Weak Transient Signal Detection Via a Polynomial Eigenvalue Decomposition

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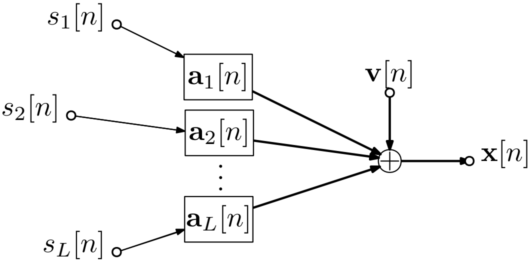
Problem & Model

- A number of broadband stationary sources $s_\ell[n]$, $\ell = 1, \ldots, L$, illuminate an M-element sensor array;
- each transfer path is modelled by a vector of impulse responses $\mathbf{a}_\ell[n] \in \mathbb{C}^M$;
- stationary additive, spatially and temporally uncorrelated noise $\mathbf{v}[n] \in \mathbb{C}^M$;



$$\mathbf{x}[n] = \sum_{\ell=1}^{L} \mathbf{a}_\ell[n] * s_\ell[n] + \mathbf{v}[n]$$

Problem & Model

University of Strathclyde Engineering

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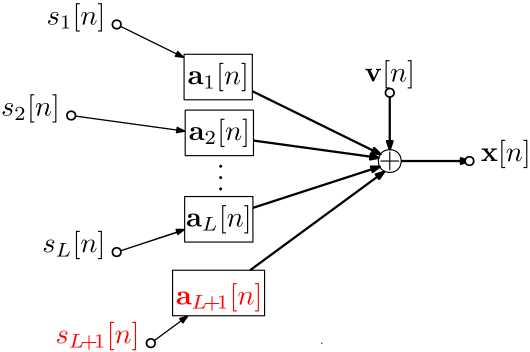
- each transfer path is modelled by a vector of impulse responses $\mathbf{a}_\ell[n] \in \mathbb{C}^M$;
- stationary additive, spatially and temporally uncorrelated noise $\mathbf{v}[n] \in \mathbb{C}^M$;
- a broadband transient signal $s_{L+1}[n]$ enters the scene at some point in time;
- aim: we want to detect the onset of this transient signal, which may be weak in power [10];
- assumption: $M > L$.

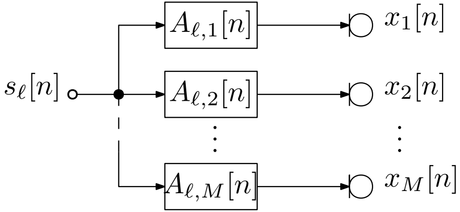


$$\mathbf{x}[n] = \sum_{\ell=1}^{L+1} \mathbf{a}_\ell[n] * s_\ell[n] + \mathbf{v}[n]$$

Model

- Each source, $s_\ell[n]$, contributes to the data vector $\mathbf{x}[n] = [x_1[n], \;\ldots\;, x_M[n]]^{\mathrm{T}}$ via a steering vector $\mathbf{a}_\ell[n] = [A_{\ell,1}[n], \ldots A_{\ell,M}[n]]^{\mathrm{T}}$;
- compact with $\mathbf{A}[n] = [\mathbf{a}_1[n] \ldots \mathbf{a}_L[n]]$ and $\mathbf{s}[n] = [s_1[n], \ldots, s_L[n]]^{\mathrm{T}}$:

$$\mathbf{x}[n] = \mathbf{A}[n] * \mathbf{s}[n] + \mathbf{v}[n] \;;$$



- space-time covariance: $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$:

$$\mathbf{R}[\tau] = \mathbf{A}[\tau] * \mathcal{E}\{\mathbf{s}[n]\mathbf{s}^{\mathrm{H}}[n-\tau]\} * \mathbf{A}^{\mathrm{H}}[-\tau] + \mathcal{E}\{\mathbf{v}[n]\mathbf{v}^{\mathrm{H}}[n-\tau]\} \quad (1)$$

$$= \mathbf{A}[\tau] * \boldsymbol{\Gamma}[\tau] * \mathbf{A}^{\mathrm{H}}[-\tau] + \sigma_v^2 \mathbf{I}_M \delta[\tau] \;. \quad (2)$$

