Stabilization and Destabilization of Hybrid Systems by Periodic Stochastic Controls

Xiaoyue Li, Wei Liu, Xuerong Mao, Junsheng Zhao

***Abstract*— This paper aims to determine whether or not a periodic stochastic feedback control can stabilize or destabilize a given nonlinear hybrid system. New methods are developed and sufficient conditions on the stability and instability for hybrid nonlinear systems with periodic stochastic perturbations are provided. These results are then used to examine stabilization and destabilization by periodic stochastic feedback controls, including intermittent stochastic controls.**

Key Words. Nonlinear hybrid differential equation, periodic stochastic control, Brownian motion, Markov chain, stationary distribution, stabilization.

1. Introduction

Stochastic modelling and control have played a crucial role in many applications. Since systems in the real world often need to run for a long period of time, an important problem concerns stability of such systems. It is not surprising that noise (or stochastic perturbation) can destabilize a stable system. But, as everything has two sides, noise can also be used to form a stochastic feedback control to stabilize a given unstable system. The pioneering work was due to Hasminskii [15, p.229], who stabilized a system by using two white noise sources. Later, Arnold et al. [4] showed, in particular, that the linear system $\dot{x}(t) = Ax(t)$ can be stabilized by zero mean stationary parameter noise if and only if trace$(A) < 0$. In the nonlinear case, Scheutzow [28] provided us with some examples on stabilization and destabilization in the plane, and Mao [21] developed a general theory on stabilization and destabilization by Brownian motion. The stochastic stabilization theory was then extended to functional differential equations, difference equations, partial differential equations by many authors (see, e.g. [2], [3], [8], [9]).

For the past 30 years, the hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters, for examples, electric power systems, the control system of a solar thermal central receiver, manufacturing systems, financial systems etc. (see, e.g., [5], [12], [13], [29], [31], [33], [34]). One of the important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the asymptotic analysis of stability (see, e.g., [6], [10], [14], [23], [24], [27], [30], [32], [36]. Naturally, stochastic stabilization and destabilization theory has also been generalised to hybrid systems (see, e.g., [11], [18], [25], [26]). In particular, the theory established can also be applicable to hybrid systems with asynchronous Markovian switching (see, e.g., [18]).

Typically, a hybrid system driven by a continuous-time Markov chain is described by

$$\dot{x}(t) = f(x(t), t, r(t)), \tag{1.1}$$

where $x(t)$ is in general referred to as the state and $r(t)$ is regarded as the mode and is modelled by a Markov chain on a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$. (We will explain the notation in more detail next section). The stochastic stabilization and destabilization theory is concerned with its stochastically perturbed system

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dB(t) \tag{1.2}$$

and to determine if this perturbed system becomes stable or unstable when (1.1) is unstable or stable, respectively, where $B(t)$ is a Brownian motion. Although both coefficients f and g are time-inhomogeneous, most existing results in this area assume they are bounded by time-homogeneous functions, say, for example, for each $i \in \mathbb{S}$, there are constants $\alpha_i, \rho_i, \sigma_i$ such that

$$x^T f(x, t, i) \le \alpha_i |x|^2, \ |g(x, t, i)| \le \rho_i |x|, \quad \text{and} \quad |x^T g(x, t, i)| \ge \sigma_i |x|^2 \tag{1.3}$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ (see, [25]). Such conditions do not make use of the time-inhomogeneous property

This work is entirely theoretical and the results can be reproduced using the methods described in this paper.

X. Li is with the School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, China. lixy209@nenu.edu.cn.

W. Liu is with the Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China. Corresponding author. weiliu@shnu.edu.cn.

X. Mao is with the Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK. x.mao@strath.ac.uk

J. Zhao is with the School of Mathematical Science, Liaocheng University, Liaocheng 252059, China. zhaojunsheng@lcu.edu.cn.

and hence the criteria obtained on stabilization and destabilization are conservative. We may ask

- **Question**: Could we make use of the time-inhomogeneous property of both coefficients f and g to establish better criteria on stabilization and destabilization?

For example, both coefficients f and g may be bounded by periodic functions

$$x^T f(x,t,i) \le \alpha_i(t)|x|^2, \ |g(x,t,i)| \le \rho_i(t)|x|, \\ \text{and} \quad |x^T g(x,t,i)| \ge \sigma_i(t)|x|^2 \qquad (1.4)$$

for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, where $\alpha_i(t), \rho_i(t), \sigma_i(t)$ ($i \in \mathbb{S}$) are all periodic functions of time t. Typically, this happens when both f and g are periodic functions with a common period in time. However, we should emphasize that it is not necessary to require f and g to be periodic in order for (1.4) to hold. We will explain the later situation in more detail when we discuss stochastic intermittent control.

Our key aim in this paper is to show the positive answer to the question. To achieve this aim, we first establish certain sufficient conditions on the stability and instability for stochastically perturbed hybrid systems in Sections 3 and 4. We then discuss the stabilization and destabilization for a class of nonlinear hybrid systems by periodic stochastic feedback controls, including stochastic intermittent controls, in Sections 5 and 6, For these purposes, we present some preliminaries on the hybrid stochastic differential equations (SDEs) in Section 2 first.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notation. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. For $T > 0$, denote by $\mathcal{K}_T$ the family of periodic functions $\kappa : \mathbb{R}_+ \to \mathbb{R}$ which are right continuous with left limits and have their period T.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). For a subset Ω_1 of Ω, denote its compliment by Ω_1^c. Let $B(t)$, $t \ge 0$, be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$. We assume that $B(t)$ and $r(t)$ are independent, and they are $\mathcal{F}_t$ adapted. As a standing hypothesis we assume in this paper that the Markov chain is *irreducible*. The algebraic interpretation of irreducibility is rank$(\Gamma) = N-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving the following linear equation $\pi\Gamma = 0$ subject to $\sum_{j=1}^N \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$. We assume that the initial distribution of the Markov chain (i.e., of $r(0)$) is fixed arbitrarily and we will not mention this any more.

Let us consider the hybrid SDE, or SDE with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dB(t) \quad (2.1)$$

on $t \ge 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^n$, where

$$f : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}.$$

Throughout the paper, both f and g satisfy the local Lipschitz condition and grow at most linearly (the precise growth condition will be given in the subsequent section). Under these conditions, equation (2.1) has a unique solution (see, e.g., Mao [26]). Denote the unique solution by $x(t; x_0)$ on $t \ge 0$. For the purpose of stability study in this paper we also assume that

$$f(0,t,i) \equiv 0 \quad \text{and} \quad g(0,t,i) \equiv 0 \ \text{ for each } \ i \in \mathbb{S}.$$

As a result, (2.1) admits a trivial solution $x(t; 0) \equiv 0$. It is also useful to recall an important property (see Mao [23, Lemma 2.1]):

$$\mathbb{P}\{x(t; x_0) \ne 0 \text{ on } t \ge 0\} = 1 \quad \forall x_0 \ne 0. \qquad (2.2)$$

That is, almost all the sample paths of any solution of equation (2.1) starting from a nonzero state will never reach the origin.

