

# Boundary value problems for elliptic partial differential operators on bounded domains

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## Abstract

For a symmetric operator or relation  $A$  with infinite deficiency indices in a Hilbert space we develop an abstract framework for the description of symmetric and self-adjoint extensions  $A_\Theta$  of  $A$  as restrictions of an operator or relation  $T$  which is a core of the adjoint  $A^*$ . This concept is applied to second order elliptic partial differential operators on smooth bounded domains, and a class of elliptic problems with eigenvalue dependent boundary conditions is investigated.

*Key words:* boundary triple, self-adjoint extension, Weyl function,  $M$ -operator, Dirichlet-to-Neumann map, Krein's formula, elliptic differential operator, boundary value problem.

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## 1 Introduction

Boundary value problems for ordinary differential equations of the form

$$(\ell f)(x) - \lambda f(x) = g(x), \quad \lambda \in \mathbb{C}, \quad x \in \Omega, \quad (1.1)$$

where

$$(\ell f)(x) = -(pf')'(x) + q(x)f(x),$$

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$\Omega = (a, b)$  is a finite interval,  $f, g \in L^2(\Omega)$ , and  $p$  and  $q$  are real continuous functions on  $\overline{\Omega}$ , are closely connected with symmetric operators in  $L^2(\Omega)$  and the spectral properties of their extensions. Namely, the solution  $f$  of the problem (1.1) satisfying, e.g. the boundary conditions  $f(a) = f(b) = 0$  is obtained by applying the resolvent  $(A_D - \lambda)^{-1}$  of the self-adjoint operator

$$A_D f = \ell f, \quad \text{dom } A_D = \{f \in H^2(\Omega) \mid f(a) = f(b) = 0\},$$

in  $L^2(\Omega)$  to the right-hand side of (1.1). Here the Sobolev space  $H^2(\Omega)$  is the domain of the usual maximal operator associated with  $\ell$  in  $L^2(\Omega)$ . This maximal operator coincides with the adjoint  $A^*$  of the minimal operator

$$A f = \ell f, \quad \text{dom } A = \{f \in H^2(\Omega) \mid f(a) = f(b) = (pf')(a) = (pf')(b) = 0\},$$

which is a symmetric operator in  $L^2(\Omega)$  with deficiency indices  $(2, 2)$ . We emphasize that the functions in  $\text{dom } A^* = H^2(\Omega)$  have boundary values at the endpoints  $a$  and  $b$  of the interval  $\Omega = (a, b)$  and therefore all self-adjoint extensions of  $A$  in  $L^2(\Omega)$  can be described with the help of boundary conditions for functions in  $\text{dom } A^*$  (and the resolvents of these extensions then yield unique solutions of (1.1) subject to certain boundary conditions).

The abstract theory of boundary triples and associated Weyl functions developed in the last decades by V. A. Derkach, V. I. Gorbachuk, M. L. Gorbachuk and M. M. Malamud (see, e.g. [15,16,25,31]) can be applied to parametrize the self-adjoint extensions of the minimal operator  $A$  in  $L^2(\Omega)$  and to describe their spectral properties. Such a boundary triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  consists of an auxiliary Hilbert space  $\mathcal{G}$  and two mappings  $\Gamma_0, \Gamma_1 : \text{dom } A^* \rightarrow \mathcal{G}$  such that  $\Gamma_0 \times \Gamma_1$  is surjective onto  $\mathcal{G} \times \mathcal{G}$  and that the ‘‘abstract Green identity’’

$$(A^* f, g)_{\mathcal{H}} - (f, A^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

holds for all  $f, g \in \text{dom } A^*$ . A possible choice for a boundary triple for the Sturm–Liouville operator  $A^*$  from above is  $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ , where

$$\Gamma_0 f = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix} \quad \text{and} \quad \Gamma_1 f = \begin{pmatrix} (pf')(a) \\ -(pf')(b) \end{pmatrix}, \quad f \in \text{dom } A^* = H^2(\Omega).$$

The corresponding Weyl function  $M$  in this case is a  $2 \times 2$ -matrix valued Nevanlinna function holomorphic on the resolvent set of the self-adjoint extension  $A_D = A^* \upharpoonright \ker \Gamma_0$ .

Let now  $\Omega \subset \mathbb{R}^m$ ,  $m > 1$ , be a bounded domain with smooth boundary  $\partial\Omega$  and consider an elliptic differential equation of the type

$$(\mathcal{L}f)(x) - \lambda f(x) = g(x), \quad \lambda \in \mathbb{C}, \quad x \in \Omega,$$

with  $f, g \in L^2(\Omega)$ , where

$$(\mathcal{L}f)(x) := - \sum_{j,k=1}^m (D_j a_{jk} D_k f)(x) + \sum_{j=1}^m (a_j D_j f - D_j \bar{a}_j f)(x) + a(x) f(x),$$

and  $a_{jk}, a_j, a \in C^\infty(\bar{\Omega})$ ,  $a_{jk} = \bar{a}_{kj}$  and  $a$  is real. By  $f|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$  and  $\frac{\partial f}{\partial\nu}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$  we denote the trace and the conormal derivative, respectively, of a function in  $H^2(\Omega)$ . Although it is well known that the Dirichlet operator

$$A_D f = \mathcal{L}f, \quad \text{dom } A_D = \{f \in H^2(\Omega) \mid f|_{\partial\Omega} = 0\},$$

and the Neumann operator

$$A_N f = \mathcal{L}f, \quad \text{dom } A_N = \{f \in H^2(\Omega) \mid \frac{\partial f}{\partial\nu}|_{\partial\Omega} = 0\},$$

are self-adjoint operators in  $L^2(\Omega)$ , it is less clear which boundary conditions

$$\Theta \frac{\partial f}{\partial\nu} \Big|_{\partial\Omega} = f|_{\partial\Omega}, \quad f \in H^2(\Omega), \quad (1.2)$$

where  $\Theta$  is a linear operator (or even a relation) in  $L^2(\partial\Omega)$ , lead to self-adjoint operators in  $L^2(\Omega)$ . As above the minimal operator  $A$  associated with  $\mathcal{L}$  in  $L^2(\Omega)$  is defined on  $\text{dom } A = \{f \in H^2(\Omega) \mid f|_{\partial\Omega} = \frac{\partial f}{\partial\nu}|_{\partial\Omega} = 0\}$ , but in contrast to ordinary differential operators,  $H^2(\Omega)$  is a proper subset of the domain

$$\text{dom } A^* = \{f \in L^2(\Omega) \mid \mathcal{L}f \in L^2(\Omega)\}$$

of the maximal operator  $A^* f = \mathcal{L}f$ . In particular, the functions (and their conormal derivatives) from  $\text{dom } A^*$  do not have  $L^2(\partial\Omega)$ -boundary values in general and boundary conditions of the form (1.2) with an operator  $\Theta$  in  $L^2(\partial\Omega)$  can not be imposed for the maximal operator. Thus if the boundary values are restricted to be in  $L^2(\partial\Omega)$ , then the boundary mappings  $\Gamma_0 f = f|_{\partial\Omega}$  and  $\Gamma_1 f = -\frac{\partial f}{\partial\nu}|_{\partial\Omega}$  can only be defined on a core of  $\text{dom } A^*$ , e.g.  $H^2(\Omega)$ , and therefore the triple  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  is not a boundary triple in the classical sense. We note that an abstract boundary triple  $\{\mathcal{N}_\mu, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ , where  $\mu \in \mathbb{C} \setminus \mathbb{R}$  and  $\mathcal{N}_\mu = \ker(A^* - \mu)$  is a defect subspace of  $A$ , can always be constructed, but then the self-adjoint extensions of  $A$  cannot be described with  $L^2(\partial\Omega)$ -boundary values. A similar abstract approach is due to W.N. Everitt and L. Markus and was applied to elliptic partial differential operators in [21] and [22]. In their terminology  $\text{dom } A^* / \text{dom } A$  is a complex symplectic space and the self-adjoint extensions of  $A$  correspond to complete Lagrangian subspaces.

We emphasize that usually one extends the trace map and the conormal derivative onto  $\text{dom } A^*$  such that  $f|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$  and  $\frac{\partial f}{\partial\nu}|_{\partial\Omega} \in H^{-3/2}(\partial\Omega)$ , see, e.g. [30]. Then it follows from general results obtained by G. Grubb that the self-adjoint extensions of  $A$  can be described with the help of the

Dirichlet-to-Neumann map  $f|_{\partial\Omega} \mapsto \frac{\partial f}{\partial\nu}|_{\partial\Omega}$ ,  $f \in \ker A^*$ , and self-adjoint operators defined on closed subspaces of  $H^{-1/2}(\partial\Omega)$ , see [26]. For the extension and spectral theory of general elliptic differential operators we refer the reader to the fundamental paper [35] of M. I. Višik, to [3,5,8,10,11,24,26,28,30,32] and to [2,12,18,21,22,25] for more abstract approaches. For other types of problems, e.g. parabolic problems or problems with a block matrix structure, see [7,6,13,20,33].

The basic aim of this paper is to introduce a generalization of the boundary triple concept and to apply it to boundary value problems for elliptic second order differential operators with  $L^2(\partial\Omega)$ -boundary values. For this we consider the following abstract setting in Section 2. Let  $A$  be a closed symmetric operator in a Hilbert space  $\mathcal{H}$ , let  $T$  be a restriction of  $A^*$  such that  $\overline{T} = A^*$  and let  $\Gamma_0, \Gamma_1$  be mappings into an auxiliary Hilbert space  $\mathcal{G}$ , the boundary space, such that

$$(A^*f, g)_{\mathcal{H}} - (f, A^*g)_{\mathcal{H}} = (\Gamma_1f, \Gamma_0g)_{\mathcal{G}} - (\Gamma_0f, \Gamma_1g)_{\mathcal{G}}$$

for all  $f, g \in \text{dom } T$  and  $\text{ran}(\Gamma_0 \times \Gamma_1)$  is dense in  $\mathcal{G} \times \mathcal{G}$  (later  $A$  and  $T$  can even be multivalued, i.e., linear relations; but in the Introduction we restrict ourselves to the operator case). The triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is called a quasi boundary triple for  $A^*$  if in addition  $\ker \Gamma_0$  is the domain of a self-adjoint operator. We note that this definition extends the notion of boundary triples and generalized boundary triples (see Section 3). Moreover, a quasi boundary triple is in general not a boundary relation, cf. [18]. The boundary mappings can be used to define a “defect function”  $\gamma$  and an abstract Weyl function  $M$ , which for elliptic operators coincides with the Dirichlet-to-Neumann map, cf. Section 2.3. The values of the Weyl function are densely defined operators in  $\mathcal{G}$  which can be unbounded and are not necessarily closed. Within the framework of quasi boundary triples one can describe symmetric and self-adjoint extensions, although not all self-adjoint extensions, and in a similar way as for classical boundary triples a Krein formula can be proved, which expresses the resolvent of a canonical extension in terms of a fixed self-adjoint extension, the Weyl function and the boundary condition.

In Section 4.1 it is shown that  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ , where the boundary mappings  $\Gamma_0f = f|_{\partial\Omega}$  and  $\Gamma_1f = -\frac{\partial f}{\partial\nu}|_{\partial\Omega}$  are defined on the Sobolev space  $H^2(\Omega)$ , is a quasi boundary triple for an elliptic second order differential expression  $A \subset A^*$  of the type considered above; here we have  $T = A^* \upharpoonright H^2(\Omega)$ . The corresponding Weyl function  $M$ , i.e., the Dirichlet-to-Neumann map, is studied and a general criterion for self-adjointness (and maximal dissipativity and accumulativity) is given with the help of Krein’s formula. We note that the extensions of  $A$  described with the quasi boundary triple  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  are in general not closed. As a simple example we show that in the case  $n = 2$ ,  $\mathcal{L} = -\Delta$  and  $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ , the Krein-von-Neumann extension or

“soft” extension of  $A$ , a self-adjoint realization of  $A$  that seemingly cannot be described by  $L^2(\partial\Omega)$ -boundary values, is the closure of the extension of  $A$  corresponding to the parameter  $M(0)$  in  $L^2(\partial\Omega)$ , see [26] for the general case and the recent papers [21,23], where this special self-adjoint realization was called the harmonic Laplacian.

The boundary mappings  $\Gamma_0$  and  $\Gamma_1$  can also be defined on a larger space  $\mathcal{D}_1(\Omega)$ , which was introduced and studied by W. G. Bade, R. Beals and R. S. Freeman in [5,8,24] and recently appeared in a paper by W. O. Amrein and D. B. Pearson in connection with Weyl–Titchmarsh theory for elliptic differential operators of a similar type we study here. In this case the mapping

$$\Gamma_1 : \mathcal{D}_1(\Omega) \rightarrow L^2(\partial\Omega), \quad f \mapsto -\frac{\partial f}{\partial\nu}\Big|_{\partial\Omega},$$

is surjective and the quasi boundary triple  $\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$  becomes a generalized boundary triple in the sense of [16]. The values of the corresponding Weyl function are compact operators in  $L^2(\partial\Omega)$ , and with Krein’s formula a Fredholm argument implies that self-adjoint operators or relations  $\Theta = \Theta^*$  with  $0 \notin \sigma_{\text{ess}}(\Theta)$  in  $L^2(\partial\Omega)$  yield self-adjoint extensions

$$A_\Theta f = \mathcal{L}f, \quad \text{dom } A_\Theta = \left\{ f \in \mathcal{D}_1(\Omega) \mid f|_{\partial\Omega} = \Theta \frac{\partial f}{\partial\nu}\Big|_{\partial\Omega} \right\},$$

in  $L^2(\Omega)$  with compact resolvent

$$(A_\Theta - \lambda)^{-1} = (A_N - \lambda)^{-1} + \gamma(\lambda) \left( \Theta - M(\lambda) \right)^{-1} \gamma(\bar{\lambda})^*, \quad (1.3)$$

$\lambda \in \rho(A_N) \cap \rho(A_\Theta)$ , cf. Theorem 4.8. In a similar way one gets maximal dissipative extensions if  $\Theta$  is maximal dissipative and  $0 \notin \sigma_{\text{ess}}(\Theta)$ , cf. Theorem 4.10.

In Section 5 we study a class of elliptic boundary value problems with eigenvalue dependent boundary conditions of the form

$$\mathcal{L}f - \lambda f = g, \quad \tau(\lambda) \frac{\partial f}{\partial\nu}\Big|_{\partial\Omega} + f|_{\partial\Omega} = 0, \quad (1.4)$$

with the help of the abstract framework of quasi boundary triples and associated Weyl functions. Here  $\lambda \mapsto \tau(\lambda)$  is assumed to be an operator-valued Nevanlinna function. A unique solution  $f \in \mathcal{D}_1(\Omega)$  of this problem is obtained with the help of the compressed resolvent of a self-adjoint extension  $\tilde{A}$  of the minimal operator  $A$  which acts in a larger Hilbert space  $L^2(\Omega) \times \mathcal{K}$ . It is shown that the compressed resolvent  $P_{L^2}(\tilde{A} - \lambda)^{-1}|_{L^2}$  is given by the usual Krein–Naimark formula, i.e., similarly to the right hand side of (1.3)  $P_{L^2}(\tilde{A} - \lambda)^{-1}|_{L^2}$  is expressed in terms of a fixed canonical resolvent, the Weyl function  $M$  of the quasi boundary triple  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  and the parameter function  $\tau$ . For the special case of  $\lambda$ -linear boundary conditions one can choose  $\mathcal{K} = L^2(\partial\Omega)$ . In this situation we retrieve some results from [10,20], where other methods were used to investigate  $\lambda$ -linear problems of the type (1.4).