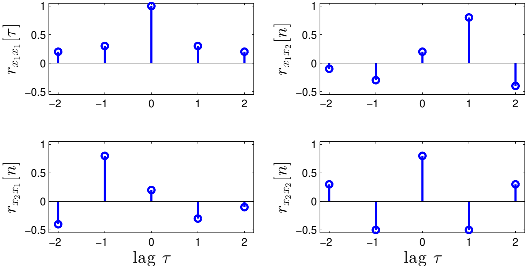
Cross-Spectral Density Matrix

- Transfer function matrix $\boldsymbol{A}(z) = \sum_n \mathbf{A}[n] z^{-n}$ (short $\boldsymbol{A}(z) \bullet\!\!-\!\!\circ\ \mathbf{A}[n]$) is a polynomial in $z \in \mathbb{C}$;
- cross-spectral density $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\ \mathbf{R}[\tau]$:

$$\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{\Gamma}(z)\boldsymbol{A}^{\mathrm{P}}(z) + \sigma_v^2 \mathbf{I}_M \ ;$$

- parahermitian property:

$$\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*) = \boldsymbol{R}(z) \ ;$$



- when evaluated for a specific normalised angular frequency Ω_0: $\mathbf{R}_0 = \boldsymbol{R}(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega_0}}$;
- $\mathbf{R}_0$ is a constant matrix that describes a *narrowband* problem;
- $\mathbf{R}_0$ is Hermitian $\longrightarrow$ eigenvalue decomposition (EVD) $\mathbf{R}_0 = \mathbf{Q}_0 \boldsymbol{\Lambda}_0 \mathbf{Q}_0^{\mathrm{H}}$.

Narrowband EVD and Subspace Decomposition

- We assume an ordered EVD $\mathbf{R}_0 = \mathbf{Q}_0 \mathbf{\Lambda}_0 \mathbf{Q}_0^{\mathrm{H}}$, where $\mathbf{\Lambda}_0 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_M\}$ with $\lambda_\ell \geq \lambda_{\ell+1}$, $\ell = 1, \ldots, (M-1)$;
- partitioning enables a subspace decomposition:

$$\mathbf{R}_0 = \left[\begin{array}{c:c} \mathbf{Q}_{\mathrm{s}} & \mathbf{Q}_{\mathrm{n}} \end{array}\right] \left[\begin{array}{c:c} \mathbf{\Lambda}_s + \sigma_v^2 \mathbf{I}_L & \\ \hdashline & \sigma_v^2 \mathbf{I}_{M-L} \end{array}\right] \left[\begin{array}{c} \mathbf{Q}_{\mathrm{s}}^{\mathrm{H}} \\ \hdashline \mathbf{Q}_{\mathrm{n}}^{\mathrm{H}} \end{array}\right]$$

- source enumeration: eigenvalues above noise floor = number of uncorrelated sources;
- $\mathbf{y}[n] = \mathbf{Q}_{\mathrm{n}}^{\mathrm{H}} \mathbf{x}[n] \in \mathbb{C}^{M-L}$ only contains noise;
- power in $\mathbf{y}[n]$: $\mathcal{E}\{\|\mathbf{y}[n]\|_2^2\} = (M-L)\sigma_v^2$ because of orthonormality of $\mathbf{Q}$.