The purpose of this paper is to discuss the almost surely exponential stability and instability of the hybrid SDE (2.1). Let us recall the following definition (see, e.g., [19], [20], [22]).

Definition 2.1: The trivial solution of equation (2.1), or simply, equation (2.1) is said to be almost surely exponential stable if for any $x_0 \in \mathbb{R}^n$,

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t; x_0)|) < 0 \quad a.s.$$

It is said to be almost surely exponentially unstable if for any $x_0 \neq 0$

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t;x_0)|) > 0 \quad a.s.$$

3. Nonlinear Hybrid SDEs

To discuss the stability of the nonlinear hybrid SDE (2.1), we impose the following assumption.

Assumption 3.1: For each $i \in \mathbb{S}$, there are functions $\alpha_i(\cdot)$, $\rho_i(\cdot)$, and $\sigma_i(\cdot)$ in $\mathcal{K}_T$ such that

$$x^T f(x,t,i) \le \alpha_i(t)|x|^2, \;\; |g(x,t,i)| \le \rho_i(t)|x|, \quad \text{and} \quad |x^T g(x,t,i)| \ge \sigma_i(t)|x|^2 \tag{3.1}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Before we state our first theorem in this paper, let us make a useful remark.

Remark 3.2: The reader may wonder why we do NOT assume explicitly that both coefficients $f(x,t,i)$ and $g(x,t,i)$ are periodic functions in t with their period T. In general, it is natural to think both f and g should be periodic. However, this is not absolutely necessary. A typical example will be seen when we discuss the periodic intermittent control later. We should also point out that all $\rho_i(\cdot)$ and $\sigma_i(\cdot)$ are nonnegative but $\alpha_i(\cdot)$ may not.

Theorem 3.3: Under Assumption 3.1, the solution of equation (2.1) satisfies

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} \log(|x(t;x_0)|) &(3.2)\\ &\le \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T [\alpha_i(s) + 0.5\rho_i^2(s) - \sigma_i^2(s)]ds, \text{ a.s.}\end{aligned}$$

for all $x_0 \in \mathbb{R}^n$. In particular, the nonlinear hybrid SDE (2.1) is almost surely exponentially stable, if

$$\sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T [\alpha_i(s) + 0.5\rho_i^2(s) - \sigma_i^2(s)]ds < 0. \tag{3.3}$$

To prove the theorem, let us present two useful lemmas.

Lemma 3.4: For any $t \ge 0$, $v > 0$ and $i \in \mathbb{S}$,

$$\mathbb{P}(r(s) \neq i \text{ for some } s \in [t, t+v] \big| r(t) = i) \le 1 - e^{-\hat{\gamma} v}, \tag{3.4}$$

where $\hat{\gamma} = \max_{i\in S}(-\gamma_{ii})$.

Proof. Given $r(t) = i$, define the stopping time

$$\zeta_i = \inf\{s \ge t : r(s) \neq i\},$$

where and throughout this paper we set $\inf \emptyset = \infty$ (in which $\emptyset$ denotes the empty set as usual). It is well known (see, e.g., [1]) that $\zeta_i - t$ has the exponential distribution with parameter $-\gamma_{ii}$. Hence

$$\begin{aligned}&\mathbb{P}(r(s) \neq i \text{ for some } s \in [t, t+v] | r(t) = i)\\ &= \mathbb{P}(\zeta_i - t \le v | r(t) = i)\\ &= \int_0^v (-\gamma_{ii}) e^{\gamma_{ii} s} ds = 1 - e^{\gamma_{ii} v} \le 1 - e^{-\hat{\gamma} v} \qquad (3.5)\end{aligned}$$

as desired. □

Lemma 3.5: Let $\kappa_i(\cdot) \in \mathcal{K}_T$ for $i \in \mathbb{S}$. Then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \kappa_{r(s)}(s) ds = \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T \kappa_i(s) ds \;\; a.s. \tag{3.6}$$

Proof. We shall omit writing a.s. after inequalities or equalities in the proof. It is clear that

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} \int_0^t \kappa_{r(s)}(s) ds &(3.7)\\ &= \limsup_{k\to\infty} \Big(\frac{1}{(k+1)T} \sum_{j=0}^k \int_{jT}^{(j+1)T} \kappa_{r(s)}(s) ds \Big).\end{aligned}$$

Let $\bar{\kappa}$ be the bound for $\kappa_i(\cdot)$, namely

$$|\kappa_i(t)| \le \bar{\kappa} \quad \text{for all } t \ge 0 \text{ and } i \in \mathbb{S}. \tag{3.8}$$

Let $\varepsilon \in (0,1)$ be arbitrary and let $\delta = T/\bar{m}$ for a sufficiently large integer $\bar{m}$ so that $\delta < \varepsilon$. It then follows from (3.7) that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \kappa_{r(s)}(s) ds \le \sum_{u=0}^{\bar{m}-1} J_u, \tag{3.9}$$

where

$$J_u = \limsup_{k\to\infty} \frac{1}{(k+1)T} \sum_{j=0}^k \int_{jT+u\delta}^{jT+(u+1)\delta} \kappa_{r(s)}(s) ds. \tag{3.10}$$

Let us first estimate

$$J_0 = \limsup_{k\to\infty} \frac{1}{(k+1)T} \sum_{j=0}^k \int_{jT}^{jT+\delta} \kappa_{r(s)}(s) ds.$$