## 2 Quasi boundary triples and associated Weyl functions

### 2.1 Preliminaries

Throughout this paper let  $(\mathcal{H}, (\cdot, \cdot))$  and  $(\mathcal{K}, (\cdot, \cdot))$  be Hilbert spaces. The linear space of bounded linear operators defined on  $\mathcal{H}$  with values in  $\mathcal{K}$  will be denoted by  $\mathcal{L}(\mathcal{H}, \mathcal{K})$ . If  $\mathcal{H} = \mathcal{K}$ , we simply write  $\mathcal{L}(\mathcal{H})$ . We shall often deal with (closed) linear relations in  $\mathcal{H}$ , that is, (closed) linear subspaces of  $\mathcal{H} \times \mathcal{H}$ . The set of closed linear relations in  $\mathcal{H}$  will be denoted by  $\tilde{\mathcal{C}}(\mathcal{H})$ , and for elements in a relation we usually use a vector notation. Linear operators  $T$  in  $\mathcal{H}$  are viewed as linear relations via their graphs  $\left\{ \begin{pmatrix} f \\ Tf \end{pmatrix} \mid f \in \text{dom } T \right\}$ . The domain, the range, the kernel, the multivalued part and the inverse of a relation  $T$  in  $\mathcal{H}$  are respectively defined by

$$\begin{aligned} \text{dom } T &:= \left\{ f \in \mathcal{H} \mid \exists f' \text{ with } \begin{pmatrix} f \\ f' \end{pmatrix} \in T \right\}, \\ \text{ran } T &:= \left\{ f' \in \mathcal{H} \mid \exists f \text{ with } \begin{pmatrix} f \\ f' \end{pmatrix} \in T \right\}, \\ \text{ker } T &:= \left\{ f \in \mathcal{H} \mid \begin{pmatrix} f \\ 0 \end{pmatrix} \in T \right\}, \\ \text{mul } T &:= \left\{ f' \in \mathcal{H} \mid \begin{pmatrix} 0 \\ f' \end{pmatrix} \in T \right\}, \\ T^{-1} &:= \left\{ \begin{pmatrix} f' \\ f \end{pmatrix} \mid \begin{pmatrix} f \\ f' \end{pmatrix} \in T \right\}. \end{aligned}$$

The sum of two relations  $T_1$  and  $T_2$  is defined by

$$T_1 + T_2 = \left\{ \begin{pmatrix} f \\ f_1 + f_2 \end{pmatrix} \mid f \in \text{dom } T_1 \cap \text{dom } T_2, \begin{pmatrix} f \\ f_1 \end{pmatrix} \in T_1, \begin{pmatrix} f \\ f_2 \end{pmatrix} \in T_2 \right\},$$

and the sum and direct sum of linear relations considered as subspaces of  $\mathcal{H} \times \mathcal{H}$  will be denoted by  $\dagger$  and  $\dot{+}$ , respectively. For further details see, e.g. [19].

Let  $S$  be a closed linear relation in  $\mathcal{H}$ . The *resolvent set*  $\rho(S)$  of  $S$  is the set of all  $\lambda \in \mathbb{C}$  such that  $(S - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ ; the *spectrum*  $\sigma(S)$  of  $S$  is the complement of  $\rho(S)$  in  $\mathbb{C}$ . A point  $\lambda \in \mathbb{C}$  is an *eigenvalue* of a linear relation  $S$  if  $\text{ker}(S - \lambda) \neq \{0\}$ ; we write  $\lambda \in \sigma_p(S)$ . We say that  $\lambda \in \mathbb{C}$  belongs to the *continuous spectrum*  $\sigma_c(S)$  (the *residual spectrum*  $\sigma_r(S)$ ) of  $S \in \tilde{\mathcal{C}}(\mathcal{H})$  if  $\text{ker}(S - \lambda) = \{0\}$  and  $\text{ran}(S - \lambda)$  is dense in  $\mathcal{H}$  but not equal to  $\mathcal{H}$  (if  $\text{ker}(S - \lambda) = \{0\}$  and  $\text{ran}(S - \lambda)$  is not dense in  $\mathcal{H}$ , respectively).

We define an indefinite inner product  $[\![\cdot, \cdot]\!]_{\mathcal{H}^2}$  on  $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$  by

$$[\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} = i((f, g') - (f', g)), \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix}, \quad \hat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in \mathcal{H}^2.$$

Then  $(\mathcal{H}^2, [\![\cdot, \cdot]\!]_{\mathcal{H}^2})$  is a Krein space and  $\mathcal{J} = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}^2)$  is a corresponding fundamental symmetry. This means that  $\mathcal{H}^2$  is the direct sum of a Hilbert and an anti-Hilbert space and that  $[\![\mathcal{J}\cdot, \cdot]\!]_{\mathcal{H}^2}$  is a positive definite inner product on  $\mathcal{H}^2$ . For a linear relation  $S$  in  $\mathcal{H}$  the *adjoint relation*  $S^* \in \tilde{\mathcal{C}}(\mathcal{H})$  is defined as the orthogonal companion of  $S$  in  $(\mathcal{H}^2, [\![\cdot, \cdot]\!]_{\mathcal{H}^2})$ , i.e.,

$$S^* := S^{\perp\perp}_{\mathcal{H}^2} = \left\{ \hat{f} \in \mathcal{H}^2 \mid [\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2} = 0 \text{ for all } \hat{g} \in S \right\}.$$

Note that this definition extends the usual definition of the adjoint of a densely defined operator. A linear relation  $S$  in  $\mathcal{H}$  is said to be *symmetric* (*self-adjoint*) if  $S \subset S^*$  ( $S = S^*$ , respectively). Recall that a symmetric relation is self-adjoint if and only if  $\text{ran}(S - \lambda_{\pm}) = \mathcal{H}$  holds for some (and hence for all)  $\lambda_{\pm} \in \mathbb{C}^{\pm}$ . We say that  $S$  is *dissipative* (*accumulative*) if  $\text{Im}(f', f) \geq 0$  ( $\text{Im}(f', f) \leq 0$ , respectively) for all  $(f, f')^{\top} \in S$  and  $S$  is said to be *maximal dissipative* (*maximal accumulative*) if  $S$  is dissipative (accumulative, respectively) and has no proper dissipative (accumulative, respectively) extensions in  $\mathcal{H}$ . A dissipative (accumulative) relation  $S$  in  $\mathcal{H}$  is maximal dissipative (maximal accumulative, respectively) if and only if  $\text{ran}(S - \lambda_-) = \mathcal{H}$  ( $\text{ran}(S - \lambda_+) = \mathcal{H}$ , respectively) for some (and hence for all)  $\lambda_- \in \mathbb{C}^-$  ( $\lambda_+ \in \mathbb{C}^+$ , respectively).

In [21] W. N. Everitt and L. Markus considered the symplectic product

$$[\hat{f} : \hat{g}] = i[\![\hat{f}, \hat{g}]\!]_{\mathcal{H}^2}$$

on the graph of an operator and discussed the relation between self-adjoint realizations and Lagrangian subspaces.

For a self-adjoint relation  $S = S^*$  in  $\mathcal{H}$  the multivalued part  $\text{mul } S$  is the orthogonal complement of  $\text{dom } S$  in  $\mathcal{H}$ . Setting  $\mathcal{H}_{\text{op}} := \overline{\text{dom } S}$  and  $\mathcal{H}_{\infty} = \text{mul } S$  one verifies that  $S$  can be written as the direct orthogonal sum of a self-adjoint operator  $S_{\text{op}}$  in the Hilbert space  $\mathcal{H}_{\text{op}}$  and the “pure” relation  $S_{\infty} = \left\{ \begin{pmatrix} 0 \\ f' \end{pmatrix} \mid f' \in \text{mul } S \right\}$  in the Hilbert space  $\mathcal{H}_{\infty}$ ,

$$S = S_{\text{op}} \oplus S_{\infty}$$

with respect to the decomposition  $\mathcal{H} = \mathcal{H}_{\text{op}} \oplus \mathcal{H}_{\infty}$ . Similarly, a maximal dissipative (maximal accumulative) relation  $S$  can be written as the orthogonal sum of a maximal dissipative (maximal accumulative, respectively) operator  $S_{\text{op}}$  in the Hilbert space  $\mathcal{H}_{\text{op}} = (\text{mul } S)^{\perp}$  and a “pure” relation  $S_{\infty}$  in

$\mathcal{H}_\infty = \text{mul } S$ , see e.g. [18, §2.2]. We say that a point  $\lambda \in \mathbb{R}$  belongs to the *essential spectrum*  $\sigma_{\text{ess}}(S)$  of the self-adjoint (maximal dissipative, maximal accumulative) relation  $S$  if  $\lambda \in \sigma_{\text{ess}}(S_{\text{op}})$ . The essential spectrum of an operator  $T$  is the set of  $\lambda \in \mathbb{C}$  such that  $T - \lambda$  is not a Fredholm operator.

## 2.2 Quasi boundary triples

The next definition generalizes the concepts of “ordinary” boundary triples (cf. [14–16, 25, 31] and Section 3.1) and so-called generalized boundary triples (cf. [16, 18] and Section 3.2). We emphasize that a quasi boundary triple is in general not a boundary relation in the sense of [18].

**Definition 2.1** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$ . We say that  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is a quasi boundary triple for  $A^*$  if  $\Gamma_0$  and  $\Gamma_1$  are linear mappings defined on a dense subspace  $T$  of  $A^*$  with values in the Hilbert space  $(\mathcal{G}, (\cdot, \cdot))$  such that  $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : T \rightarrow \mathcal{G} \times \mathcal{G}$  has dense range,  $\ker \Gamma_0$  is self-adjoint and the identity*

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2} \quad (2.1)$$

holds for all  $\hat{f}, \hat{g} \in T$ .

Explicitly, equation (2.1) means

$$(f', g)_{\mathcal{H}} - (f, g')_{\mathcal{H}} = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{G}} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{G}}$$

for all  $\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix}, \hat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in T$ .

We note that a quasi boundary triple for  $A^*$  exists if and only if the deficiency indices  $n_{\pm}(A) = \dim \ker(A^* \mp i)$  of  $A$  coincide. This follows, e.g. from the fact that every boundary triple is also a quasi boundary triple, see Section 3.1.

**Proposition 2.2** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  a quasi boundary triple for  $A^*$ . Then  $A = \ker \Gamma$  and  $\Gamma$  regarded as a mapping from  $\mathcal{H} \times \mathcal{H}$  into  $\mathcal{G} \times \mathcal{G}$  is closable.*

*Proof.* Assume that  $\hat{f}$  belongs to  $\ker \Gamma$  and let  $\hat{g} \in A^*$ . Let  $\hat{g}_n \in T, n = 1, 2, \dots$ , such that  $\hat{g}_n \rightarrow \hat{g}$  for  $n \rightarrow \infty$ . Then

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \lim_{n \rightarrow \infty} \llbracket \hat{f}, \hat{g}_n \rrbracket_{\mathcal{H}^2} = \lim_{n \rightarrow \infty} \llbracket \Gamma \hat{f}, \Gamma \hat{g}_n \rrbracket_{\mathcal{G}^2} = 0$$

implies  $\hat{f} \in (A^*)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} = A^{**} = A$ .

Let  $\hat{f} \in A$ . Then for all  $\hat{g} \in T \subset A^*$  we have  $\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = 0$  and hence  $\hat{f}$  belongs to  $T^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} = (\text{dom } \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}}$ . Since  $A_0 = \ker \Gamma_0 \subset T = \text{dom } \Gamma$  is a self-adjoint

extension of  $A$ , we obtain

$$T^* = (\text{dom } \Gamma)^* = (\text{dom } \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} \subset A_0 \subset \text{dom } \Gamma = T$$

and therefore  $\hat{f} \in \text{dom } \Gamma$ . Hence  $0 = \llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2}$  for all  $\hat{g} \in T$ ; since  $\text{ran } \Gamma$  is dense in  $\mathcal{G} \times \mathcal{G}$ , this yields  $\hat{f} \in \ker \Gamma$ .

The closability of the mapping  $\Gamma$  follows from relation (2.1) and the fact that  $\text{ran } \Gamma$  is dense in  $\mathcal{G} \times \mathcal{G}$ .  $\square$

The following theorem will be useful in Section 4 where quasi boundary triples for elliptic differential operators are constructed. The proof of Theorem 2.3 makes use of some recent results from [18] on isometric operators and relations between Krein spaces, see also [4,34].

**Theorem 2.3** *Let  $\mathcal{H}$  and  $\mathcal{G}$  be Hilbert spaces and let  $T$  be a linear relation in  $\mathcal{H}$ . Assume that  $\Gamma_0, \Gamma_1 : T \rightarrow \mathcal{G}$  are linear mappings such that the following conditions are satisfied:*

- (a)  $\ker \Gamma_0$  contains a self-adjoint relation;
- (b)  $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : T \rightarrow \mathcal{G} \times \mathcal{G}$  has dense range;
- (c)  $\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2}$  for all  $\hat{f}, \hat{g} \in T$ .

Then the following assertions hold.

- (i)  $A := \ker \Gamma$  is a closed symmetric relation in  $\mathcal{H}$  and  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is a quasi boundary triple for  $A^*$ .
- (ii)  $T = A^*$  if and only if  $\text{ran } \Gamma = \mathcal{G} \times \mathcal{G}$ .

*Proof.* (i) We regard  $\Gamma$  as an operator from the Krein space  $(\mathcal{H}^2, \llbracket \cdot, \cdot \rrbracket_{\mathcal{H}^2})$  into the Krein space  $(\mathcal{G}^2, \llbracket \cdot, \cdot \rrbracket_{\mathcal{G}^2})$  and denote the adjoint of  $\Gamma$  by  $\Gamma^{\llbracket + \rrbracket}$ . Then condition (c) implies that  $\Gamma$  is isometric, that is, the relation  $\Gamma^{-1}$  satisfies  $\Gamma^{-1} \subset \Gamma^{\llbracket + \rrbracket}$ . In fact, since an element  $\begin{pmatrix} \hat{h} \\ \hat{f} \end{pmatrix}$  belongs to  $\Gamma^{\llbracket + \rrbracket}$  if and only if

$$\llbracket \hat{\Gamma} \hat{g}, \hat{h} \rrbracket_{\mathcal{G}^2} = \llbracket \hat{g}, \hat{f} \rrbracket_{\mathcal{H}^2} \quad \text{for all} \quad \begin{pmatrix} \hat{g} \\ \Gamma \hat{g} \end{pmatrix} \in \Gamma,$$

it follows that all  $\begin{pmatrix} \hat{h} \\ \hat{f} \end{pmatrix} \in \Gamma^{-1}$  (i.e.,  $\hat{h} = \Gamma \hat{f}$ ,  $\hat{f} \in T$ ) belong to  $\Gamma^{\llbracket + \rrbracket}$ . By condition (a) there exists a self-adjoint relation  $A_0 = A_0^*$  in  $\mathcal{H}$  such that the inclusions  $A_0 \subset \ker \Gamma_0 \subset T = \text{dom } \Gamma$  hold and therefore

$$T^* = (\text{dom } \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} \subset A_0^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} = A_0 \subset \text{dom } \Gamma = T.$$

From assumption (b) we immediately conclude that  $(\text{ran } \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{G}^2}} = \{0\}$ . Now the same argument as in the proof of Proposition 2.2 (see also [18, Proposi-

tion 2.5]) shows that

$$\ker \Gamma = (\operatorname{dom} \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}}.$$

For the closed symmetric relation  $A := \ker \Gamma$  this implies

$$A^* = A^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} = (\ker \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} = \overline{\operatorname{dom} \Gamma} = \overline{T}.$$

For  $\hat{f}, \hat{g} \in \ker \Gamma_0$  condition (c) yields  $\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = 0$  and therefore  $\ker \Gamma_0$  is a symmetric relation in  $\mathcal{H}$ . Hence by (a)  $\ker \Gamma_0$  coincides with the self-adjoint relation  $A_0$  and it follows that  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is a quasi boundary triple for  $A^*$ .