Broadband EVD

- Space-time covariance $\mathbf{R}[\tau]$ or equivalently the CSD matrix $\boldsymbol{R}(z)$ are only diagonalised by the EVD for a specific value τ or z;
- for an analytic $\boldsymbol{R}(z)$ that is not derived from multiplexed data, there exists a parahermitian matrix EVD [12, 11]

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z) \; ; \tag{3}$$

- $\boldsymbol{\Lambda}(z)$ is diagonal, parahermitian, analytic, and unique;
- eigenvectors in $\boldsymbol{Q}(z)$ are paraunitary, analytic, and unique up to an arbitrary allpass function;
- paraunitarity $\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \boldsymbol{Q}^{\mathrm{P}}(z)\boldsymbol{Q}(z) = \mathbf{I}$ implies losslessness;
- a number of algorithms can approximate (3) [6, 7, 8, 15, 13, 14].

Broadband Subspace Decomposition

- The parahermitian matrix EVD $\boldsymbol{R}(z) = \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)$ enables a broadband subspace decomposition:

$$\boldsymbol{R}(z) = \left[\begin{array}{c:c} \boldsymbol{Q}_{\mathrm{s}}(z) & \boldsymbol{Q}_{\mathrm{n}}(z) \end{array}\right] \left[\begin{array}{c:c} \boldsymbol{\Lambda}_s(z) + \sigma_v^2 \mathbf{I}_L & \\ \hdashline & \sigma_v^2 \mathbf{I}_{M-L} \end{array}\right] \left[\begin{array}{c} \boldsymbol{Q}_{\mathrm{s}}^{\mathrm{P}}(z) \\ \hdashline \boldsymbol{Q}_{\mathrm{n}}^{\mathrm{P}}(z) \end{array}\right]$$

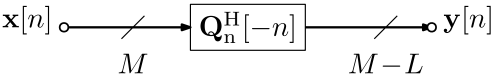
- $\mathbf{Q}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{Q}(z)$ describes a lossless filter bank;
- data vector component in the noise-only subspace: $\mathbf{y}[n] = \mathbf{Q}_{\mathrm{n}}^{\mathrm{H}}[-n] * \mathbf{x}[n]$;
- again, it can be shown that ideally $\mathcal{E}\{\|\mathbf{y}[n]\|_2^2\} = (M-L)\sigma_v^2$.

'Syndrome' Idea

- We estimate $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau]$ over a window of data, with $L < M$ stationary sources present;
- compute parahermitian matrix EVD, perform source enumeration, and determine the eigenvectors spanning the noise-only subspace, $\boldsymbol{Q}_{\mathrm{n}}(z)$;
- if an additional source $s_{L+1}[n]$ enters the scene, it will likely protrude into the noise-only subspace;
- we therefore monitor the syndrome vector

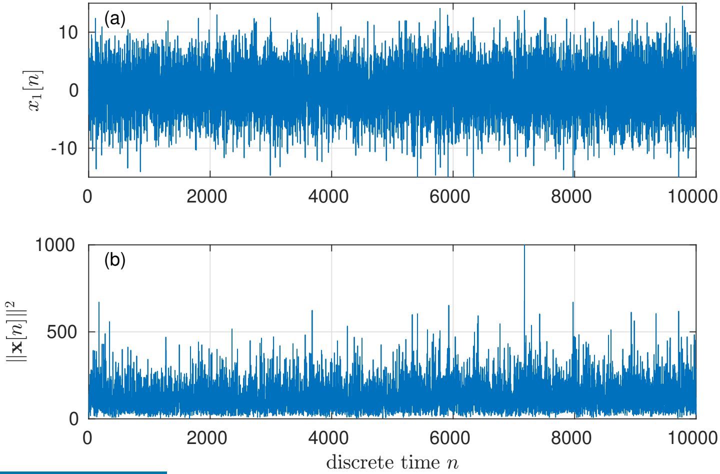
$$\mathbf{y}[n] = \mathbf{Q}_{\mathrm{n}}^{\mathrm{H}}[-n] * \mathbf{x}[n] \tag{4}$$

for a change in power, or for any structured / correlated components.