For each $i \in \mathbb{S}$, define

$$\tau_0^i = \inf\{j \ge 0 : r(jT) = i\}$$

and

$$\tau_k^i = \inf\{j > \tau_{k-1}^i : r(jT) = i\} \text{ for } k \ge 1.$$

Then $\tau_k^i (k \ge 0)$ are all finite stopping times with respect to filtration $\{\mathcal{F}_{jT}\}_{j\ge 0}$ such that $0 \le \tau_0^i < \cdots \tau_k^i \to \infty$ a.s. Set

$$\mathbb{S}_k^i = \{j \ge 0 : \tau_j^i \le k\},$$

and denote T_k^i be the total number of nonnegative integers which $\mathbb{S}_k^i$ contains. By the ergodic property of the Markov chain (see, e.g., [1]), we have

$$\lim_{k\to\infty} \frac{T_k^i}{k+1} = \pi_i.$$

Moreover, we can derive

$$
\begin{aligned}
J_0 &= \limsup_{k\to\infty} \frac{1}{(k+1)T} \sum_{i\in\mathbb{S}} \sum_{j\in\mathbb{S}_k^i} \int_{jT}^{jT+\delta} \kappa_{r(s)}(s)ds \\
&= \limsup_{k\to\infty} \sum_{i\in\mathbb{S}} \frac{T_k^i}{(k+1)T} \Big(\frac{1}{T_k^i} \sum_{j\in\mathbb{S}_k^i} \int_{jT}^{jT+\delta} \kappa_{r(s)}(s)ds \Big) \\
&\le \sum_{i\in\mathbb{S}} \frac{1}{T} \left(\limsup_{k\to\infty} \frac{T_k^i}{k+1} \right) \\
&\quad \times \left(\limsup_{k\to\infty} \frac{1}{T_k^i} \sum_{j\in\mathbb{S}_k^i} \int_{jT}^{jT+\delta} \kappa_{r(s)}(s)ds \right) \\
&= \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big(\limsup_{k\to\infty} \frac{1}{T_k^i} \sum_{j\in\mathbb{S}_k^i} \int_{jT}^{jT+\delta} \kappa_{r(s)}(s)ds \Big) \\
&= \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big(\limsup_{k\to\infty} \frac{1}{k+1} \sum_{j=0}^{k} \xi_j^i \Big), \qquad (3.11)
\end{aligned}
$$

where

$$
\xi_j^i = \int_{\tau_j^i T}^{\tau_j^i T+\delta} \kappa_{r(s)}(s)ds.
$$

But, by the strong Markov property, $\{r(\tau_j^i T + t)\}_{t\ge 0}$ forms a Markov chain with the same generator Γ which starts from state i and is independent of $\{r(t)\}_{0\le t\le \tau_j^i T}$ for each $j \ge 0$. We hence see that $\{\xi_j^i\}_{j\ge 0}$ are i.i.d. with their mean $\mathbb{E}\xi_0^i$. By the large number theory,

$$
\lim_{k\to\infty} \frac{1}{k+1} \sum_{j=0}^{k} \xi_j^i = \mathbb{E}\xi_0^i. \qquad (3.12)
$$

To estimate

$$
\mathbb{E}\xi_0^i = \mathbb{E} \int_{\tau_0^i T}^{\tau_0^i T+\delta} \kappa_{r(s)}(s)ds,
$$

we observe that $r(\tau_0^i T) = i$ a.s. Applying Lemma 3.4 and recalling $\delta < \varepsilon$, we see that

$$
\mathbb{P}(\Omega_1) \le 1 - e^{-\hat{\gamma}\delta} \le \hat{\gamma}\varepsilon,
$$

where $\Omega_1 = \{\omega \in \Omega : r(s,\omega) \ne i \text{ for some } s \in [\tau_0^i T, \tau_0^i T + \delta]\}$. Using (3.8) and the Hölder inequality, we then derive that

$$
\begin{aligned}
\mathbb{E}\xi_0^i &= \mathbb{E}\Big(I_{\Omega_1} \int_{\tau_0^i T}^{\tau_0^i T+\delta} \kappa_{r(s)}(s)ds \Big) \\
&\quad + \mathbb{E}\Big(I_{\Omega_1^c} \int_{\tau_0^i T}^{\tau_0^i T+\delta} \kappa_{r(s)}(s)ds \Big) \\
&\le \left(\mathbb{E}\left(I_{\Omega_1}^2 \right) \right)^{1/2} \times \left(\mathbb{E}\left(\left(\int_{\tau_0^i T}^{\tau_0^i T+\delta} \kappa_{r(s)}(s)ds \right)^2 \right) \right)^{1/2} \\
&\quad + \mathbb{E}\Big(I_{\Omega_1^c} \int_{\tau_0^i T}^{\tau_0^i T+\delta} \kappa_{r(s)}(s)ds \Big) \\
&\le \bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2}\delta + \mathbb{E}\Big(I_{\Omega_1^c} \int_{\tau_0^i T}^{\tau_0^i T+\delta} \kappa_i(s)ds \Big) \\
&= \bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2}\delta + \mathbb{E}\Big(I_{\Omega_1^c} \int_0^{\delta} \kappa_i(s)ds \Big) \\
&\le \bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2}\delta + \int_0^{\delta} \kappa_i(s)ds. \qquad (3.13)
\end{aligned}
$$

Substituting this into (3.12) and then into (3.11) we obtain

$$
J_0 \le \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big(\bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2}\delta + \int_0^{\delta} \kappa_i(s)ds \Big). \qquad (3.14)
$$

Similarly, we can show

$$
J_u \le \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big(\bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2}\delta + \int_{u\delta}^{(u+1)\delta} \kappa_i(s)ds \Big), \qquad (3.15)
$$

for $u = 1, 2, \cdots, \bar{m}$. Substituting these into (3.9) we get

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \int_0^t \kappa_{r(s)}(s)ds \\
&\le \sum_{u=0}^{\bar{m}-1} \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \Big(\bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2}\delta + \int_{u\delta}^{(u+1)\delta} \kappa_i(s)ds \Big) \\
&= \bar{\kappa}(\hat{\gamma}\varepsilon)^{1/2} + \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T \kappa_i(s)ds. \qquad (3.16)
\end{aligned}
$$

As ε is arbitrary, we must have

$$
\limsup_{t\to\infty} \frac{1}{t} \int_0^t \kappa_{r(s)}(s)ds \le \sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T \kappa_i(s)ds. \qquad (3.17)
$$

On the other hand, we can show in the same fashion that

$$
\liminf_{t\to\infty} \frac{1}{t} \int_0^t \kappa_{r(s)}(s)ds \ge \sum_{u=0}^{\bar{m}-1} \bar{J}_u, \qquad (3.18)
$$

where

$$
\bar{J}_u = \liminf_{k\to\infty} \frac{1}{(k+1)T} \sum_{j=0}^{k} \int_{jT+u\delta}^{jT+(u+1)\delta} \kappa_{r(s)}(s)ds. \qquad (3.19)
$$

And then show

$$\bar{J}_u \geq \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\Big(-\bar{\kappa}\hat{\gamma}\varepsilon\delta + \int_{u\delta}^{(u+1)\delta} \kappa_i(s)ds\Big), \quad (3.20)$$

for $u = 0, 1, \cdots, \bar{m}-1$. Combining these together and then letting $\varepsilon \to 0$, we get

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \kappa_{r(s)}(s)ds \geq \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T \kappa_i(s)ds. \quad (3.21)$$