(ii) Assume that  $T = A^*$  holds. We claim that under this assumption  $\Gamma^{\llbracket + \rrbracket} = \Gamma^{-1}$ . By part (i) of the proof we have  $\Gamma^{-1} \subset \Gamma^{\llbracket + \rrbracket}$ . Let now  $\begin{pmatrix} \hat{k} \\ \hat{f} \end{pmatrix} \in \Gamma^{\llbracket + \rrbracket}$ ,  $\hat{k} = \begin{pmatrix} k \\ k' \end{pmatrix} \in \mathcal{G}^2$ ,  $\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in \mathcal{H}^2$ . We have

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \hat{k}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2} \quad \text{for all} \quad \begin{pmatrix} \hat{g} \\ \Gamma \hat{g} \end{pmatrix} \in \Gamma, \quad \hat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in T. \quad (2.2)$$

Hence for  $\hat{g} \in A = \ker \Gamma$  we have  $\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = 0$  and this implies that  $\hat{f}$  belongs to  $A^{\llbracket \perp \rrbracket_{\mathcal{H}^2}} = A^* = T$ . By condition (c) we then have

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2}$$

for all  $\hat{g} \in T$  and combining this with (2.2) we obtain

$$\llbracket \Gamma \hat{f} - \hat{k}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2} = 0.$$

Now condition (b) implies  $\hat{k} = \Gamma \hat{f}$  and therefore  $\begin{pmatrix} \hat{k} \\ \hat{f} \end{pmatrix} \in \Gamma^{-1}$ , which shows  $\Gamma^{-1} = \Gamma^{\llbracket + \rrbracket}$ . As  $\operatorname{dom} \Gamma = T = A^*$  is closed, we can now apply [18, Proposition 2.3]. It follows that  $\operatorname{ran} \Gamma$  is closed, i.e.,  $\operatorname{ran} \Gamma = \mathcal{G} \times \mathcal{G}$ .

Now let us prove the converse implication in (ii). Assume that  $\operatorname{ran} \Gamma = \mathcal{G} \times \mathcal{G}$ . Then  $\operatorname{dom} \Gamma^{\llbracket + \rrbracket} \subset \operatorname{ran} \Gamma$ . By part (i) of the proof  $\ker \Gamma = (\operatorname{dom} \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{H}^2}}$  holds and [18, Proposition 2.5] yields  $\Gamma^{-1} = \Gamma^{\llbracket + \rrbracket}$ . An application of [18, Proposition 2.3] shows that  $\operatorname{dom} \Gamma = T$  is closed, hence  $T = A^*$ .  $\square$

Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$ . For a linear relation  $\Theta \subset \mathcal{G} \times \mathcal{G}$  we define

$$A_\Theta := \left\{ \hat{f} \in T \mid \Gamma \hat{f} \in \Theta \right\} = \Gamma^{-1}(\Theta \cap \operatorname{ran} \Gamma). \quad (2.3)$$

If  $\Theta \subset \mathcal{G} \times \mathcal{G}$  is an operator, then obviously  $A_\Theta$  is given by  $A_\Theta = \ker(\Gamma_1 - \Theta \Gamma_0)$ .

**Proposition 2.4** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$ , let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$  and let  $\Theta$  be a linear relation in  $\mathcal{G}$ . Then the following holds.*

- (i)  $\Theta \subset \Theta^*$  implies  $A_\Theta \subset A_\Theta^*$ .
- (ii) If  $\overline{A_\Theta^* \cap T} = A_\Theta^*$  and  $\overline{\Theta \cap \text{ran } \Gamma} = \Theta$ , then  $\Theta = \Theta^*$  implies  $\overline{A_\Theta} = A_\Theta^*$ .
- (iii) If  $\overline{\Theta \cap \text{ran } \Gamma} = \Theta$ , then  $A_\Theta \subset A_\Theta^*$  implies  $\Theta \subset \Theta^*$ .
- (iv) If  $\overline{\Theta \cap \text{ran } \Gamma} = \Theta$  and  $\overline{\Theta^* \cap \text{ran } \Gamma} = \Theta^*$ , then  $A_\Theta^* = \overline{A_\Theta}$  implies  $\overline{\Theta} = \Theta^*$ .

*Proof.* (i) Let  $\hat{f}, \hat{g} \in A_\Theta$ . Then  $\Gamma \hat{f}, \Gamma \hat{g} \in \Theta \subset \Theta^*$  implies

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2} = 0$$

and therefore  $A_\Theta \subset A_\Theta^*$ .

(ii) From part (i) we have  $A_\Theta \subset A_\Theta^*$  and therefore  $\overline{A_\Theta} \subset A_\Theta^*$ . Let  $\hat{f} \in A_\Theta^*$ , then by our assumptions we can choose a sequence  $(\hat{f}_n) \in A_\Theta^* \cap T$  with  $\hat{f}_n \rightarrow \hat{f}$  for  $n \rightarrow \infty$ . For all  $\hat{g} \in A_\Theta$  and  $n \in \mathbb{N}$  we have

$$0 = \llbracket \hat{f}_n, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}_n, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2}$$

and therefore

$$\Gamma \hat{f}_n \in (\Theta \cap \text{ran } \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{G}^2}} = (\overline{\Theta \cap \text{ran } \Gamma})^* = \Theta^*, \quad n \in \mathbb{N},$$

where we have used  $\overline{\Theta \cap \text{ran } \Gamma} = \Theta$ . From  $\Gamma \hat{f}_n \in \Theta^* \cap \text{ran } \Gamma = \Theta \cap \text{ran } \Gamma$  we conclude  $\hat{f}_n \in A_\Theta$  and hence  $\hat{f} \in \overline{A_\Theta}$ .

(iii) For  $\hat{x}, \hat{y} \in \Theta \cap \text{ran } \Gamma$  we choose  $\hat{f}, \hat{g} \in A_\Theta$  such that  $\hat{x} = \Gamma \hat{f}$  and  $\hat{y} = \Gamma \hat{g}$ . From  $A_\Theta \subset A_\Theta^*$  we obtain

$$0 = \llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2} = \llbracket \hat{x}, \hat{y} \rrbracket_{\mathcal{G}^2};$$

hence  $(\Theta \cap \text{ran } \Gamma) \subset (\Theta \cap \text{ran } \Gamma)^*$ . The assumption  $\overline{\Theta \cap \text{ran } \Gamma} = \Theta$  implies  $\Theta \subset \Theta^*$ .

(iv) From part (iii) we have  $\overline{\Theta} \subset \Theta^*$ . Let  $\hat{x} \in \Theta^* \cap \text{ran } \Gamma$  and choose  $\hat{f} \in T$  with  $\Gamma \hat{f} = \hat{x}$ . We claim that  $\hat{f} \in \overline{A_\Theta}$ . In fact, if  $\hat{g} \in A_\Theta$  and  $\hat{y} = \Gamma \hat{g}$ , then  $\hat{y} \in \Theta$  implies

$$\llbracket \hat{g}, \hat{f} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{g}, \Gamma \hat{f} \rrbracket_{\mathcal{G}^2} = \llbracket \hat{y}, \hat{x} \rrbracket_{\mathcal{G}^2} = 0$$

and we conclude that  $\hat{f} \in A_\Theta^* = \overline{A_\Theta}$ .

Let  $\hat{f}_n \in A_\Theta$  such that  $\hat{f}_n \rightarrow \hat{f}$  for  $n \rightarrow \infty$  and let  $\hat{z} \in \Theta^* \cap \text{ran } \Gamma$ . Then as above there exists an  $\hat{h} \in A_\Theta^*$  with  $\Gamma \hat{h} = \hat{z}$  and we obtain

$$\llbracket \hat{z}, \hat{x} \rrbracket_{\mathcal{G}^2} = \llbracket \Gamma \hat{h}, \Gamma \hat{f} \rrbracket_{\mathcal{G}^2} = \llbracket \hat{h}, \hat{f} \rrbracket_{\mathcal{H}^2} = \lim_{n \rightarrow \infty} \llbracket \hat{h}, \hat{f}_n \rrbracket_{\mathcal{H}^2} = 0;$$

therefore  $\overline{\Theta^* \cap \text{ran } \Gamma} = \Theta^*$  implies

$$\hat{x} \in (\Theta^* \cap \text{ran } \Gamma)^{\llbracket \perp \rrbracket_{\mathcal{G}^2}} = (\overline{\Theta^* \cap \text{ran } \Gamma})^{\llbracket \perp \rrbracket_{\mathcal{G}^2}} = \Theta^{*\llbracket \perp \rrbracket_{\mathcal{G}^2}} = \Theta^{**} = \overline{\Theta},$$

and we obtain  $\Theta^* \cap \text{ran } \Gamma \subset \overline{\Theta}$ , i.e.,  $\Theta^* \subset \overline{\Theta}$ .  $\square$

Later we will particularly make use of the fact that a symmetric relation  $\Theta$  in  $\mathcal{G}$  induces a symmetric extension  $A_\Theta$  in  $\mathcal{H}$  via (2.3). For completeness we note that similarly dissipative (accumulative) relations  $\Theta$  in  $\mathcal{G}$  induce dissipative (accumulative, respectively) extensions  $A_\Theta$  of  $A$ .

### 2.3 Weyl functions and $\gamma$ -fields associated to quasi boundary triples

Let again  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for the adjoint relation  $A^*$ . We set

$$\mathcal{G}_0 := \text{ran } \Gamma_0 \quad \text{and} \quad \mathcal{G}_1 := \text{ran } \Gamma_1.$$

The inclusion  $\text{ran } \Gamma \subset \mathcal{G}_0 \times \mathcal{G}_1$  implies that  $\mathcal{G}_0$  and  $\mathcal{G}_1$  are dense subspaces of  $\mathcal{G}$ . As  $A_0 = \ker \Gamma_0 \subset T = \text{dom } \Gamma$  is a self-adjoint extension of  $A$  in  $\mathcal{H}$ , the decomposition

$$A^* = A_0 \dot{+} \hat{\mathcal{N}}_{\lambda, A^*}, \quad \hat{\mathcal{N}}_{\lambda, A^*} := \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \mid f_\lambda \in \ker(A^* - \lambda) \right\},$$

holds for all  $\lambda \in \rho(A_0)$  and hence also

$$T = A_0 \dot{+} \hat{\mathcal{N}}_{\lambda, T}, \quad \hat{\mathcal{N}}_{\lambda, T} = \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \mid f_\lambda \in \ker(T - \lambda) \right\} = \hat{\mathcal{N}}_{\lambda, A^*} \cap T \quad (2.4)$$

for all  $\lambda \in \rho(A_0)$ . Therefore the mapping

$$\hat{\gamma}(\lambda) := (\Gamma_0|_{\hat{\mathcal{N}}_{\lambda, T}})^{-1} : \mathcal{G}_0 \rightarrow \hat{\mathcal{N}}_{\lambda, T}, \quad \lambda \in \rho(A_0), \quad (2.5)$$

is well defined and bijective. The  $\gamma$ -field and Weyl function of  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  can now be defined as in [16,18] for generalized boundary triples and boundary relations.

**Definition 2.5** *Let  $A \subset A^*$  and  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be as above and denote the orthogonal projection in  $\mathcal{H} \oplus \mathcal{H}$  onto the first component of  $\mathcal{H} \oplus \mathcal{H}$  by  $\pi_1$ . The  $\gamma$ -field  $\gamma$  and the Weyl function  $M$  corresponding to the quasi boundary triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  are defined by*

$$\gamma(\lambda) := \pi_1 \hat{\gamma}(\lambda) \quad \text{and} \quad M(\lambda) := \Gamma_1 \hat{\gamma}(\lambda), \quad \lambda \in \rho(A_0). \quad (2.6)$$

The next proposition collects some properties of the  $\gamma$ -field and the Weyl function of a quasi boundary triple. In the special case of a boundary triple

the statements are well known. We note that the values of the Weyl function corresponding to a quasi boundary triple are not necessarily closed operators (as it is the case for a Weyl function or Weyl family of a generalized boundary triple or boundary relation, respectively), cf. Proposition 2.6 (v)-(vi) and Section 4.1.

**Proposition 2.6** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$  with  $\gamma$ -field  $\gamma$  and Weyl function  $M$ . For  $\lambda, \mu \in \rho(A_0)$  the following assertions hold.*

- (i)  $\gamma(\lambda)$  is a densely defined bounded operator from  $\mathcal{G}$  into  $\mathcal{H}$  with domain  $\text{dom } \gamma(\lambda) = \mathcal{G}_0$ , the function  $\lambda \mapsto \gamma(\lambda)g$  is holomorphic on  $\rho(A_0)$  for every  $g \in \mathcal{G}_0$ , and the relation

$$\gamma(\lambda) = \left( I + (\lambda - \mu)(A_0 - \lambda)^{-1} \right) \gamma(\mu) \quad (2.7)$$

holds.

- (ii)  $\gamma(\bar{\lambda})^*$  is a bounded mapping defined on  $\mathcal{H}$  with values in  $\mathcal{G}_1 \subset \mathcal{G}$  and for all  $h \in \mathcal{H}$  we have

$$\gamma(\bar{\lambda})^* h = \Gamma_1 \begin{pmatrix} (A_0 - \lambda)^{-1} h \\ (I + \lambda(A_0 - \lambda)^{-1}) h \end{pmatrix}.$$

- (iii)  $M(\lambda)$  maps  $\mathcal{G}_0$  into  $\mathcal{G}_1$ . If, in addition,  $\ker \Gamma_1 \subset T$  is a self-adjoint relation in  $\mathcal{H}$  and  $\lambda \in \rho(\ker \Gamma_1)$ , then  $M(\lambda)$  maps  $\mathcal{G}_0$  onto  $\mathcal{G}_1$ .
- (iv)  $M(\lambda)\Gamma_0 \hat{f}_\lambda = \Gamma_1 \hat{f}_\lambda$  for all  $\hat{f}_\lambda \in \hat{\mathcal{N}}_{\lambda, T}$ .
- (v)  $M(\lambda) \subset M(\bar{\lambda})^*$  and  $M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^* \gamma(\lambda)$ . The function  $\lambda \mapsto M(\lambda)$  is holomorphic in the sense that it can be written as the sum of the possibly unbounded operator  $\text{Re } M(\mu)$  and a bounded holomorphic operator function,

$$\begin{aligned} M(\lambda) &= \text{Re } M(\mu) \\ &+ \gamma(\mu)^* \left( (\lambda - \text{Re } \mu) + (\lambda - \mu)(\lambda - \bar{\mu})(A_0 - \lambda)^{-1} \right) \gamma(\mu). \end{aligned} \quad (2.8)$$

- (vi)  $\text{Im } M(\lambda) = \frac{1}{2i}(M(\lambda) - M(\bar{\lambda}))$  is a densely defined bounded operator in  $\mathcal{G}$ . For  $\lambda \in \mathbb{C}^+(\mathbb{C}^-)$  the operator  $\text{Im } M(\lambda)$  is positive (negative, respectively).

