency responses of the mistuned system are different for the six blades, and the maximum frequency response can be either larger or smaller than its counterpart for the tuned system. Also, the frequency responses of the blades contain harmonics dependent on all the nodal diameters. This phenomenon is also discussed in reference [29].

![Diagram](https://via.placeholder.com/150)

(a) Tuned system

(b) Mistuned system

Figure 12 The forced frequency response of the (a) tuned and (b) mistuned systems
A trade-off study is carried out to identify the required NoET for the adaptive indicator. Figure 13 shows the ratio between CPU times when using the NEM approach and matrix factorization versus the number of expansion terms considered for the Neumann expansion. NEM becomes less efficient than matrix factorization when more than 15 terms have to be employed to achieve a relative error of 0.01. Hence fifteen terms are considered as the threshold for the adaptive indicator to switch the matrix inversion techniques. It is worth noticing that, also in the case of the full-scale finite element model, in computational terms the advantage of using the Neumann expansion is similar to the one observed for the low-order model (Figure 4).

![Figure 13 Ratio of the computing time between NEM and LU factorization for different numbers of expansion terms in the case of the high-order bladed disc](image)

**Accuracy and computational effort**

Figure 14 shows the comparison of the normalized frequency responses of three blades (blade 1, 2, 3 shown in Figure 9) at the NF perturbation level of 2% and damping of 1%. The frequencies are separately calculated using the hybrid expansion method and the LU factorization. One can observe in general a very good agreement across the frequency range for all the three blades. The consistency of the response close to the main resonance can be appreciated in the enlarged view (a). Similar to the first case study, the largest error occurs when the frequencies are close to the ones for which the switching the methods is necessary. The enlarged views (b) and (c) on the two sides across the major resonance provide a clear view on the largest departures of the responses at 42 Hz and 85Hz, respectively. Also in the high-order model, the average response error corresponding to the switch frequencies is less than 0.1%. This behaviour further confirms that the use of the hybrid method ensures a high accuracy when compared with results from classic factorization methods.
Figure 14 Comparisons of the frequency response in three blades by using the hybrid expansion method against the results obtained from using the benchmark techniques.

Figure 15 shows the 2D map plot of the ratios of the CPU times required by the hybrid expansion and the LU factorization to perform 50 Monte Carlo simulations at different damping coefficients (from 0.1% to 1%) and perturbation levels (from 0.1% to 1%), respectively. Also in this case, the hybrid method simulations use less CPU time (around 55% on average) than the LU factorization for all the mistuning and damping levels considered. The advantage of using the hybrid method becomes more obvious when the structural damping is large, or the mistuning levels decrease. This behaviour can be explained observing that high damping levels can increase the magnitude of the coefficients in the nominal dynamic stiffness matrix. This improves the convergence of the Neumann expansion. When the mistuning levels decrease, lower perturbation levels lead to smaller entries in the perturbation matrix, which reduce the spectral radius of the corresponding iterative matrix.

Figure 15 The ratios between hybrid method and LU factorization CPU times for 50 Monte Carlo simulations when varying the damping and mistuning levels.
Conclusions

It is well recognized that mistuning phenomenon due to the manufacturing, assembling tolerance and operational wear and tear compromises the high-cycle fatigue resistance of bladed disks, and also reduces the endurance and the reliability of the whole engine. Predicting the effects of uncertainties on the maximum dynamic response of the blades is therefore extremely important at the design stage. However, repeated inversions of the dynamic stiffness matrix are very expensive from a computational point of view, even when classic reduced-order methods like CMS and SNM are considered. In this paper, an efficient matrix-inversion technique has been proposed for improving the computational efficiency of the stochastic analysis of mistuned bladed disc systems. The method is based on using a hybrid combination of Neumann expansion and matrix factorization techniques, and identifying a suitable index (at no additional CPU time cost) to select automatically the most efficient numerical method to be used within the frequency range of interest. We considered two case studies with different mass mistuning patterns and log-normally distributed stiffness perturbations. Monte Carlo simulations showed a significant decrease of the required CPU times compared to classical matrix factorization techniques, while maintaining maximum relative errors below 0.1%. This increase in computational efficiency has been obtained for both small-scale and large-scale bladed disc models.

Acknowledgements

The authors would like to acknowledge the support of Rolls-Royce plc for the support of this research through the Composites University Technology Centre (UTC) at the University of Bristol, UK. Special acknowledgement goes also to the Strategic Investment in Low carbon Engine Technology (SILOET) programme supported by Rolls-Royce plc and Technology Strategy Board (TSB), and to the China Scholarship Council.

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