The required assertion (3.6) now follows from (3.17) and (3.21). The proof is therefore complete. □

We can now begin to prove Theorem 3.3

Proof of Theorem 3.3. If $x_0 = 0$, the solution $x(t; 0) \equiv 0$ and hence assertion (3.2) holds. Fix any $x_0 \neq 0$ and write $x(t; x_0) = x(t)$. Recalling that this solution $x(t)$ will never reach zero with probability one, we can apply the Itô formula (see, e.g., [22], [26]) to obtain that

$$\begin{aligned} &d[\log(|x(t)|^2)] \\ &= \frac{2x^T(t)}{|x(t)|^2}\Big[f(x(t),t,r(t))dt + g(x(t),t,r(t))dB(t)\Big] \\ &\quad + \frac{1}{2}\Big[\frac{2|g(x(t),t,r(t))|^2}{|x(t)|^2} - \frac{4|x^T(t)g(x(t),t,r(t))|^2}{|x(t)|^4}\Big]dt. \end{aligned} \quad (3.22)$$

By Assumption 3.1, we obtain that

$$\begin{aligned} &\log(|x(t)|^2) \\ &\leq \log(|x_0|^2) + \int_0^t \Big[2\alpha_{r(s)}(s) + \rho^2_{r(s)}(s) - 2\sigma^2_{r(s)}(s)\Big]ds \\ &\quad + M(t), \end{aligned} \quad (3.23)$$

where

$$M(t) = \int_0^t \frac{2}{|x(s)|^2} x^T(s)g(x(s),s,r(s))dB(s),$$

which is a continuous martingale vanishing at $t = 0$. The quadratic variation of the martingale is given by

$$\begin{aligned} \langle M(t), M(t)\rangle &= \int_0^t \frac{4}{|x(s)|^4}|x^T(s)g(x(s),s,r(s))|^2 ds \\ &\leq 4t \max_{1\leq i\leq N}\sup_{0\leq s\leq T} \rho_i^2(s). \end{aligned}$$

By the strong law of the large numbers for local martingales (see, e.g., [17], [26]),

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad a.s. \quad (3.24)$$

Moreover, by Lemma 3.5,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t}\int_0^t \Big[2\alpha_{r(s)}(s) + \rho^2_{r(s)}(s) - 2\sigma^2_{r(s)}(s)\Big]ds \\ &= \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T [2\alpha_i(s) + \rho_i^2(s) - 2\sigma_i^2(s)]ds \quad a.s. \end{aligned} \quad (3.25)$$

Dividing both sides of (3.23) by $2t$, letting $t \to \infty$ and making use of two equalities above we get the required assertion (3.2). The proof is therefore complete. □

To discuss the instability we impose the following assumption.

Assumption 3.6: For each $i \in \mathbb{S}$, there are functions $\alpha_i(\cdot)$, $\rho_i(\cdot)$, and $\sigma_i(\cdot)$ in $\mathcal{K}_T$ such that

$$\begin{aligned} &x^T f(x,t,i) \geq \alpha_i(t)|x|^2, \ |g(x,t,i)| \geq \rho_i(t)|x|, \\ &|x^T g(x,t,i)| \leq \sigma_i(t)|x|^2 \end{aligned} \quad (3.26)$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Comparing this assumption with Assumption 3.1 we observe that the inequalities given in (3.1) and (3.26) have different directions.

Theorem 3.7: Under Assumption 3.6, the solution of equation (2.1) satisfies

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t}\log(|x(t)|) \\ &\geq \sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T [\alpha_i(s) + 0.5\rho_i^2(s) - \sigma_i^2(s)]ds \text{ a.s.} \end{aligned} \quad (3.27)$$

as long as the initial value $x_0 \neq 0$. In particular, the nonlinear hybrid SDE (2.1) is almost surely exponentially unstable if

$$\sum_{i\in\mathbb{S}} \frac{\pi_i}{T}\int_0^T [\alpha_i(s) + 0.5\rho_i^2(s) - \sigma_i^2(s)]ds > 0.$$

Proof. Fix any $x_0 \neq 0$ and write $x(t; x_0) = x(t)$ again. By (3.26), we can show from (3.22) that

$$\begin{aligned} &\log(|x(t)|^2) \\ &\geq \log(|x_0|^2) + \int_0^t \Big[2\alpha_{r(s)}(s) + \rho^2_{r(s)(s)} - 2\sigma^2_{r(s)}(s)\Big]ds \\ &\quad + M(t), \end{aligned} \quad (3.28)$$

where $M(t)$ is the same continuous martingale as defined in the proof of Theorem 3.3 but its quadratic variation is now estimated as

$$\begin{aligned} \langle M(t), M(t)\rangle &= \int_0^t \frac{4}{|x(s)|^4}|x^T(s)g(x(s),s,r(s))|^2 ds \\ &\leq 4t \max_{1\leq i\leq N}\sup_{0\leq s\leq T} \sigma_i^2(s). \end{aligned}$$

Thus, by the strong law of the large numbers for local martingales, we still have (3.24). Making use of (3.24) and (3.25) we can easily obtain assertion (3.27) from (3.28). The proof is therefore complete. □

4. Stochastic Stabilization

Let us now begin with the discussion of the stochastic stabilization for the hybrid ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t), t, r(t)). \tag{4.1}$$

As before, f satisfies the local Lipschitz condition and

$$x^T f(x, t, i) \le \alpha_i(t)|x|^2 \tag{4.2}$$

for all $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, where $\alpha_i(\cdot) \in \mathcal{K}_T$. Assume that this given hybrid ODE is not stable and we are required to stochastic feedback control (i.e., stochastic perturbation) to make the controlled stochastic system

$$dx(t) = f(x(t), t, r(t))dt + u(x(t), t, r(t))dB(t) \tag{4.3}$$

become almost surely exponentially stable, where $u : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}$. In this paper we only consider the periodic linear feedback control of the form

$$u(x, t, i) = \beta_i(t)(A_{1,i}x, A_{2,i}x, \cdots, A_{m,i}x), \tag{4.4}$$

where $\beta_i(\cdot) \in \mathcal{K}_T$ and $A_{k,i} \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{S}$ and $k = 1, 2, \cdots, m$. Thus the controlled system (4.3) becomes

$$dx(t) = f(x(t), t, r(t))dt + \sum_{k=1}^{m} \beta_{r(t)}(t)A_{k,r(t)}x(t)dB_k(t). \tag{4.5}$$

Our stabilization problem is therefore to design $\beta_i(\cdot)$ and $A_{k,i}$ in order for equation (4.5) to be almost surely exponentially stable. The following theorem describes the procedure.