*Proof.* (i)-(ii) Let  $x \in \text{dom } \gamma(\bar{\lambda}) = \mathcal{G}_0 \subset \mathcal{G}$  and set  $\hat{g} := \hat{\gamma}(\bar{\lambda})x$ ,  $\hat{g} = \begin{pmatrix} g \\ \bar{\lambda}g \end{pmatrix}$ . For  $h \in \mathcal{H}$  we define

$$\Psi(\lambda)h := \begin{pmatrix} (A_0 - \lambda)^{-1} h \\ (I + \lambda(A_0 - \lambda)^{-1}) h \end{pmatrix} \in A_0.$$

Since  $\Gamma$  is closable and  $\Psi(\lambda)$  is bounded, the mapping  $\Gamma\Psi(\lambda) : \mathcal{H} \rightarrow \mathcal{G} \times \mathcal{G}$  is closable and everywhere defined, hence bounded and therefore also the mapping  $\Gamma_1\Psi(\lambda) : \mathcal{H} \rightarrow \mathcal{G}$  is bounded. Then

$$\begin{aligned} (\gamma(\bar{\lambda})x, h) &= (g, h) = (g, (I + \lambda(A_0 - \lambda)^{-1})h) - (\bar{\lambda}g, (A_0 - \lambda)^{-1}h) \\ &= -i \left[ \hat{g}, \Psi(\lambda)h \right]_{\mathcal{H}^2} = -i \left[ \Gamma\hat{g}, \Gamma\Psi(\lambda)h \right]_{\mathcal{G}^2} \\ &= (\Gamma_0\hat{g}, \Gamma_1\Psi(\lambda)h) = (x, \Gamma_1\Psi(\lambda)h) \end{aligned}$$

proves assertion (ii). Replacing  $\bar{\lambda}$  by  $\lambda$  we find that  $\gamma(\lambda)^{**} = \overline{\gamma(\bar{\lambda})} \supset \gamma(\lambda)$  is a bounded operator. It is straightforward to verify the relation

$$\gamma(\lambda)^* - \gamma(\mu)^* = \gamma(\mu)^*(\bar{\lambda} - \bar{\mu})(A_0 - \bar{\lambda})^{-1}. \quad (2.9)$$

By taking the adjoint of (2.9) we find (2.7) and it follows that  $\lambda \mapsto \gamma(\lambda)$  is holomorphic on  $\rho(A_0)$ .

The assertions (iii) and (iv) follow immediately from the definition of the Weyl function and the decomposition  $T = \ker \Gamma_1 \dot{+} \hat{\mathcal{N}}_{\lambda, T}$ ,  $\lambda \in \rho(\ker \Gamma_1)$ .

(v) Let  $\hat{f}_{\bar{\lambda}} \in \hat{\mathcal{N}}_{\bar{\lambda}, T}$  and  $\hat{g}_{\lambda} \in \hat{\mathcal{N}}_{\lambda, T}$ . By (2.1) we have

$$0 = \left[ \hat{f}_{\bar{\lambda}}, \hat{g}_{\lambda} \right]_{\mathcal{H}^2} = \left[ \Gamma\hat{f}_{\bar{\lambda}}, \Gamma\hat{g}_{\lambda} \right]_{\mathcal{G}^2},$$

and therefore

$$(M(\bar{\lambda})\Gamma_0\hat{f}_{\bar{\lambda}}, \Gamma_0\hat{g}_{\lambda}) = (\Gamma_1\hat{f}_{\bar{\lambda}}, \Gamma_0\hat{g}_{\lambda}) = (\Gamma_0\hat{f}_{\bar{\lambda}}, \Gamma_1\hat{g}_{\lambda}) = (\Gamma_0\hat{f}_{\bar{\lambda}}, M(\lambda)\Gamma_0\hat{g}_{\lambda})$$

holds. This implies  $\mathcal{G}_0 \subset \text{dom } M(\bar{\lambda})^*$  and  $M(\lambda) \subset M(\bar{\lambda})^*$ .

Analogously, for  $\hat{h}_{\lambda} = \begin{pmatrix} h_{\lambda} \\ \lambda h_{\lambda} \end{pmatrix} \in \hat{\mathcal{N}}_{\lambda, T}$  and  $\hat{k}_{\mu} = \begin{pmatrix} k_{\mu} \\ \mu k_{\mu} \end{pmatrix} \in \hat{\mathcal{N}}_{\mu, T}$  we obtain

$$\begin{aligned} (\lambda - \bar{\mu})(h_{\lambda}, k_{\mu}) &= i \left[ \hat{h}_{\lambda}, \hat{k}_{\mu} \right]_{\mathcal{H}^2} = i \left[ \Gamma\hat{h}_{\lambda}, \Gamma\hat{k}_{\mu} \right]_{\mathcal{G}^2} \\ &= (M(\lambda)\Gamma_0\hat{h}_{\lambda}, \Gamma_0\hat{k}_{\mu}) - (\Gamma_0\hat{h}_{\lambda}, M(\mu)\Gamma_0\hat{k}_{\mu}) \end{aligned}$$

from (2.1). Let  $x_{\lambda} := \Gamma_0\hat{h}_{\lambda}$  and  $y_{\mu} := \Gamma_0\hat{k}_{\mu}$ . From the definition of  $\gamma$  and  $\mathcal{G}_0 \subset \text{dom } M(\mu)^*$  we conclude that

$$(\lambda - \bar{\mu})(\gamma(\lambda)x_{\lambda}, \gamma(\mu)y_{\mu}) = \left( (M(\lambda) - M(\mu)^*)x_{\lambda}, y_{\mu} \right).$$

Since  $\text{dom } \gamma(\mu)^* = \mathcal{H}$  and  $\mathcal{G}_0 = \text{ran } \Gamma_0$  is dense in  $\mathcal{G}$ , the second equality in (v) holds. Making use of (2.7),  $M(\mu) - M(\mu)^* = (\mu - \bar{\mu})\gamma(\mu)^*\gamma(\mu)$  and the fact that  $\text{Re } M(\mu)^* = \frac{1}{2}(M(\mu)^* + \overline{M(\mu)})$  is an extension of  $\text{Re } M(\mu)$ , it is not difficult to verify relation (2.8).

(vi) It follows from (v) that  $(\operatorname{Im} M(\lambda)x, x) = \operatorname{Im} \lambda \|\gamma(\lambda)x\|^2$ , which is positive (negative) for  $\lambda \in \mathbb{C}^+$  ( $\mathbb{C}^-$ , respectively) and  $x \neq 0$ . By (i) and (ii)

$$\operatorname{Im} M(\lambda) = (\operatorname{Im} \lambda) \gamma(\lambda)^* \gamma(\lambda)$$

is densely defined and bounded.  $\square$

**Example 2.7** Let  $K$  be a non-negative compact operator in the Hilbert space  $\mathcal{H}$  with  $0 \in \sigma_c(K)$  and let

$$T := \left\{ \begin{pmatrix} f \\ f' \end{pmatrix} \mid f \in \operatorname{ran} K, f' \in \mathcal{H} \right\}.$$

The adjoint of the (trivial) relation  $A := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \in \tilde{\mathcal{C}}(\mathcal{H})$  is given by

$$A^* = \left\{ \begin{pmatrix} f \\ f' \end{pmatrix} \mid f, f' \in \mathcal{H} \right\} = \bar{T}$$

and  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ , where  $\mathcal{G} = \mathcal{H}$ ,

$$\Gamma_0 \hat{f} := K^{-\frac{1}{2}} f \quad \text{and} \quad \Gamma_1 \hat{f} := K^{\frac{1}{2}} f', \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in T,$$

is a quasi boundary triple for  $A^*$ . Here  $A_0 = \ker \Gamma_0 = \left\{ \begin{pmatrix} 0 \\ f' \end{pmatrix} \mid f' \in \mathcal{H} \right\}$  is a purely multivalued relation and therefore the  $\gamma$ -field  $\gamma$  and the Weyl function  $M$  corresponding to  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  are defined for all  $\lambda \in \mathbb{C}$ . We remark that  $\ker \Gamma_1 = \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix} \mid f \in \operatorname{ran} K \right\}$  is not self-adjoint but essentially self-adjoint. From

$$\hat{\mathcal{N}}_{\lambda, T} = \left\{ \begin{pmatrix} f \\ \lambda f \end{pmatrix} \mid f \in \operatorname{ran} K \right\},$$

(2.5) and Definition 2.5 we obtain  $\gamma(\lambda)h = K^{\frac{1}{2}}h$ ,  $\lambda \in \mathbb{C}$ ,  $h \in \mathcal{H}$ , and the Weyl function  $M$  is given by  $\lambda \mapsto M(\lambda) = \lambda K$ ,  $\lambda \in \mathbb{C}$ . Note that in contrast to Weyl functions corresponding to ordinary boundary triples (cf. Section 3.1), here there is no  $\lambda \in \mathbb{C}^+$  ( $\lambda \in \mathbb{C}^-$ ) such that  $\operatorname{Im} M(\lambda) = (\operatorname{Im} \lambda)K$  is uniformly positive (uniformly negative, respectively).

The next theorem is a variant of Krein's formula for the resolvents of canonical extensions. In the framework of ordinary boundary triples formula (2.10) is well known and a more precise description of the spectrum of the canonical extensions in terms of the Weyl function and the parameter  $\Theta$  can be given (see e.g. [14–16,31] and Section 3.1). For the convenience of the reader we give a complete proof of Theorem 2.8 which is similar to the proofs in [14–16,31].

**Theorem 2.8** Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$  with  $\gamma$ -field  $\gamma$  and Weyl function  $M$ . Let  $\Theta$  be a linear relation in  $\mathcal{G}$  and let  $A_\Theta$  be the corresponding extension defined in (2.3). Then for all  $\lambda \in \rho(A_0)$  the following assertions (i)-(iii) hold.

- (i) A point  $\lambda \in \rho(A_0)$  belongs to  $\sigma_p(A_\Theta)$  if and only if  $0 \in \sigma_p(\Theta - M(\lambda))$ .  
(ii) If  $\Theta - M(\lambda)$  is injective and  $\gamma(\bar{\lambda})^*g \in \text{ran}(\Theta - M(\lambda))$ , then

$$(A_\Theta - \lambda)^{-1}g = (A_0 - \lambda)^{-1}g + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*g \quad (2.10)$$

holds. If, in particular,  $\Theta - M(\lambda)$  is injective and  $\text{ran}(\Theta - M(\lambda)) \supset \mathcal{G}_1$ , then  $A_\Theta - \lambda$  is bijective and formula (2.10) holds for all  $g \in \mathcal{H}$ .

- (iii) If  $\{0\} \times \mathcal{G}_1 \subset \text{ran} \Gamma$  and  $A_\Theta - \lambda$  is bijective, then  $\Theta - M(\lambda)$  is injective and  $\text{ran}(\Theta - M(\lambda)) \supset \mathcal{G}_1$  holds.

*Proof.* (i) Assume that  $\begin{pmatrix} f \\ 0 \end{pmatrix} \in A_\Theta - \lambda$  for some  $f \neq 0$ . Then  $\hat{f} := \begin{pmatrix} f \\ \lambda f \end{pmatrix}$  belongs to  $A_\Theta \cap \hat{\mathcal{N}}_{\lambda, T}$  and as  $\Gamma \hat{f} \in \Theta$ , we obtain

$$\begin{pmatrix} \Gamma_0 \hat{f} \\ 0 \end{pmatrix} = \begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} - M(\lambda) \Gamma_0 \hat{f} \end{pmatrix} \in \Theta - M(\lambda).$$

Moreover,  $\Gamma_0 \hat{f} \neq 0$  as otherwise  $\hat{f} \in A_0 \cap \hat{\mathcal{N}}_{\lambda, T}$  would imply  $f = 0$ .

If  $\begin{pmatrix} y \\ 0 \end{pmatrix} \in \Theta - M(\lambda)$ , then  $\begin{pmatrix} y \\ M(\lambda)y \end{pmatrix} \in \Theta$  and for  $\hat{f} := \hat{\gamma}(\lambda)y \in \hat{\mathcal{N}}_{\lambda, T}$  we obtain

$$\begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix} = \begin{pmatrix} y \\ M(\lambda)y \end{pmatrix} \in \Theta.$$

Therefore  $\hat{f} \in A_\Theta$ , i.e.,  $\begin{pmatrix} \gamma(\lambda)y \\ 0 \end{pmatrix} \in A_\Theta - \lambda$ .

(ii) Assume that  $\Theta - M(\lambda)$  is injective and let  $\gamma(\bar{\lambda})^*g \in \text{ran}(\Theta - M(\lambda))$  for some  $g \in \mathcal{H}$ . By part (i) of the theorem  $A_\Theta - \lambda$  is injective. We show that  $g \in \text{ran}(A_\Theta - \lambda)$  and formula (2.10) holds. By Proposition 2.6 (ii) we have

$$\gamma(\bar{\lambda})^*g = \Gamma_1 \begin{pmatrix} (A_0 - \lambda)^{-1}g \\ (I + \lambda(A_0 - \lambda)^{-1})g \end{pmatrix} \in \mathcal{G}_1$$

and since  $\gamma(\bar{\lambda})^*g \in \text{ran}(\Theta - M(\lambda))$ , we conclude that  $(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*g$  belongs to  $\mathcal{G}_0$ . We claim that

$$\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} := \hat{\gamma}(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*g + \begin{pmatrix} (A_0 - \lambda)^{-1}g \\ (I + \lambda(A_0 - \lambda)^{-1})g \end{pmatrix} \quad (2.11)$$

belongs to  $A_\Theta$ . In fact, as

$$\begin{aligned} \Gamma_0 \hat{f} &= (\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*g \\ \Gamma_1 \hat{f} &= M(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*g + \gamma(\bar{\lambda})^*g \end{aligned}$$

and

$$\begin{pmatrix} (\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^* g \\ \gamma(\bar{\lambda})^* g \end{pmatrix} \in \Theta - M(\lambda)$$

we obtain

$$\begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix} = \begin{pmatrix} (\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^* g \\ M(\lambda) (\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^* g + \gamma(\bar{\lambda})^* g \end{pmatrix} \in M(\lambda) + (\Theta - M(\lambda)) = \Theta,$$

that is,  $\hat{f} \in A_\Theta$ . From the definition of  $\hat{f}$  we find  $g = f' - \lambda f$ , hence

$$\begin{pmatrix} f \\ g \end{pmatrix} \in A_\Theta - \lambda \quad (2.12)$$

and therefore  $g \in \text{ran}(A_\Theta - \lambda)$ . It follows from (2.11) and (2.12) that

$$(A_\Theta - \lambda)^{-1} g = f = (A_0 - \lambda)^{-1} g + \gamma(\lambda) (\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^* g$$

holds.

In the case  $\mathcal{G}_1 \subset \text{ran}(\Theta - M(\lambda))$  each  $g \in \mathcal{H}$  satisfies  $\gamma(\bar{\lambda})^* g \in \text{ran}(\Theta - M(\lambda))$ . Therefore the arguments above imply that  $A_\Theta - \lambda$  is bijective and formula (2.10) is valid for all  $g \in \mathcal{H}$ .