Theorem 4.1: Let (4.2) hold. Choose $\beta_i(\cdot) \in \mathcal{K}_T$ and nonnegative constants a_i, b_i with $0.5a_i \le b_i \le a_i$ for $i \in \mathbb{S}$ such that

$$\sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T \alpha_i(s)ds < \sum_{i\in\mathbb{S}} \pi_i(b_i - 0.5a_i)\frac{1}{T}\int_0^T (\beta_i(s))^2 ds. \tag{4.6}$$

Design the matrices $A_{k,i}$ for the following conditions

$$\sum_{k=1}^{m} |A_{k,i}x|^2 \le a_i|x|^2 \quad \text{and} \quad \sum_{k=1}^{m} |x^T A_{k,i}x|^2 \ge b_i|x|^4 \tag{4.7}$$

to hold for all $(x, i) \in \mathbb{R}^n \times \mathbb{S}$. Then the controlled system (4.5) is almost surely exponentially stable.

The proof is a simple application of Theorem 3.3 so is omitted. The questions are: (i) could we find $\beta_i(\cdot), a_i, b_i$ for (4.6) to hold; and (ii) once they are chosen, could we further find the matrices $A_{k,i}$ to satisfy (4.7)? The answer to question (i) is yes. For example, choosing any nonnegative constants a_i, b_i with $0.5a_i < b_i \le a_i$ and then letting $\beta_i(t) = \sqrt{(|\alpha_i(t)| + 1)/(b_i - 0.5a_i)}$, we see (4.6) is satisfied. Of course, this is only one of lots of choices. Let us now answer question (ii) positively. It is sufficient if we could show that a couple of classes of matrices $A_{k,i}$ can satisfy (4.7) for any given nonnegative constants a_i, b_i with $0.5a_i \le b_i \le a_i$ ($i \in \mathbb{S}$). First of all, let

$$A_{k,i} = \theta_{k,i}Q_n, \quad 1 \le k \le m, \ i \in \mathbb{S},$$

where Q_n is the $n \times n$ identity matrix and $\theta_{k,i}$ are constants. Then

$$\sum_{k=1}^{m} |A_{k,i}x|^2 = \Big(\sum_{k=1}^{m} \theta_{k,i}^2\Big)|x|^2$$

and

$$\sum_{k=1}^{m} |x^T A_{k,i}x|^2 = \Big(\sum_{k=1}^{m} \theta_{k,i}^2\Big)|x|^4 \quad \forall x \in \mathbb{R}^n.$$

If we choose $\theta_{k,i}$ so that $\sum_{k=1}^m \theta_{k,i}^2 = a_i$, then (4.7) holds. As one more example, for these $i \in \mathbb{S}$ with $a_i = 0$, set $A_{k,i} = 0$ for $1 \le k \le m$, while for other i, k, choose symmetric positive definite matrices $A_{k,i}$ such that

$$\sum_{k=1}^{m} \|A_{k,i}\|^2 = a_i$$

and

$$(x^T A_{k,i}x)^2 \ge \frac{b_i}{a_i}\|A_{k,i}\|^2|x|^4 \quad \forall x \in \mathbb{R}^n.$$

Obviously, there are many such matrices. It is then easy to see such matrices satisfy (4.7).

Let us discuss an example to compare our new results with existing ones, typically, those in [25]. We will use a simple hybrid ODE in order to avoid unnecessary calculations but the advantages of our new results will be explained clearly.

Example 4.2: Consider a scalar hybrid ODE $\dot{x}(t) = f(x(t), t, r(t))$, where the Markov chain $r(t)$ has its state space $\mathbb{S} = \{1, 2\}$ and generator

$$\Gamma = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$$

while

$$f(x, t, i) = \begin{cases} (1 + \sin(t))x, & \text{if } i = 1, \\ (2 + \cos(t))\sin(x), & \text{if } i = 2 \end{cases}$$

for $(x, t, i) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}$. It is clear that the solution of the system does not tend to 0 as the time advances. We will use a scalar Brownian motion $B(t)$ as a source of noise to form the stochastic control $\sigma_{r(t)}x(t)dB(t)$ so that the controlled system has the form

$$dx(t) = f(x(t), t, r(t))dt + \sigma_{r(t)}x(t)dB(t), \tag{4.8}$$

where $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 2$. The stationary distribution of the Markov chain is $\pi_1 = 0.25$ and $\pi_2 = 0.75$. To apply Theorem 3.3 in [25] or Corollary 1 in [18] (with $\tau = 0$ there), we note

$$xf(x, t, i) \le \alpha_i x^2, \ \forall(x, i, t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S},$$

where $\alpha_1 = 2$ and $\alpha_2 = 3$. Condition (3.3) in [25] or Condition (a) in Corollary 1 of [18] needs

$$\sum_{i=1}^{2} \pi_i(\alpha_i - 0.5\sigma_i^2) < 0,$$

but the left-hand-side term $= 7/8$ so the condition does not hold and we can NOT apply Theorem 3.3 in [25] or Corollary 1 in [18] to conclude that the controlled SDE (4.8) is almost surely exponentially stable. On the other hand, we can apply our Theorem 4.1 to show it is. In fact, f is a periodic function of t with period 2π. Observe that

$$xf(x,t,i) \le \alpha_i(t)x^2, \ \forall (x,i,t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S},$$

where $\alpha_1(t) = 1 + \sin(t)$ and $\alpha_2(t) = 2 + \cos(t)$. It is also easy to see from the stochastic control $\sigma_{r(t)}x(t)dB(t)$ that the corresponding $\beta_i(\cdot)$ in (4.5) have the form $\beta_1(t) = \beta_2(t) = 1$ which can be regarded as periodic functions of course. Condition (4.7) holds with $a_1 = b_1 = 3$ and $a_2 = b_2 = 4$ while condition (4.6) becomes

$$\frac{7}{4} = \sum_{i=1}^{2} \frac{\pi_i}{2\pi} \int_0^{2\pi} \alpha_i(s)ds < \sum_{i=1}^{2} \pi_i(b_i - 0.5a_i) = \frac{15}{8},$$

which is true. By Theorem 4.1 we can conclude that the controlled SDE (4.8) is almost surely exponentially stable.

Let further consider a situation where the state $x(t)$ is only observable in mode 2 but not in mode 1 so the stochastic control could only be made in mode 2. In terms of mathematics, we have to set $\sigma_1 = 0$. In this situation, we need to increase the noise intensity σ_2 from 2 to 2.5. It is not difficult to see that Theorem 3.3 in [25] fails but our new Theorem 4.1 shows that the controlled SDE (4.8) is almost surely exponentially stable when $\sigma_1 = 0$ and $\sigma_2 = 2.5$.

This simple example shows clearly that our Theorem 4.1 is an improvement of Theorem 3.3 in [25] by taking the periodic property into account. To show another advantage of our new results, we will discuss stochastic intermittent controls, which are one of useful classes of controls (see, e.g., [35]). In terms of mathematics, $\beta_i(\cdot)$, $i \in \mathbb{S}$, in equation (4.5) become

$$\beta_i(t) = \sum_{q=0}^{\infty} I_{[qT, qT+\delta_i)}(t), \quad t \ge 0, \tag{4.9}$$

where $\delta_i \in (0, T]$ and $I_{[qT,qT+\delta_i)}(t)$ is the indicator function of $[qT, qT + \delta_i)$, namely it takes 1 when $t \in [qT, qT+\delta_i)$ and 0 otherwise. In operation, the stochastic control (or perturbation) is switched on whenever time $t \in [qT, qT+\delta_i)$ while the system is in mode i. Clearly, if $\delta_i = 0$, then there is no control in mode i but if $\delta_i = T$, then the control is always on in mode i. A special case is the situation where δ_i is independent of i, namely $\delta_i = \delta$ for all $i \in \mathbb{S}$, then the stochastic control is switched on during time periods $[0, \delta)$, $[T, T + \delta)$, $[2T, 2T + \delta)$, $\cdots$, while off during $[\delta, T)$, $[T + \delta, 2T)$, $[2T + \delta, 3T)$, $\cdots$ for all modes. One of the practical reasons for such an intermittent control is because a controller needs a rest periodically. To the best of our knowledge, none of the existing results including those in [25], [35] can be applied to the controlled system (4.5) with $\beta_i(\cdot)$ being defined by (4.9). However, the following corollary follows from our new Theorem 4.1 easily.

Corollary 4.3: Let (4.2) hold. Choose $\delta_i \in [0, T]$ and nonnegative constants a_i, b_i with $0.5a_i \le b_i \le a_i$ for $i \in \mathbb{S}$ such that

$$\sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T \alpha_i(s)ds < \frac{1}{T} \sum_{i\in\mathbb{S}} \pi_i\delta_i(b_i - 0.5a_i). \tag{4.10}$$

Design the matrices $A_{k,i}$ for (4.7) to hold. Then the controlled system (4.5) with $\beta_i(\cdot)$ being defined by (4.9) is almost surely exponentially stable.

Remark 4.4: For the purpose of comparison, we recall that [35] investigates the stabilization by an intermittent stochastic perturbation for a given system without regime switching, namely, the stability of the SDE

$$dx(t) = f(x(t), t)dt + \sum_{k=1}^{m} \beta_{r(t)}(t)A_k x(t)dB_k(t), \tag{4.11}$$

with $\beta(t) = \sum_{q=0}^{\infty} I_{[qT,qT+\delta)}(t)$ on $t \ge 0$. Theorem 3 in [35] states that under the conditions

$$x^T f(x,t) \le \alpha|x|^2, \quad \sum_{k=1}^{m} |A_k x|^2 \le a|x|^2$$
$$\text{and} \quad \sum_{k=1}^{m} |x^T A_k x|^2 \ge b|x|^4 \tag{4.12}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, the controlled system (4.11) is almost surely exponentially stable if $\delta(b - 0.5a) > \alpha T$. Our Corollary 4.3 is better than this even in the case of no regime switching (namely, $\mathbb{S} = \{1\}$). For an example, consider a scalar SDE (4.11 with $f(x,t) = (2 + \cos(t))\sin(x)$, $T = 2\pi$ and choose appropriate numbers A_k such that the last two inequalities hold with $a = b = 4$. Then, by Theorem 3 in [35] the controlled system (4.11) is almost surely exponentially stable if $\delta > 3\pi$ while by our new Corollary 4.4 we only require $\delta > 2\pi$.

Continuation of Example 4.2 Let us return to Example 4.2 but we will use an intermittent control. That is, the stochastically controlled system is of the form

$$dx(t) = f(x(t), t, r(t))dt + \beta_{r(t)}(t)\sigma_{r(t)}x(t)dB(t), \tag{4.13}$$

where $\beta_i(t)$ $(i = 1, 2)$ are defined by (4.9) with $T = 2\pi$. We only consider the case where the state $x(t)$ is observable in mode 2 but not in mode 1. In terms of mathematics, we have to set $\sigma_1 = 0$ and $\delta_1 = 0$. Moreover, we set $\sigma_2 = 3$. Then condition (4.10) becomes

$$\frac{7}{4} = \sum_{i=1}^{2} \frac{\pi_i}{2\pi} \int_0^{2\pi} \alpha_i(s)ds < \frac{27\delta_2}{16\pi}, \tag{4.14}$$

namely, $\delta_2 > 28\pi/27$. By Corollary 4.3, we can conclude that the controlled system (4.13) is almost surely exponentially stable if $\sigma_1 = 0$, $\delta_1 = 0$, $\sigma_2 = 3$ and $\delta_2 > 28\pi/27$.

For illustration, we perform a computer simulation. We set the initial data $x(0) = 2$ and $r(0) = 1$ while let $\delta_2 = 1.5\pi$, and use the Euler-Maruyama method (see, e.g. [16], [26]) with the step size $10^{-4}\pi$. Figure 4.1 shows the sample paths of the Markov chain and the state, which illustrate our theory well.

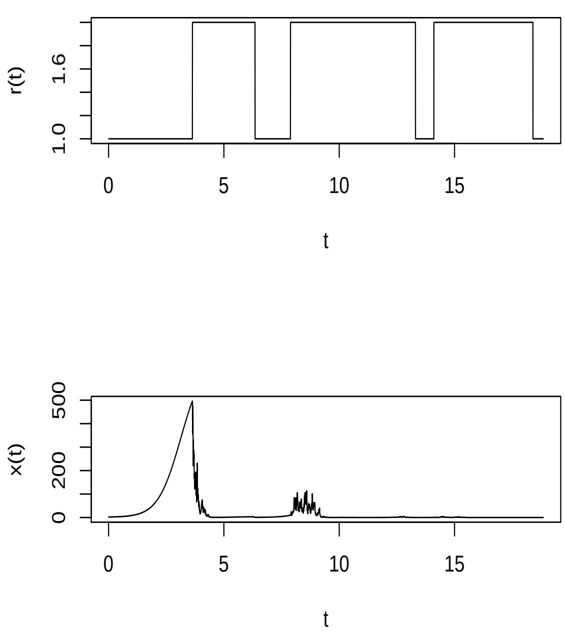

Figure 4.1: The computer simulation of the sample paths of the Markov chain and the controlled system (4.13).