(iii) Assume that  $\{0\} \times \mathcal{G}_1 \subset \text{ran} \Gamma$  and that  $A_\Theta - \lambda$  is bijective. Since  $A_\Theta - \lambda$  is injective it follows from part (i) that  $\Theta - M(\lambda)$  is injective and it remains to show that  $\mathcal{G}_1$  is a subset of  $\text{ran}(\Theta - M(\lambda))$ . For  $y' \in \mathcal{G}_1$  there exists an element  $\hat{g} = \begin{pmatrix} g \\ y' \end{pmatrix} \in T$  such that  $\Gamma_0 \hat{g} = 0$  and  $\Gamma_1 \hat{g} = y'$  holds. For

$$f := (A_\Theta - \lambda)^{-1} (g' - \lambda g)$$

we have  $\hat{f} := \begin{pmatrix} f \\ g' + \lambda(f - g) \end{pmatrix} \in A_\Theta$  and  $\hat{f} - \hat{g} \in \hat{\mathcal{N}}_{\lambda, T}$ . From  $\begin{pmatrix} \Gamma_0(\hat{f} - \hat{g}) \\ \Gamma_1 \hat{f} \end{pmatrix} = \Gamma \hat{f} \in \Theta$  we obtain

$$\begin{aligned} \begin{pmatrix} \Gamma_0(\hat{f} - \hat{g}) \\ y' \end{pmatrix} &= \begin{pmatrix} \Gamma_0(\hat{f} - \hat{g}) \\ \Gamma_1 \hat{f} - \Gamma_1(\hat{f} - \hat{g}) \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_0(\hat{f} - \hat{g}) \\ \Gamma_1 \hat{f} - M(\lambda) \Gamma_0(\hat{f} - \hat{g}) \end{pmatrix} \in \Theta - M(\lambda), \end{aligned}$$

and this shows  $\mathcal{G}_1 \subset \text{ran}(\Theta - M(\lambda))$ .  $\square$

### 3 Special quasi boundary triples

#### 3.1 Ordinary boundary triples

The notion of (ordinary) boundary triples is basic in the extension theory of symmetric operators and relations in Hilbert and Krein spaces, see, e.g. [15,16,25,31]. In the following we recall the definition and we show that quasi boundary triples are a natural generalization of this concept.

**Definition 3.1** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$ . Then  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is said to be an ordinary boundary triple for  $A^*$  if  $(\mathcal{G}, (\cdot, \cdot))$  is a Hilbert space and  $\Gamma_0, \Gamma_1 : A^* \rightarrow \mathcal{G}$  are linear mappings such that  $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : A^* \rightarrow \mathcal{G} \times \mathcal{G}$  is surjective and the identity*

$$\llbracket \hat{f}, \hat{g} \rrbracket_{\mathcal{H}^2} = \llbracket \Gamma \hat{f}, \Gamma \hat{g} \rrbracket_{\mathcal{G}^2}$$

holds for all  $\hat{f}, \hat{g} \in A^*$ .

If  $A$  is a closed symmetric relation and  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is an ordinary boundary triple for  $A^*$ , then  $A_0 := \ker \Gamma_0$  and  $A_1 := \ker \Gamma_1$  are self-adjoint extensions of  $A$ . Therefore Theorem 2.3 yields the following corollary.

**Corollary 3.2** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$ . Then (i)-(iii) are equivalent.*

- (i)  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is an ordinary boundary triple.
- (ii)  $\text{dom } \Gamma = A^*$ .
- (iii)  $\text{ran } \Gamma = \mathcal{G} \times \mathcal{G}$ .

Quasi boundary triples that are not boundary triples at the same time can only appear in the case of infinite deficiency indices as the following proposition shows.

**Proposition 3.3** *Let  $A$  be a closed symmetric relation with finite deficiency indices  $n_+(A) = n_-(A) < \infty$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$ . Then  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is an ordinary boundary triple.*

*Proof.* By Proposition 2.2 we have  $\ker \Gamma = A$  and as  $\dim(A^*/A)$  is finite, also  $\dim(T/\ker \Gamma) < \infty$  and therefore  $\text{ran } \Gamma$  is closed, i.e.,  $\text{ran } \Gamma = \mathcal{G} \times \mathcal{G}$ . From Corollary 3.2 we obtain that  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is an ordinary boundary triple.  $\square$

If  $A$  is a closed symmetric relation and  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is an ordinary boundary triple for  $A^*$ , then (2.3) establishes a one-to-one correspondence between the closed extensions  $A_\Theta \subset A^*$  of  $A$  and the set of closed linear relations  $\Theta \in \tilde{\mathcal{C}}(\mathcal{G})$ .

Proposition 2.4 reduces to the well-known fact that a closed extension  $A_\Theta$  is symmetric (self-adjoint) in  $\mathcal{H}$  if and only if  $\Theta \in \tilde{\mathcal{C}}(\mathcal{G})$  is symmetric (self-adjoint, respectively) in  $\mathcal{G}$ . The  $\gamma$ -field  $\gamma$  and the Weyl function  $M$  corresponding to an ordinary boundary triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  are defined as in Definition 2.5. Here we have

$$\gamma(\lambda) \in \mathcal{L}(\mathcal{G}, \mathcal{H}) \quad \text{and} \quad M(\lambda) = M(\bar{\lambda})^* \in \mathcal{L}(\mathcal{G}), \quad \lambda \in \rho(A_0),$$

and  $\gamma$  and  $M$  are holomorphic on  $\rho(A_0)$ . It follows that the Weyl function  $M$  is an  $\mathcal{L}(\mathcal{G})$ -valued Nevanlinna function with the additional property that  $\text{Im } M(\lambda)$  is uniformly positive (uniformly negative) if  $\lambda \in \mathbb{C}^+$  ( $\mathbb{C}^-$ , respectively). Such Nevanlinna functions  $M$  are sometimes called *uniformly strict*. In Section 5 we make use of the following converse statement, see, e.g. [15, Theorem 1], [29, Theorem 2.2] and [18, §5.1].

**Theorem 3.4** *Let  $M$  be an  $\mathcal{L}(\mathcal{G})$ -valued Nevanlinna function with the additional property  $0 \in \rho(\text{Im } M(\lambda))$  for some (and hence for all)  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Then there exists a Hilbert space  $\mathcal{K}$ , a simple symmetric operator  $S$  in  $\mathcal{K}$  and a boundary triple  $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$  for  $S^*$  such that  $M$  is the corresponding Weyl function.*

It is worth to state Theorem 2.8 for the special case of an ordinary boundary triple, cf. [14–16, 31].

**Corollary 3.5** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be an ordinary boundary triple for  $A^*$  with  $\gamma$ -field  $\gamma$  and Weyl function  $M$ . Let  $\Theta$  be a closed linear relation in  $\mathcal{G}$  and let  $A_\Theta$  be the corresponding extension. Then for all  $\lambda \in \rho(A_0)$  the assertions (i) and (ii) hold.*

- (i)  $\lambda \in \sigma_i(A_\Theta)$  if and only if  $0 \in \sigma_i(M(\lambda) - \Theta)$ ,  $i = p, c, r$ .
- (ii)  $\lambda \in \rho(A_\Theta)$  if and only if  $0 \in \rho(M(\lambda) - \Theta)$ . For all  $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$  the formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda) (\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^*$$

holds.

### 3.2 Generalized boundary triples

The notion of generalized boundary triples has been introduced by V. A. Derkach and M. M. Malamud in [16] in order to realize larger subclasses of Nevanlinna functions than those in Theorem 3.4 as Weyl functions, cf. [16, 18].

**Definition 3.6** *Let  $A$  be a closed symmetric operator or relation in  $\mathcal{H}$ . Then  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is said to be a generalized boundary triple for  $A^*$  if  $\Gamma_0$  and  $\Gamma_1$*

are linear mappings defined on a dense subspace  $T$  of  $A^*$  with values in the Hilbert space  $(\mathcal{G}, (\cdot, \cdot))$  such that  $\text{ran } \Gamma_0 = \mathcal{G}$ ,  $\ker \Gamma_0$  is self-adjoint and the identity (2.1) holds for all  $\hat{f}, \hat{g} \in T$ .

The Weyl function  $M$  (as defined in (2.6)) corresponding to a generalized boundary triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is an  $\mathcal{L}(\mathcal{G})$ -valued Nevanlinna function with the additional property  $\ker \text{Im } M(\lambda) = \{0\}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Conversely, every  $\mathcal{L}(\mathcal{G})$ -valued Nevanlinna function  $M$  with  $\ker \text{Im } M(\lambda) = \{0\}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , can be realized as the Weyl function of some generalized boundary triple, see [16, 18].

The definition of a generalized boundary triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  yields that  $\text{ran } \Gamma$  is dense in  $\mathcal{G} \times \mathcal{G}$  (see [16, Lemma 6.1]). This and Proposition 2.6 imply the next corollary.

**Corollary 3.7** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$ .*

- (i) *Each generalized boundary triple for  $A^*$  is also a quasi boundary triple for  $A^*$ .*
- (ii) *If  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is a quasi boundary triple for  $A^*$  such that  $\text{ran } \Gamma_0 = \mathcal{G}$ , then  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  is a generalized boundary triple for  $A^*$ . The corresponding Weyl function  $M$  is an  $\mathcal{L}(\mathcal{G})$ -valued Nevanlinna function with the property  $\ker \text{Im } M(\lambda) = \{0\}$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ .*

We note that in an implicit way generalized boundary triples appeared in connection with second order elliptic differential operators recently in a paper of W. O. Amrein and D. B. Pearson, see [3] and Section 4.2.

## 4 Elliptic Differential Operators

### 4.1 Boundary mappings defined on $H^2$

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^m$  with  $C^\infty$  boundary  $\partial\Omega$  and closure  $\bar{\Omega}$ . We study the differential expression

$$(\mathcal{L}f)(x) := - \sum_{j,k=1}^m (D_j a_{jk} D_k f)(x) + \sum_{j=1}^m (a_j D_j f - D_j \bar{a}_j f)(x) + a(x) f(x), \quad (4.1)$$

$x \in \Omega$ , with coefficients  $a_{jk}, a_j, a \in C^\infty(\bar{\Omega})$ . We assume that  $a_{jk}(x) = \overline{a_{kj}(x)}$  holds for all  $x \in \bar{\Omega}$  and  $j, k = 1, \dots, m$  and that  $a$  is real valued. Moreover, we assume that there exists  $C > 0$  such that

$$\sum_{j,k=1}^m a_{jk}(x) \xi_j \xi_k \geq C \sum_{k=1}^m \xi_k^2$$

holds for all  $x \in \bar{\Omega}$  and all  $\xi = (\xi_1, \dots, \xi_m)^\top \in \mathbb{R}^m$ , i.e.,  $\mathcal{L}$  is a uniformly elliptic differential expression which is symmetric or “formally self-adjoint”. We note that the following results remain valid under weaker assumptions on the domain  $\Omega$  and the functions  $a_{jk}$ ,  $a_j$  and  $a$ , but since the most general setting is not our main objective here, we restrict ourselves to the  $C^\infty$  case.

Define the operator  $T$  in  $\mathcal{H} = L^2(\Omega)$  by

$$Tf = \mathcal{L}f, \quad \text{dom } T = H^2(\Omega),$$

where  $H^k(\Omega)$  denotes the Sobolev space of  $k$ th order. Moreover, let  $\mathcal{G} = L^2(\partial\Omega)$ . In the following we denote by  $\hat{f}$ ,  $\hat{g}$  the elements  $\begin{pmatrix} f \\ Tf \end{pmatrix}$ ,  $\begin{pmatrix} g \\ Tg \end{pmatrix} \in T$ ,  $f, g \in H^2(\Omega)$ . For  $\hat{f} \in T$  we set

$$\Gamma_0 \hat{f} := f|_{\partial\Omega} \quad \text{and} \quad \Gamma_1 \hat{f} := -\frac{\partial f}{\partial \nu} \Big|_{\partial\Omega}, \quad (4.2)$$

where  $f|_{\partial\Omega}$  denotes the image of  $f$  under the trace operator, which is defined on  $H^2(\Omega)$  and has images in  $H^{3/2}(\partial\Omega) \subset L^2(\partial\Omega)$ , and  $\frac{\partial f}{\partial \nu} \Big|_{\partial\Omega}$  is the conormal derivative defined by

$$\frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} := \sum_{j,k=1}^m a_{jk} n_j (D_k f)|_{\partial\Omega} + \sum_{j=1}^m \bar{a}_j n_j f|_{\partial\Omega};$$

here  $n(x) = (n_1(x), \dots, n_m(x))^\top$  is the unit vector at the point  $x \in \partial\Omega$  pointing out of  $\Omega$ .

As in Section 2 we set  $\Gamma = (\Gamma_0, \Gamma_1)^\top$ ; then  $\text{dom } \Gamma = T$ . The scalar products in  $L^2(\Omega)$  and  $L^2(\partial\Omega)$  are denoted by  $(\cdot, \cdot)_\Omega$  and  $(\cdot, \cdot)_{\partial\Omega}$ , respectively. By [30, Section 2.2 and Section 1.8.2] (cf. also [10, Section 2]) we have

$$(Tf, g)_\Omega - (f, Tg)_\Omega = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\partial\Omega} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\partial\Omega}, \quad f, g \in \text{dom } T,$$

and  $\text{ran } \Gamma = \text{ran } \Gamma_0 \times \text{ran } \Gamma_1 = H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ , which is dense in the space  $L^2(\partial\Omega) \times L^2(\partial\Omega)$ . Moreover,  $\ker \Gamma_0$  is the *Dirichlet operator*  $A_D$ ,

$$A_D f = \mathcal{L}f, \quad \text{dom } A_D = \left\{ f \in H^2(\Omega) \mid f|_{\partial\Omega} = 0 \right\} = H^2(\Omega) \cap H_0^1(\Omega),$$

which is self-adjoint. Here  $H_0^k(\Omega)$  is the closure of smooth functions with compact support in  $H^k(\Omega)$ . Now the following proposition follows directly from Theorem 2.3.

**Proposition 4.1** *Define the operator  $A$  in  $L^2(\Omega)$  by*

$$Af = \mathcal{L}f, \quad \text{dom } A = \left\{ f \in H^2(\Omega) \mid f|_{\partial\Omega} = \frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} = 0 \right\} = H_0^2(\Omega), \quad (4.3)$$

and let  $T$ ,  $\Gamma_0$  and  $\Gamma_1$  be as above. Then  $A = \ker \Gamma$  is a densely defined closed symmetric operator in  $L^2(\Omega)$  with infinite deficiency indices,  $\overline{T} = A^*$  and  $T$  is not closed. Moreover,  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  is a quasi boundary triple for  $A^*$ .

The operator  $A_N := \ker \Gamma_1$  is also self-adjoint, cf. [11, Theorem 5], the *Neumann operator*. Its domain is given by  $\{f \in H^2(\Omega) \mid \frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0\}$ . We note that  $A$ , the *minimal operator*, is the closure of the operator  $\mathcal{L}$  with domain  $C_0^\infty(\Omega)$ . The adjoint operator  $A^*$  of  $A$  in (4.3) is the usual *maximal operator*,

$$A^*f = \mathcal{L}f, \quad \text{dom } A^* = \left\{f \in L^2(\Omega) \mid \mathcal{L}f \in L^2(\Omega)\right\}.$$

In the next proposition we collect some properties of the Weyl function corresponding to the quasi boundary triple  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ .

**Proposition 4.2** *Let  $T = \mathcal{L} \upharpoonright H^2(\Omega)$ , let  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  be the quasi boundary triple for  $A^*$  from above and let  $M$  be the corresponding Weyl function in  $L^2(\partial\Omega)$ . Then the following holds.*

- (i) *For  $\lambda \in \rho(A_D)$  we have  $\text{dom } M(\lambda) = H^{3/2}(\partial\Omega)$ ,  $\text{ran } M(\lambda) \subset H^{1/2}(\partial\Omega)$  and*

$$M(\lambda)(f_\lambda|_{\partial\Omega}) = \frac{\partial f_\lambda}{\partial \nu}|_{\partial\Omega}, \quad (4.4)$$

*where  $f_\lambda \in H^2(\Omega)$  is a solution of  $\mathcal{L}f_\lambda = \lambda f_\lambda$ . If, in addition  $\lambda \in \rho(A_N)$ , then  $\text{ran } M(\lambda) = H^{1/2}(\partial\Omega)$ .*

- (ii) *The operator  $\overline{M(\lambda)}$ ,  $\lambda \in \rho(A_D)$ , in (4.4) is unbounded and closable. The closure  $\overline{M(\lambda)}$  of  $M(\lambda)$  in  $L^2(\partial\Omega)$  is defined on  $H^1(\partial\Omega)$ .*
- (iii)  *$\{L^2(\partial\Omega), -\Gamma_1, \Gamma_0\}$  is also a quasi boundary triple for  $A^*$ . The values of the corresponding Weyl function  $-M^{-1}$  are bounded operators in  $L^2(\partial\Omega)$  defined on  $H^{1/2}(\partial\Omega)$  and their closures are compact operators in  $L^2(\partial\Omega)$ .*

*Proof.* Assertion (i) follows from Proposition 2.6 and assertions (ii) and (iii) will be easy consequences of Proposition 4.6 in the next section.  $\square$

With the help of Krein's formula (see Theorem 2.8) we give a sufficient condition on the parameter  $\Theta$  such that the corresponding extension  $A_\Theta$  of  $A$  via (2.3) is self-adjoint.

**Proposition 4.3** *Let  $T = \mathcal{L} \upharpoonright H^2(\Omega)$ ,  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  and  $M$  be as above, let  $\Theta$  be a symmetric relation in  $L^2(\partial\Omega)$  and assume that for some  $\lambda_\pm \in \mathbb{C}^\pm$  the condition  $H^{1/2}(\partial\Omega) \subset \text{ran}(\Theta - M(\lambda_\pm))$  holds. Then*

$$A_\Theta f = \mathcal{L}f, \quad \text{dom } A_\Theta = \left\{f \in H^2(\Omega) \mid \begin{pmatrix} \Gamma_0 f \\ \Gamma_1 f \end{pmatrix} \in \Theta, \hat{f} = \begin{pmatrix} f \\ T f \end{pmatrix}\right\} \quad (4.5)$$

*is a self-adjoint extension of  $A$  in  $L^2(\Omega)$ .*

*Proof.* By Proposition 2.4 (i) the operator  $A_\Theta$  in assertion (ii) is symmetric. The assumption  $H^{1/2}(\partial\Omega) \subset \text{ran}(\Theta - M(\lambda_\pm))$  and Theorem 2.8 (ii) imply that  $\text{ran}(A_\Theta - \lambda_\pm) = L^2(\Omega)$  for some  $\lambda_\pm \in \mathbb{C}^\pm$ . Hence  $A_\Theta$  is a closed symmetric operator and  $A_\Theta - \lambda_\pm$  is surjective, that is,  $A_\Theta$  is self-adjoint.  $\square$

For completeness we state a variant of Proposition 4.3 for maximal dissipative and maximal accumulative extensions  $A_\Theta$ .

**Proposition 4.4** *Let  $T = \mathcal{L} \upharpoonright H^2(\Omega)$ ,  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  and  $M$  be as above, let  $\Theta$  be a dissipative (accumulative) relation in  $L^2(\partial\Omega)$  and assume that for some  $\lambda_- \in \mathbb{C}^-$  ( $\lambda_+ \in \mathbb{C}^+$ , respectively) the condition  $H^{1/2}(\partial\Omega) \subset \text{ran}(\Theta - M(\lambda_-))$  ( $H^{1/2}(\partial\Omega) \subset \text{ran}(\Theta - M(\lambda_+))$ , respectively) holds. Then the operator  $A_\Theta$  in (4.5) is a maximal dissipative (maximal accumulative, respectively) extension of  $A$  in  $L^2(\Omega)$ .*

Let again  $T = \mathcal{L} \upharpoonright H^2(\Omega)$  and  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  be the quasi boundary triple from above with corresponding Weyl function  $M$ . We note that in general the extensions

$$A_\Theta = \left\{ \hat{f} \in T \mid \Gamma \hat{f} \in \Theta \right\} = \Gamma^{-1}(\Theta \cap \text{ran } \Gamma)$$

corresponding to a linear relation  $\Theta$  in  $L^2(\partial\Omega)$  are not closed. Let us consider the simple case  $n = 2$ ,  $\mathcal{L} = -\Delta$  and  $\Omega = \mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ . Then the Krein-von-Neumann extension, sometimes also called the “soft” extension of  $A$  is the self-adjoint operator  $\tilde{A}$  given by

$$\tilde{A}f = -\Delta f, \quad \text{dom } \tilde{A} = H_0^2(\mathbb{D}) \dot{+} \left\{ f \in L^2(\mathbb{D}) \mid \Delta f = 0 \text{ on } \mathbb{D} \right\}.$$

Obviously 0 belongs to the essential spectrum  $\sigma_{\text{ess}}(\tilde{A})$  of  $\tilde{A}$  and all harmonic functions on  $\mathbb{D}$  that belong to  $L^2(\mathbb{D})$  are in  $\text{dom } \tilde{A}$  and hence do not possess boundary values belonging to  $L^2(\partial\mathbb{D})$  in general. See [26,27] for a characterization of the Krein-von-Neumann extension and its spectral asymptotics in a general setting and [21,23] where  $\tilde{A}$  is called the harmonic Laplacian. In view of [26, III. Theorem 1.2 (iii)] the next statement is not surprising.

**Proposition 4.5** *Let  $\{L^2(\partial\mathbb{D}), \Gamma_0, \Gamma_1\}$  be the quasi boundary triple for  $A^*$  defined above and let  $M$  be the corresponding Weyl function. Then  $M$  is holomorphic in a neighbourhood of 0 and the extension  $A_\Theta$  corresponding to the parameter  $\Theta := M(0)$  satisfies  $\overline{A_\Theta} = \tilde{A}$ .*

*Proof.* It is well known that the spectrum of  $A_{\mathbb{D}} = \ker \Gamma_0$  consists of point spectrum only and the maximum principle for harmonic functions implies  $0 \notin \sigma_p(A_{\mathbb{D}})$ . Hence the Weyl function  $M$  is holomorphic in a neighbourhood of 0, see Proposition 2.6. The operator  $A_\Theta = A_{M(0)}$  is the restriction of the maximal operator  $A^* = -\Delta$  to

$$\text{dom } A_\Theta = \left\{ f \in H^2(\mathbb{D}) \mid \hat{f} = \begin{pmatrix} f \\ T f \end{pmatrix}, \Gamma_1 \hat{f} = M(0) \Gamma_0 \hat{f} \right\},$$

where  $T = A^* \upharpoonright H^2(\mathbb{D})$ . From the decomposition  $T = A_0 \dot{+} \hat{\mathcal{N}}_{0,T}$  (see (2.4)) we obtain

$$\text{dom } A_\Theta = H_0^2(\mathbb{D}) \dot{+} \{f \in H^2(\mathbb{D}) \mid \Delta f = 0\}.$$

Therefore  $A_\Theta \subset \tilde{A}$  and as  $\tilde{A}$  is closed, also  $\overline{A_\Theta} \subset \tilde{A}$  holds. In order to show  $\tilde{A} \subset \overline{A_\Theta}$  it remains to verify that the set  $\{f \in H^2(\mathbb{D}) \mid \Delta f = 0\}$  is dense in  $\{f \in L^2(\mathbb{D}) \mid \Delta f = 0\}$  with respect to the graph norm of  $A^*$ . On the latter space the graph norm of  $A^*$  coincides with the  $L^2$  norm. Since every harmonic function can be written as a sum of an analytic and an anti-analytic function, it is sufficient to show that  $\{f \in H^2(\mathbb{D}) \mid f \text{ analytic}\}$  is dense in the Bergman space  $B^2(\mathbb{D}) := \{f \in L^2(\mathbb{D}) \mid f \text{ analytic}\}$ . Because for  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  the  $L^2$  norm is given by  $\|f\|^2 = \pi \sum_{n=0}^{\infty} |c_n|^2 / (n+1)$  (as one can show easily), it is clear that the set of polynomials in  $z$  is dense in  $B^2(\mathbb{D})$ . Since polynomials are clearly in  $H^2(\mathbb{D})$ , the assertion of the proposition follows.  $\square$

#### 4.2 Boundary mappings defined on a Beals space

In this subsection we consider the same differential expression  $\mathcal{L}$  as in the previous subsection, but we define boundary mappings on a larger domain than  $H^2(\Omega)$ . This space defined below was introduced by Bade, Freeman and Beals, cf. [5,8,24], and recently considered in [3]. Let  $n(x)$  be the outward normal vector at  $x \in \partial\Omega$ . Since the boundary  $\partial\Omega$  is  $C^\infty$ , there exists an  $\varepsilon_0 > 0$  such that for  $0 \leq \varepsilon \leq \varepsilon_0$  the map  $x \mapsto x - \varepsilon n(x)$  is a homeomorphism from  $\partial\Omega$  onto the set  $\{x - \varepsilon n(x) \mid x \in \partial\Omega\}$ , cf., e.g. [5, Theorem 2.1].

For  $f \in H_{\text{loc}}^2(\Omega)$  define  $f_\varepsilon(x) := f(x - \varepsilon n(x))$  for  $x \in \partial\Omega$ ; then  $f_\varepsilon \in L^2(\partial\Omega)$ . We say that  $f$  has  $L^2$  boundary value on  $\partial\Omega$  if  $\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon$  exists as a limit in  $L^2(\partial\Omega)$ . In this case we write  $f|_{\partial\Omega} := \lim_{\varepsilon \rightarrow 0^+} f_\varepsilon$ .

The Beals space  $\mathcal{D}_1(\Omega)$  is now defined by

$$\mathcal{D}_1(\Omega) := \left\{ f \in L^2(\Omega) \mid \mathcal{L}f \in L^2(\Omega); f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m} \text{ have } L^2 \text{ boundary values on } \partial\Omega \right\}.$$

Note that  $f \in L^2(\Omega)$  and  $\mathcal{L}f \in L^2(\Omega)$  imply  $f \in H_{\text{loc}}^2(\Omega)$ . On  $\mathcal{D}_1(\Omega)$  the following boundary mappings are well defined:

$$\begin{aligned} \Gamma_0 \hat{f} &:= \frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} := \sum_{j,k=1}^m a_{jk} n_j \frac{\partial f}{\partial x_k} \Big|_{\partial\Omega} + \sum_{j=1}^m \bar{a}_j n_j f \Big|_{\partial\Omega}, \\ \Gamma_1 \hat{f} &:= f \Big|_{\partial\Omega} \end{aligned}$$

for  $\hat{f} = \begin{pmatrix} f \\ \mathcal{L}f \end{pmatrix}$ . Note that we have  $\Gamma_0, \Gamma_1$  as in Proposition 4.2 (iii). On the smaller domain

$$\Omega_\varepsilon := \Omega \setminus \{x - \varepsilon' n(x) \mid x \in \partial\Omega, 0 < \varepsilon' \leq \varepsilon\}$$

Green's identity

$$(\mathcal{L}f, g)_{\Omega_\varepsilon} - (f, \mathcal{L}g)_{\Omega_\varepsilon} = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\partial\Omega_\varepsilon} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\partial\Omega_\varepsilon}$$

holds for  $f, g \in \mathcal{D}_1(\Omega)$ ,  $\hat{f} = \begin{pmatrix} f \\ \mathcal{L}f \end{pmatrix}$ ,  $\hat{g} = \begin{pmatrix} g \\ \mathcal{L}g \end{pmatrix}$  since such  $f$  and  $g$  are in  $H^2(\Omega_\varepsilon)$ . By letting  $\varepsilon \rightarrow 0$ , it follows that Green's identity is also true on  $\Omega$ , i.e.,

$$(\mathcal{L}f, g)_\Omega - (f, \mathcal{L}g)_\Omega = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\partial\Omega} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\partial\Omega}$$

for all  $f, g \in \mathcal{D}_1(\Omega)$ . In [8] it was shown that  $H^2(\Omega) \subset \mathcal{D}_1(\Omega) \subset H^{3/2}(\Omega)$  and

$$\text{ran } \Gamma_0 = L^2(\partial\Omega), \quad \text{ran } \Gamma_1 = H^1(\partial\Omega), \quad (4.6)$$

cf. [8, Lemma 4.1, Lemma 4.4 and Corollary to Theorem 4.1]. As  $\Gamma$  is an extension of the boundary mappings in Proposition 4.2 (iii),  $\text{ran } \Gamma$  is dense in  $\mathcal{G} \times \mathcal{G} = L^2(\partial\Omega) \times L^2(\partial\Omega)$ .

**Proposition 4.6** *Let  $T$  be the restriction of  $\mathcal{L}$  to  $\mathcal{D}_1(\Omega)$  and  $\Gamma_0, \Gamma_1$  be as above, defined on  $\text{dom } \Gamma = T = \mathcal{L} \upharpoonright \mathcal{D}_1(\Omega)$ . Then  $A := \ker \Gamma$  is the same closed symmetric operator as in (4.3) and the quasi boundary triple  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  is even a generalized boundary triple for  $A^*$  with  $\text{dom } \Gamma = \mathcal{L} \upharpoonright \mathcal{D}_1(\Omega)$ . Moreover,  $\ker \Gamma_0 = A_N$  and  $\ker \Gamma_1 = A_D$ .*

*Let  $\gamma$  be the  $\gamma$ -field and  $M$  be the Weyl function of  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ ; then  $\gamma(\lambda)$ ,  $\gamma(\lambda)^*$  and  $M(\lambda)$  are compact operators for all  $\lambda \in \rho(A_N)$ .*

*Proof.* Let  $A := \ker \Gamma$ . Since  $\ker \Gamma_0$  contains the Neumann operator  $A_N$ , which is self-adjoint, we can apply Theorem 2.3, which shows that  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  is a quasi boundary triple. From  $\text{ran } \Gamma_0 = L^2(\partial\Omega)$  and Corollary 3.7 we conclude that  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  is even a generalized boundary triple for  $A^*$ . In particular  $\ker \Gamma_0$  is self-adjoint and hence coincides with  $A_N$ ;  $\ker \Gamma_1$  is a symmetric extension of  $A_D$  and hence equal to  $A_D$ . Since  $\ker \Gamma$  is a symmetric extension of the closed symmetric operator (4.3) and both operators have the same adjoint, we conclude that  $\ker \Gamma$  and the operator in (4.3) coincide.