5. Stochastic Destabilization

Let us now turn to consider the opposite problem—stochastic destabilization. More precisely, given a nonlinear stable hybrid system (4.1), can we design a linear controller $u(x, i)$ of form (4.4) so that the controlled system (4.5) become unstable? To answer this question positively, let us state a result which follows from Theorem 3.7 directly.

Theorem 5.1: Assume that there are $\alpha_i(\cdot) \in \mathcal{K}_T, i \in \mathbb{S}$, such that

$$x^T f(x, t, i) \geq \alpha_i(t)|x|^2 \tag{5.1}$$

for all $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$. Choose $\beta_i(\cdot) \in \mathcal{K}_T$ and constants a_i, b_i with $0.5a_i \geq b_i \geq 0$ for $i \in \mathbb{S}$ such that

$$\begin{aligned} &-\sum_{i\in\mathbb{S}} \frac{\pi_i}{T} \int_0^T \alpha_i(s)ds \\ &< \sum_{i\in\mathbb{S}} \pi_i(0.5a_i - b_i)\frac{1}{T}\int_0^T (\beta_i(s))^2 ds. \end{aligned} \tag{5.2}$$

Design the matrices $A_{k,i}$ for the following conditions

$$\sum_{k=1}^{m} |A_{k,i}x|^2 \geq a_i|x|^2 \quad \text{and} \quad \sum_{k=1}^{m} |x^T A_{k,i}x|^2 \leq b_i|x|^4 \tag{5.3}$$

to hold for all $(x, i) \in \mathbb{R}^n \times \mathbb{S}$. Then the controlled system (4.5) is almost surely exponentially unstable.

There is no problem at all to find $\beta_i(\cdot)$ and a_i, b_i for (5.2) to hold. However, it is not obvious to see if we can find matrices $A_{k,i}$ to satisfy (5.3). We shall show that this is possible if the dimension of the state space is greater than or equal to 2.

First, let the dimension n of the state space be an even number. For each $i \in \mathbb{S}$, define

$$A_{1,i} = \begin{bmatrix} 0 & \sqrt{a_i} & & & \\ -\sqrt{a_i} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \sqrt{a_i} \\ & & & -\sqrt{a_i} & 0 \end{bmatrix},$$

but set $A_{k,i} = 0$ for $2 \leq k \leq m$. The controlled system (4.5) becomes

$$\begin{aligned} dx(t) &= f(x(t), t, r(t))dt \\ &\quad + \beta_{r(t)}(t)\sqrt{a_{r(t)}} \begin{bmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_n(t) \\ -x_{n-1}(t) \end{bmatrix} dB_1(t). \end{aligned} \tag{5.4}$$

Note that for each $i \in \mathbb{S}$,

$$\sum_{k=1}^{m} |A_{k,i}x|^2 = |A_{1,i}x|^2 = a_i|x|^2$$

and

$$\sum_{k=1}^{m} |x^T A_{k,i} x|^2 = |x^T A_{1,i} x|^2 = 0.$$

We see that condition (5.3) holds.

We next consider the case when the dimension n of the state space is an odd number and $n \geq 3$. Let the dimension of the Brownian motion $m \geq 2$. For each $i \in \mathbb{S}$, define

$$A_{1,i} = \begin{bmatrix} 0 & \sqrt{a_i} & & & & \\ -\sqrt{a_i} & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \sqrt{a_i} & \\ & & & -\sqrt{a_i} & 0 & \\ & & & & & 0 \end{bmatrix},$$

$$A_{2,i} = \begin{bmatrix} 0 & & & & & \\ & 0 & \sqrt{a_i} & & & \\ & -\sqrt{a_i} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sqrt{a_i} \\ & & & & -\sqrt{a_i} & 0 \end{bmatrix}$$

but set $A_{k,i} = 0$ for $3 \leq k \leq m$. So the controlled system (4.5) becomes

$$\begin{aligned} dx(t) &= f(x(t), t, r(t))dt \\ &\quad + \beta_{r(t)}(t)\sqrt{a_{r(t)}} \begin{bmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_{n-1}(t) \\ -x_{n-2}(t) \\ 0 \end{bmatrix} dB_1(t) \\ &\quad + \beta_{r(t)}(t)\sqrt{a_{r(t)}} \begin{bmatrix} 0 \\ x_3(t) \\ -x_2(t) \\ \vdots \\ x_n(t) \\ -x_{n-1}(t) \end{bmatrix} dB_2(t). \end{aligned} \quad (5.5)$$

Note that for each $i \in \mathbb{S}$,

$$\begin{aligned} &\sum_{k=1}^{m} |A_{k,i} x|^2 \\ &= |A_{1,i} x|^2 + |A_{2,i} x|^2 \\ &= a_i(x_1^2 + \cdots + x_{n-1}^2) + a_i(x_2^2 + \cdots + x_n^2) \\ &\geq a_i |x|^2 \end{aligned}$$

and

$$\sum_{k=1}^{m} |x^T A_{k,i} x|^2 = |x^T A_{1,i} x|^2 + |x^T A_{2,i} x|^2 = 0.$$

Hence, condition (5.3) is satisfied.

It is also possible to use the intermittent controls to destabilize stable systems. The following corollary describes this situation.

Corollary 5.2: Let (5.1) hold. Choose $\delta_i \in [0, T]$ and nonnegative constants a_i, b_i with $0.5a_i \geq b_i$ for $i \in \mathbb{S}$ such that

$$-\sum_{i \in \mathbb{S}} \frac{\pi_i}{T} \int_0^T \alpha_i(s) ds < \frac{1}{T} \sum_{i \in \mathbb{S}} \pi_i \delta_i (0.5a_i - b_i). \quad (5.6)$$

Design the matrices $A_{k,i}$ for (5.3) to hold. Then the controlled system (4.5) with $\beta_i(\cdot)$ being defined by (4.9) is almost surely exponentially unstable.

Let us discuss an example to illustrate our new theory on the destabilization.

Example 5.3: Consider a 2-dimensional hybrid ODE $\dot{x}(t) = f(x(t), t, r(t))$, where the Markov chain $r(t)$ is the same as in Example 4.2 but

$$f(x, t, i) = \begin{cases} (1 + \sin(t))F_1 x, & \text{if } i = 1, \\ (2 + \cos(t))F_2 x, & \text{if } i = 2, \end{cases}$$

for $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$, in which

$$F_1 = \begin{pmatrix} -1 & 0.2 \\ 0.1 & -2 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} -2 & 0.3 \\ 0.2 & -1 \end{pmatrix}.$$

It is easy to see that

$$x^T f(x, t, i) \leq \begin{cases} -0.9779847(1 + \sin(t))|x|^2, & \text{if } i = 1, \\ -0.9409832(2 + \cos(t))|x|^2, & \text{if } i = 2. \end{cases}$$

Hence the hybrid ODE is stable. To destabilize it stochastically, we observe that

$$x^T f(x, t, i) \geq \begin{cases} -2.0220153(1 + \sin(t))|x|^2, & \text{if } i = 1, \\ -2.0590170(2 + \cos(t))|x|^2, & \text{if } i = 2. \end{cases}$$

That is, condition (5.1) holds. We consider the case where the state $x(t)$ is observable in mode 2 but not in mode 1. Moreover, we will use a scalar Brownian motion $B(t)$ as a source of noise to form the stochastic intermittent control. In terms of mathematics, the stochastically controlled system has the form

$$dx(t) = f(x(t), t, r(t))dt + \beta_{r(t)}(t) A_{r(t)} x(t) dB(t), \quad (5.7)$$

where $\beta_1(t) = 0$, $A_1 = 0$,

$$\beta_2(t) = \sum_{q=0}^{\infty} I_{[2q\pi, 2q\pi + \delta_2)}(t), \quad A_2 = \begin{pmatrix} 0 & \sqrt{a_2} \\ -\sqrt{a_2} & 0 \end{pmatrix}.$$

It is easy to see that (5.6) becomes

$$\begin{aligned} 3.594029 &= -\sum_{i \in \mathbb{S}} \frac{\pi_i}{2\pi} \int_0^{2\pi} \alpha_i(s) ds \\ &< \frac{1}{2\pi} \sum_{i \in \mathbb{S}} \pi_i \delta_i (0.5a_i - b_i) = \frac{3\delta_2 a_2}{16\pi}. \end{aligned}$$

By Corollary 5.2, we can therefore conclude that if we choose $\delta_2 \in (0, 2\pi)$ and $a_2 > 0$ for $\delta_2 a_2 > 60.21853$, then controlled system (5.7) is almost surely exponentially unstable.

For computer simulations, we choose $\delta_2 = 1.5\pi$, $a_2 = 16$ while set $x_1(0) = x_2(0) = 0.001$ and $r(0) = 2$. The reason why we set the initial value of the state so small is because we want to show that the stochastic intermittent control will make the state to increase very quickly. We use the Euler-Maruyama method with the step size $10^{-4}\pi$. Figure 5.1 shows the sample paths of $r(t)$ and $x(t)$ for one period of time, namely $[0, 2\pi]$. During the time interval $[0, 1.5\pi)$, we see the stochastic control perturbs the given stable system significantly (from value of 0.001 to scale of 10^7, while during the time interval $[1.5\pi, 2\pi)$, there is no stochastic control and the system behaves stably. Figure 5.2 shows the sample paths for three period of time, namely $[0, 6\pi]$. We see the stochastic control perturbs the given stable system from value of 0.001 to scale of 10^{21}. This supports the theoretical result of exponential instability. Please note that the values of $x(t)$ are in scale of 10^7 during $[0, 2\pi)$ and 10^{14} during $[2\pi, 4\pi)$ and hence they are plotted near zero.

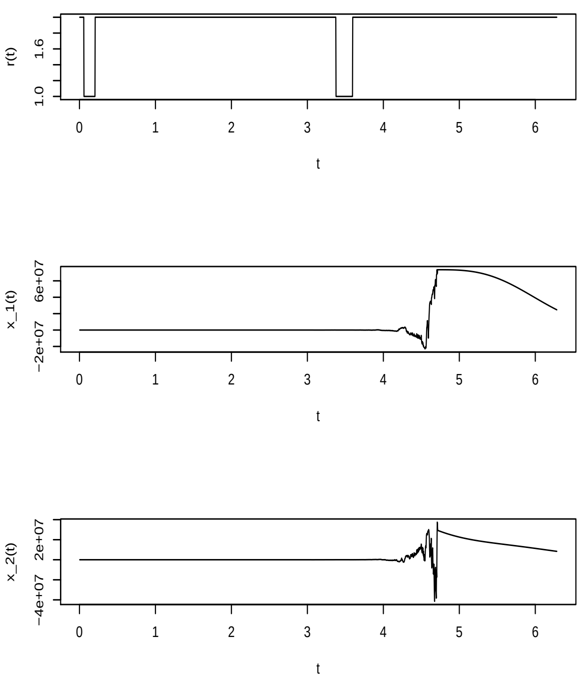

Figure 5.1: The computer simulation of the sample paths of $r(t)$ and $x(t)$ for the controlled system (5.7) for one period of time.

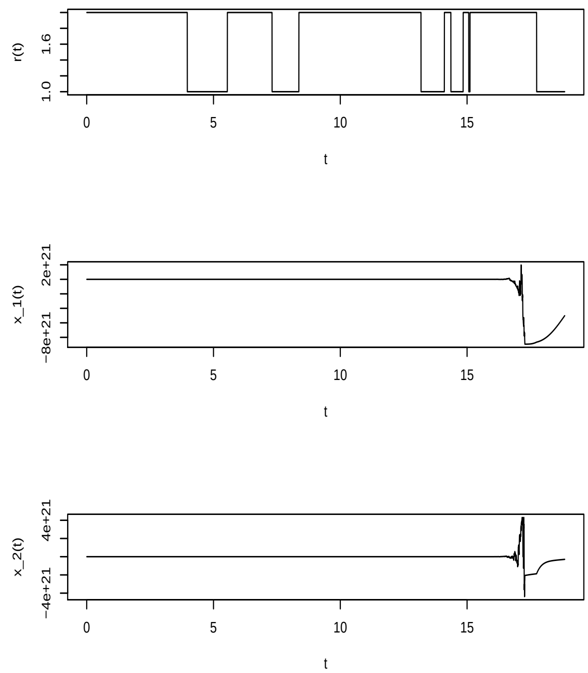

Figure 5.2: The computer simulation of the sample paths of $r(t)$ and $x(t)$ for the controlled system (5.7) for three period of time.

6. Conclusion

In this paper we pointed out that the existing results on the stochastic stabilization and destabilization for hybrid ODEs do NOT take the time-inhomogeneous property into account and hence the criteria so far are conservative. We then successfully established better criteria on stability and instability of hybrid SDEs by making use of the time-inhomogeneous properties, e.g., the periodic property, of both shift and diffusion coefficients. These results were then used to examine stabilization and destabilization by periodic stochastic feedback controls, including stochastic intermittent controls. A couple of examples were used to show the advantages of our new results in comparison with some of existing ones. These examples also illustrated our new theory well.

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