Relations (4.6) and Corollary 3.7 imply that  $M(\lambda)$  is an operator from  $L^2(\partial\Omega)$  into  $H^1(\partial\Omega)$ . Because of Proposition 2.6 (v),  $M(\lambda)$  is closed as an operator from  $L^2(\partial\Omega)$  to  $L^2(\partial\Omega)$ . But then it is also closed from  $L^2(\partial\Omega)$  to  $H^1(\partial\Omega)$ ; hence by the closed graph theorem bounded. Since  $H^1(\partial\Omega)$  is compactly embedded in  $L^2(\partial\Omega)$ , cf. [1], the operator  $M(\lambda)$  is a compact operator in  $L^2(\partial\Omega)$ . A similar argument shows that  $\gamma(\bar{\lambda})^*$  is compact from  $L^2(\Omega)$  to  $L^2(\partial\Omega)$ ; hence also  $\gamma(\lambda)$  is compact.  $\square$

**Remark 4.7** We note that here the condition  $\{0\} \times \mathcal{G}_1 \subset \text{ran } \Gamma$  in Theorem 2.8 (iii) is not satisfied. For otherwise, for every  $h \in \mathcal{G}_1 = H^1(\partial\Omega)$  one could find an  $\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in T$  (so  $f \in \mathcal{D}_1(\Omega)$ ) with  $\Gamma_0 \hat{f} = 0$  and  $\Gamma_1 \hat{f} = h$ . The former relation implies that  $f \in \text{dom } A_N \subset H^2(\Omega)$ . But then  $\Gamma_1 \hat{f} \in H^{3/2}(\partial\Omega)$ , a contradiction. This also shows that  $\text{ran } \Gamma \neq \mathcal{G}_0 \times \mathcal{G}_1$ .

In the next theorem we give a sufficient condition for the relations  $\Theta$  in  $\mathcal{G}$  such that the corresponding extension  $A_\Theta$  of  $A$  via (2.3) is self-adjoint, see also [26, III. Theorem 4.1].

**Theorem 4.8** Let  $T = \mathcal{L} \upharpoonright \mathcal{D}_1(\Omega)$  and  $A, A^*$  be as above. Let  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  be the quasi boundary triple from Proposition 4.6 and denote by  $\gamma$  and  $M$  the corresponding  $\gamma$ -field and Weyl function. Let  $\Theta$  be a self-adjoint relation in  $L^2(\partial\Omega)$  such that  $0 \notin \sigma_{\text{ess}}(\Theta)$ . Then

$$\begin{aligned} A_\Theta &= \mathcal{L} \upharpoonright \text{dom } A_\Theta = T \upharpoonright \text{dom } A_\Theta, \\ \text{dom } A_\Theta &= \left\{ f \in \mathcal{D}_1(\Omega) \mid \begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix} \in \Theta, \hat{f} = \begin{pmatrix} f \\ Tf \end{pmatrix} \right\}, \end{aligned}$$

is a self-adjoint extension of  $A$  in  $L^2(\Omega)$  and  $A_\Theta$  has a compact resolvent,

$$(A_\Theta - \lambda)^{-1} = (A_N - \lambda)^{-1} + \gamma(\lambda) \left( \Theta - M(\lambda) \right)^{-1} \gamma(\bar{\lambda})^*, \quad (4.7)$$

$\lambda \in \rho(A_\Theta) \cap \rho(A_N)$ . A point  $\lambda \in \rho(A_N)$  is an eigenvalue of  $A_\Theta$  if and only if  $\Theta - M(\lambda)$  is not injective.

*Proof.* It follows from Proposition 2.4 (i) that  $A_\Theta$  is symmetric. We have to show that  $A_\Theta - \lambda_\pm$  is surjective for some  $\lambda_+ \in \mathbb{C}^+$  and  $\lambda_- \in \mathbb{C}^-$ . By Theorem 2.8 (ii) it is sufficient to show that  $\Theta - M(\lambda_\pm)$  is bijective for some  $\lambda_\pm \in \mathbb{C}^\pm$ .

Let us decompose the self-adjoint relation  $\Theta \in \tilde{\mathcal{C}}(\mathcal{G})$  into its self-adjoint operator part and the purely multivalued part:  $\Theta = \Theta_{\text{op}} \oplus \Theta_\infty$  with a corresponding decomposition of the space  $\mathcal{G} = \mathcal{G}_{\text{op}} \oplus \mathcal{G}_\infty$ , cf. Section 2.1. Denote by  $P_{\text{op}}$  the orthogonal projection onto  $\mathcal{G}_{\text{op}}$ . Since  $0 \notin \sigma_{\text{ess}}(\Theta_{\text{op}})$  and  $M(\lambda)$  is compact for  $\lambda \in \rho(A_N)$ , the operator  $\Theta_{\text{op}} - P_{\text{op}}M(\lambda)P_{\text{op}}$  is a Fredholm operator with index 0. From

$$\begin{aligned} \text{Im} \left( (\Theta_{\text{op}} - P_{\text{op}}M(\lambda)P_{\text{op}})x, x \right)_{\mathcal{G}_{\text{op}}} &= -\text{Im}(M(\lambda)x, x) < 0, \\ x &\in \mathcal{G}_{\text{op}}, x \neq 0, \lambda \in \mathbb{C}^+, \end{aligned}$$

it follows that  $\Theta_{\text{op}} - P_{\text{op}}M(\lambda)P_{\text{op}}$  has a trivial kernel for all  $\lambda \in \mathbb{C}^+$  and similarly for  $\lambda \in \mathbb{C}^-$ . But then it is boundedly invertible in  $\mathcal{G}_{\text{op}}$ . By [29, p. 137] we have

$$\left( \Theta - M(\lambda) \right)^{-1} = \left( \Theta_{\text{op}} - P_{\text{op}}M(\lambda)P_{\text{op}} \right)^{-1} P_{\text{op}}$$

and hence  $\Theta - M(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , is boundedly invertible with an everywhere defined inverse. In particular  $\Theta - M(\lambda_{\pm})$  is bijective and therefore  $A_{\Theta}$  is self-adjoint.

If  $\Theta = 0$  and  $\lambda \in \rho(A_N) \cap \rho(A_D)$ , then by Lemma 2.6 (iii) we have  $\text{ran } M(\lambda) = \mathcal{G}_1 = H^1(\partial\Omega)$ , and for  $\lambda \in \rho(A_N) \cap \rho(A_D)$  Theorem 2.8 (i) implies that the operator  $M(\lambda)$  is injective. Hence by Theorem 2.8 (ii)

$$(A_D - \lambda)^{-1} = (A_N - \lambda)^{-1} - \gamma(\lambda)M(\lambda)^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \rho(A_N) \cap \rho(A_D).$$

It is well known that  $(A_D - \lambda)^{-1}$  is compact. Moreover, since  $M(\lambda)$  is closed the operator  $M(\lambda)^{-1}\gamma(\bar{\lambda})^*$  is closed and everywhere defined, hence bounded. The compactness of  $\gamma(\lambda)$  (see Proposition 4.6) yields the compactness of the resolvent of  $A_N$ . If now  $\Theta$  is self-adjoint in  $L^2(\partial\Omega)$  and  $0 \notin \sigma_{\text{ess}}(\Theta)$ , then  $(\Theta - M(\lambda))^{-1}$  is bounded and again Krein's formula,

$$(A_{\Theta} - \lambda)^{-1} = (A_N - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*,$$

$\lambda \in \rho(A_{\Theta}) \cap \rho(A_N)$ , and the compactness of  $\gamma(\lambda)$  or  $\gamma(\bar{\lambda})^*$  give the compactness of the resolvent of  $A_{\Theta}$ .  $\square$

**Corollary 4.9** *Let  $\Theta$  be a self-adjoint relation in  $L^2(\partial\Omega)$  with  $0 \notin \sigma_{\text{ess}}(\Theta)$ . Then for all  $g \in L^2(\Omega)$  and all  $\lambda \in \mathbb{C} \setminus \sigma(A_{\Theta})$ , where  $\sigma(A_{\Theta})$  is a discrete subset of  $\mathbb{R}$  which has no finite accumulation points, the unique solution  $f \in \mathcal{D}_1(\Omega)$  of the boundary value problem*

$$\mathcal{L}f - \lambda f = g, \quad \begin{pmatrix} \frac{\partial f}{\partial \nu} |_{\partial\Omega} \\ f |_{\partial\Omega} \end{pmatrix} \in \Theta,$$

is given by  $f = (A_{\Theta} - \lambda)^{-1}g$ . If, in addition  $\lambda \in \rho(A_N)$ , then

$$f = (A_{\Theta} - \lambda)^{-1}g = (A_N - \lambda)^{-1}g + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*g.$$

A variant of Theorem 4.8 for maximal dissipative and maximal accumulative extensions  $A_{\Theta}$  of  $A$  reads as follows. We leave it to the reader to formulate a version of Corollary 4.9 for this case.

**Theorem 4.10** *Let  $T = \mathcal{L} \upharpoonright \mathcal{D}_1(\Omega)$  and  $A, A^*$  be as above. Moreover, let  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  be the quasi boundary triple from Proposition 4.6 and denote by  $\gamma$  and  $M$  the corresponding  $\gamma$ -field and Weyl function. Let  $\Theta$  be a maximal dissipative (maximal accumulative) relation in  $L^2(\partial\Omega)$  such that  $0 \notin \sigma_{\text{ess}}(\Theta)$ . Then*

$$A_{\Theta} = \mathcal{L} \upharpoonright \text{dom } A_{\Theta} = T \upharpoonright \text{dom } A_{\Theta}, \\ \text{dom } A_{\Theta} = \left\{ f \in \mathcal{D}_1(\Omega) \mid \begin{pmatrix} \Gamma_0 f \\ \Gamma_1 f \end{pmatrix} \in \Theta, \hat{f} = \begin{pmatrix} f \\ T f \end{pmatrix} \right\},$$

is a maximal dissipative (maximal accumulative, respectively) extension of  $A$  in  $L^2(\Omega)$  and  $A_\Theta$  has a compact resolvent given by (4.7). A point  $\lambda \in \rho(A_N)$  is an eigenvalue of  $A_\Theta$  if and only if  $\Theta - M(\lambda)$  is not injective.

The boundary condition in Theorem 4.8 and Corollary 4.9 (and Theorem 4.10) can be written more explicitly. Let  $\Theta$  be as in Theorem 4.8 and let  $P_1$  be the orthogonal projection onto  $\ker \Theta$  and set  $P_2 := 1 - P_1$ . Since  $0 \notin \sigma_{\text{ess}}(\Theta)$ ,  $P_1$  has finite rank and the restriction of  $\Theta$  to  $\text{ran } P_2$  is boundedly invertible; denote the inverse of this restriction by  $B$ . With these notations we can write  $\begin{pmatrix} \Gamma_0 \hat{f} \\ \Gamma_1 \hat{f} \end{pmatrix} \in \Theta$  as

$$P_1(f|_{\partial\Omega}) = 0, \quad P_2\left(\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}\right) - P_2 B P_2(f|_{\partial\Omega}) = 0.$$

Vice versa, if  $B$  is a bounded self-adjoint operator in  $L^2(\partial\Omega)$ ,  $P_1$  an orthogonal projection of finite rank and  $P_2 = 1 - P_1$ , then this gives rise to a  $\Theta$  like in Theorem 4.8. Note that  $B$  can have an arbitrarily large kernel. The case  $P_1 = 0$  was considered in [8].

In the next proposition we show that self-adjoint relations  $\Theta$  which do not satisfy the condition  $0 \notin \sigma_{\text{ess}}(\Theta)$  in general do not yield self-adjoint or essentially self-adjoint extensions of  $A$ . Moreover, the general characterization of the self-adjoint extensions in [26, III. Theorem 4.1] suggests that also the self-adjoint extensions obtained in Theorem 4.8 are in general not defined on subspaces of  $H^2(\Omega)$ .

**Proposition 4.11** *Let  $T = \mathcal{L} \upharpoonright \mathcal{D}_1(\Omega)$ ,  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  and  $A$  be as above. Then the following assertions are true.*

- (i) *There exists a self-adjoint relation  $\Theta$  in  $L^2(\partial\Omega)$  such that  $A_\Theta$  is not essentially self-adjoint.*
- (ii) *There exists a self-adjoint relation  $\Theta$  in  $L^2(\partial\Omega)$  such that  $A_\Theta$  is self-adjoint and  $\text{dom } A_\Theta$  is not contained in  $H^2(\Omega)$ .*

*Proof.* (i) Take an element  $h \in L^2(\partial\Omega) \setminus \mathcal{G}_1$  and define  $\Theta g = (g, h)h$ . If  $\hat{f} \in A_\Theta$ , then we have  $\Gamma_1 \hat{f} \in \text{ran } \Theta = \text{span}\{h\}$ . Hence  $\Gamma_1 \hat{f} = 0$  and  $\Gamma_0 \hat{f} \in \ker \Theta$ , i.e.,  $(\Gamma_0 \hat{f}, h) = 0$ . Therefore  $A_\Theta \subset A_D = \ker \Gamma_1$  and it follows that  $\hat{f} \in A_D$  belongs to  $A_\Theta$  if and only if  $(\Gamma_0 \hat{f}, h) = 0$ . The functional  $A_D \ni \hat{f} \mapsto (\Gamma_0 \hat{f}, h)$  is bounded on  $A_D$  since  $\Gamma_0$  is bounded from  $H^2(\Omega)$  to  $L^2(\partial\Omega)$  and on  $\text{dom } A_D$  the graph norm and the  $H^2$  norm are equivalent. Hence the operator  $A_\Theta$  is a closed symmetric operator with defect  $(1, 1)$ , i.e., not essentially self-adjoint.

(ii) Let  $h \in L^2(\partial\Omega) \setminus H^{1/2}(\partial\Omega)$  and define  $\Theta$  by  $\Theta^{-1}g = (g, h)h$ . Then  $\Theta$  is a self-adjoint relation with  $0 \notin \sigma_{\text{ess}}(\Theta)$ ; hence  $A_\Theta$  is a self-adjoint extension of  $A$ . If  $\hat{f} \in A_\Theta$ , then  $\Gamma_0 \hat{f} = (\Gamma_1 \hat{f}, h)h$ . Suppose that  $\Gamma_0 \hat{f} = 0$  for all  $\hat{f} \in A_\Theta$ . Then  $\hat{f} \in A_N$ , and since  $A_\Theta$  is self-adjoint, we would have  $A_\Theta = A_N$ . Since

$\{\Gamma_1 \hat{f} \mid \hat{f} \in A_N\} = H^{3/2}(\partial\Omega)$ , there exists an  $\hat{f} \in A_\Theta$  such that  $(\Gamma_1 \hat{f}, h) \neq 0$ , which is a contradiction to  $\Gamma_0 \hat{f} = (\Gamma_1 \hat{f}, h)h$ . Hence there exists an element  $\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in A_\Theta$  such that  $\Gamma_0 \hat{f}$  is a non-zero multiple of  $h \notin H^{1/2}(\partial\Omega)$ . This implies that  $f \notin H^2(\Omega)$ .  $\square$

## 5 Elliptic boundary value problems with eigenvalue dependent boundary conditions

### 5.1 A general theorem on $\lambda$ -dependent boundary value problems

In the next theorem we investigate a class of abstract  $\lambda$ -dependent boundary value problems. We generalize the coupling method from [17] to the case of a closed symmetric relation  $A$  and a quasi boundary triple  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  for  $A^*$ . The proof is similar to the proof of [9, Theorem 4.1]. For a Nevanlinna function  $\tau$  we denote by  $\mathfrak{h}(\tau)$  the union of  $\mathbb{C} \setminus \mathbb{R}$  and the set of real points into which  $\tau$  can be continued analytically such that the continuations of the upper and lower half planes coincide.

**Theorem 5.1** *Let  $A$  be a closed symmetric relation in  $\mathcal{H}$  and let  $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$  be a quasi boundary triple for  $A^*$ ,  $A_0 = \ker \Gamma_0$ , with corresponding  $\gamma$ -field  $\gamma$  and Weyl function  $M$ . Let  $\tau$  be an  $\mathcal{L}(\mathcal{G})$ -valued Nevanlinna function such that  $0 \in \rho(\text{Im } \tau(\lambda_0))$  for some (and hence for all)  $\lambda_0 \in \mathbb{C}^+$  and choose a Hilbert space  $\mathcal{K}$ , a symmetric operator  $S$  in  $\mathcal{K}$  and an ordinary boundary triple  $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$  for  $S^*$  such that  $\tau$  is the corresponding Weyl function (see Theorem 3.4).*

If  $\text{ran}(M(\lambda_\pm) + \tau(\lambda_\pm)) = \mathcal{G}$  for some  $\lambda_\pm \in \mathbb{C}^\pm$ , then

$$\tilde{A} = \left\{ \{ \hat{f}, \hat{k} \} \in \text{dom } \Gamma \times S^* \mid \Gamma_0 \hat{f} - \Gamma'_0 \hat{k} = 0, \Gamma_1 \hat{f} + \Gamma'_1 \hat{k} = 0 \right\} \quad (5.1)$$

is a self-adjoint extension of  $A$  in  $\mathcal{H} \times \mathcal{K}$  such that for all  $\lambda \in \rho(\tilde{A}) \cap \mathfrak{h}(\tau)$  a solution of the boundary value problem

$$f' - \lambda f = g, \quad \tau(\lambda) \Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in \text{dom } \Gamma \subset A^*, \quad (5.2)$$

is given by

$$f = P_{\mathcal{H}}(\tilde{A} - \lambda)^{-1}|_{\mathcal{H}} g, \quad f' = g + \lambda f. \quad (5.3)$$

If  $\lambda \in \rho(A_0) \cap \mathfrak{h}(\tau)$ ,  $M(\lambda) + \tau(\lambda)$  is injective and  $\text{ran}(M(\lambda) + \tau(\lambda)) = \mathcal{G}$ , then

$$P_{\mathcal{H}}(\tilde{A} - \lambda)^{-1}|_{\mathcal{H}} = (A_0 - \lambda)^{-1} - \gamma(\lambda) \left( M(\lambda) + \tau(\lambda) \right)^{-1} \gamma(\bar{\lambda})^* \quad (5.4)$$

and (5.3) is the unique solution of (5.2).

*Proof.* Let us denote the self-adjoint relation  $\ker \Gamma'_0$  by  $S_0$  and the  $\gamma$ -field corresponding to the ordinary boundary triple  $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$  by  $\gamma'$ . It is obvious that  $\{\mathcal{G} \times \mathcal{G}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ , where  $\tilde{\Gamma}_0 := (\Gamma_0, \Gamma'_0)^\top$  and  $\tilde{\Gamma}_1 := (\Gamma_1, \Gamma'_1)^\top$ , is a quasi boundary triple for  $A^* \times S^*$  with  $\text{dom } \tilde{\Gamma} = \text{dom } \Gamma \times S^*$ ,  $\overline{\text{dom } \tilde{\Gamma}} = A^*$ . The  $\gamma$ -field  $\tilde{\gamma}$  and the Weyl function  $\tilde{M}$  corresponding to  $\{\mathcal{G} \times \mathcal{G}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$  are given by

$$\lambda \mapsto \tilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma'(\lambda) \end{pmatrix} \quad \text{and} \quad \lambda \mapsto \tilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix},$$

$\lambda \in \rho(A_0) \cap \rho(S_0)$ . Note that  $\tilde{\gamma}$  and  $\tilde{M}$  are defined on  $\text{ran } \Gamma_0 \times \mathcal{G}$ . By Proposition 2.4 (i) the extension  $\tilde{A}$  of  $A \times S$  defined in (2.3) corresponding to the self-adjoint relation

$$\tilde{\Theta} = \left\{ \left( \begin{pmatrix} (x, x)^\top \\ (y, -y)^\top \end{pmatrix} \mid x, y \in \mathcal{G} \right) \right\}$$

in  $\mathcal{G} \times \mathcal{G}$  is symmetric in  $\mathcal{H} \times \mathcal{K}$ . An element  $\{\hat{f}, \hat{k}\}$ ,  $\hat{f} \in \text{dom } \Gamma$ ,  $\hat{k} \in S^*$ , belongs to  $\tilde{A}$  if and only if

$$\Gamma_0 \hat{f} = \Gamma'_0 \hat{k} \quad \text{and} \quad \Gamma_1 \hat{f} = -\Gamma'_1 \hat{k} \quad (5.5)$$

and hence  $\tilde{A}$  has the form (5.1).

Let now  $\lambda_\pm \in \mathbb{C}^\pm$  be such that  $M(\lambda_\pm) + \tau(\lambda_\pm)$  is a surjective operator in  $\mathcal{G}$ . We claim that  $\tilde{M}(\lambda_+) - \tilde{\Theta}$  is injective and that  $\text{ran}(\tilde{M}(\lambda_+) - \tilde{\Theta}) = \mathcal{G} \times \mathcal{G}$  holds. In fact, since  $\text{Im}(M(\lambda_+) + \tau(\lambda_+))$  is uniformly positive, we find that  $M(\lambda_+) + \tau(\lambda_+)$  is injective and therefore

$$\tilde{M}(\lambda_+) - \tilde{\Theta} = \left\{ \left( \begin{pmatrix} (x, x)^\top \\ (M(\lambda_+)x - y, \tau(\lambda_+)x + y)^\top \end{pmatrix} \mid x \in \text{ran } \Gamma_0, y \in \mathcal{G} \right) \right\}$$

is injective. Similarly, the assumption  $\text{ran}(M(\lambda_+) + \tau(\lambda_+)) = \mathcal{G}$  implies that  $\text{ran}(\tilde{M}(\lambda_+) - \tilde{\Theta}) = \mathcal{G} \times \mathcal{G}$  holds. Analogous considerations hold for  $\lambda_- \in \mathbb{C}^-$ . Hence by Theorem 2.8 (ii)  $\tilde{A} - \lambda_\pm$  is bijective and from  $\tilde{A} \subset \tilde{A}^*$  we conclude that  $\tilde{A}$  is a self-adjoint relation in  $\mathcal{H} \times \mathcal{K}$ .

Let  $\lambda \in \rho(\tilde{A})$ . We show that  $f := P_{\mathcal{H}}(\tilde{A} - \lambda)^{-1}|_{\mathcal{H}} g$ ,  $f' := g + \lambda f$  is a solution of (5.2). Indeed, if  $k := P_{\mathcal{K}}(\tilde{A} - \lambda)^{-1}|_{\mathcal{H}} g$ , then  $\begin{pmatrix} f \\ k \end{pmatrix} = (\tilde{A} - \lambda)^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix}$  and hence

$$\begin{pmatrix} (f, k)^\top \\ (g + \lambda f, \lambda k)^\top \end{pmatrix} \in \tilde{A} \subset \text{dom } \Gamma \times S^*,$$

where

$$\hat{f} := \begin{pmatrix} f \\ g + \lambda f \end{pmatrix} \in \text{dom } \Gamma \quad \text{and} \quad \hat{k} := \begin{pmatrix} k \\ \lambda k \end{pmatrix} \in S^*.$$

By  $k \in \ker(S^* - \lambda)$  we have  $\tau(\lambda)\Gamma'_0\hat{k} = \Gamma'_1\hat{k}$  and therefore (5.5) implies

$$\tau(\lambda)\Gamma_0\hat{f} = \tau(\lambda)\Gamma'_0\hat{k} = \Gamma'_1\hat{k} = -\Gamma_1\hat{f},$$

that is,  $\hat{f} \in \text{dom } \Gamma$  is a solution of the boundary value problem (5.2).

For all  $\lambda \in \mathbb{C}$  where  $M(\lambda) + \tau(\lambda)$  is injective and  $\text{ran}(M(\lambda) + \tau(\lambda)) = \mathcal{G}$  holds we have

$$\left(\widetilde{M}(\lambda) - \widetilde{\Theta}\right)^{-1} = \begin{pmatrix} (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \\ (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \end{pmatrix}$$

and it follows from Theorem 2.8 applied to  $A_0 \times S_0$  and  $\widetilde{A}$  that for every  $\lambda \in \rho(A_0) \cap \mathfrak{h}(\tau)$  the compressed resolvent  $P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{H}}$  has the form (5.4). The uniqueness of the solution follows from  $\ker(M(\lambda) + \tau(\lambda)) = \{0\}$ . In fact, if  $\hat{l} \in \text{dom } \Gamma$  is also a solution of (5.2), then  $\hat{f} - \hat{l}$  belongs to  $\widehat{\mathcal{N}}_{\lambda, T}$  and we have

$$\left(\tau(\lambda) + M(\lambda)\right)\Gamma_0(\hat{f} - \hat{l}) = \tau(\lambda)\Gamma_0(\hat{f} - \hat{l}) + \Gamma_1(\hat{f} - \hat{l}) = 0.$$

Hence  $\hat{f} - \hat{l} \in A_0 \cap \widehat{\mathcal{N}}_{\lambda, T}$  and from  $\lambda \in \rho(A_0)$  we conclude  $\hat{f} = \hat{l}$ .  $\square$

## 5.2 Elliptic boundary value problems with $\lambda$ -dependent boundary conditions

Let  $\mathcal{L}$  be the differential expression from (4.1) and let  $\Omega$  be a bounded  $C^\infty$ -domain as in Section 4. In this section we consider boundary value problems with  $\lambda$ -dependent boundary conditions of the following type: for a given function  $g \in L^2(\Omega)$  find a function  $f$  in the Beals space  $\mathcal{D}_1(\Omega)$  (cf. Section 4.2) such that

$$(\mathcal{L} - \lambda)f = g \quad \text{and} \quad \tau(\lambda) \left( \frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} \right) + f|_{\partial\Omega} = 0 \quad (5.6)$$

holds. Here  $\tau$  is assumed to be an  $\mathcal{L}(L^2(\partial\Omega))$ -valued Nevanlinna function with the additional property  $0 \in \rho(\text{Im } \tau(\lambda_+))$  for some (and hence for all)  $\lambda_+ \in \mathbb{C}^+$ . Let  $T = \mathcal{L} \upharpoonright \mathcal{D}_1(\Omega)$  and let  $A, A^* = \overline{T}$  and  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ , the quasi boundary triple, be as in Section 4.2 with  $A_N = \ker \Gamma_0$ . Then (5.6) can be rewritten in the form

$$(T - \lambda)f = g, \quad \tau(\lambda)\Gamma_0\hat{f} + \Gamma_1\hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ Tf \end{pmatrix}. \quad (5.7)$$

As the Weyl function  $M$  corresponding to  $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$  is an  $\mathcal{L}(L^2(\partial\Omega))$ -valued Nevanlinna function and  $\text{Im } \tau(\lambda_\pm)$ ,  $\lambda_\pm \in \mathbb{C}^\pm$ , is uniformly positive (uniformly negative, respectively), the condition  $\text{ran}(M(\lambda_\pm) + \tau(\lambda_\pm)) = L^2(\partial\Omega)$  from Theorem 5.1 is fulfilled for every  $\lambda_\pm \in \mathbb{C}^\pm$ . Hence we have the following corollary.

**Corollary 5.2** *Let  $S$  be a symmetric operator in some Hilbert space  $\mathcal{K}$  such that  $\tau$  is the Weyl function corresponding to an ordinary boundary triple  $\{L^2(\partial\Omega), \Gamma'_0, \Gamma'_1\}$  for  $S^*$ , cf. Theorem 3.4. Then  $\tilde{A}$  in (5.1) is a self-adjoint extension of  $A$  in  $L^2(\Omega) \times \mathcal{K}$ , and for every  $\lambda \in \rho(A_N) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((M + \tau)^{-1})$  the unique solution of the boundary value problem (5.6) or (5.7) is given by*

$$f = P_{L^2(\Omega)} \left( \tilde{A} - \lambda \right)^{-1} \Big|_{L^2(\Omega)} g = (A_N - \lambda)^{-1} g - \gamma(\lambda) \left( (M(\lambda) + \tau(\lambda))^{-1} \gamma(\bar{\lambda}) \right)^* g.$$

### 5.2.1 $A$ $\lambda$ -linear boundary condition

We assume now that the function  $\tau$  in the boundary condition in (5.6) is given by  $\tau(\lambda) = \lambda \in \mathcal{L}(L^2(\partial\Omega))$ . Let  $S := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$  be the (trivial) linear relation in  $L^2(\partial\Omega)$ . The adjoint  $S^*$  is

$$S^* = \left\{ \begin{pmatrix} x \\ x' \end{pmatrix} \mid x, x' \in L^2(\partial\Omega) \right\} \in \tilde{\mathcal{C}}(L^2(\partial\Omega))$$

and  $\{L^2(\partial\Omega), \Gamma'_0, \Gamma'_1\}$ , where  $\Gamma'_0 \hat{k} := k$  and  $\Gamma'_1 \hat{k} := k'$ ,  $\hat{k} = \begin{pmatrix} k \\ k' \end{pmatrix} \in S^*$ , is an ordinary boundary triple for  $S^*$  with corresponding Weyl function  $\tau$ . Hence the compressed resolvent of

$$\tilde{A} = \left\{ \left\{ \begin{pmatrix} (f, k)^\top \\ (\mathcal{L}f, k')^\top \end{pmatrix} \right\} \in \text{dom } \Gamma \times S^* \mid \begin{pmatrix} \frac{\partial f}{\partial \nu} \\ f \end{pmatrix} \Big|_{\partial\Omega} = k, f|_{\partial\Omega} = -k' \right\}$$

onto  $L^2(\Omega)$  yields the solution of the problem (5.6) with  $\tau(\lambda) = \lambda$ . Here  $\tilde{A}$  is an operator in  $L^2(\Omega) \times L^2(\partial\Omega)$  and can be written in the form

$$\tilde{A} \left\{ f, \frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} \right\} = \{ \mathcal{L}f, -f|_{\partial\Omega} \}, \quad f \in \mathcal{D}_1(\Omega).$$

This coincides with the results from [10,20] where  $\lambda$ -linear boundary value problems were investigated. As a consequence of [10, Lemma 2.2] we obtain the following characterization of the Beals space  $\mathcal{D}_1(\Omega)$ , cf. [8, Chapter 4].

**Corollary 5.3** *The Beals space  $\mathcal{D}_1(\Omega)$  is the completion of  $H^2(\Omega)$  under the norm*

$$\|f\|_{L^2(\Omega)} + \|\mathcal{L}f\|_{L^2(\Omega)} + \left\| \frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} \right\|_{L^2(\partial\Omega)} + \|f|_{\partial\Omega}\|_{L^2(\partial\Omega)}, \quad f \in H^2(\Omega).$$

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