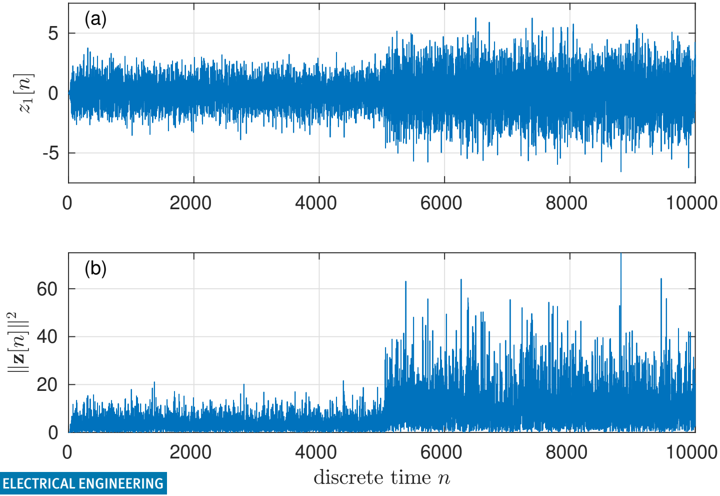
Intuitive Example I

- $M = 6$ sensors, $L = 3$ stationary sources; weak transient source at $n = 5000$;
- monitoring a sensor output $x_1[n]$:



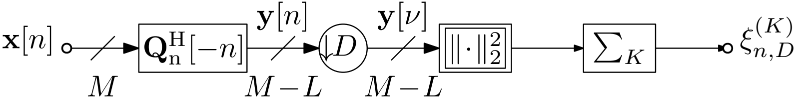
Intuitive Example II

- $M = 6$ sensors, $L = 3$ stationary sources; weak transient source at $n = 5000$;
- monitoring a syndrome element $y_1[n]$:



Proposed Approach

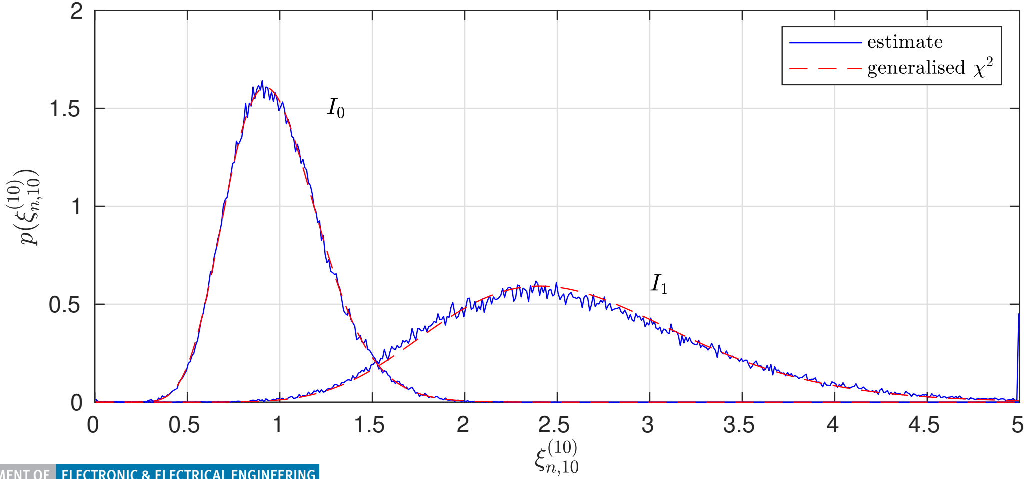
- We use the statistics and evaluated parahermitian matrix EVD of a previous time window, and utilise the broadband noise-only subspace spanned by the columns of $\boldsymbol{Q}_{\mathrm{n}}(z)$;
- being analytic, $\boldsymbol{Q}_{\mathrm{n}}(z)$ can typically be approximated well by low-order polyomials, and is relatively inexpensive to implement;



- because of the processing, elements of the syndrome vector $\mathbf{y}[n]$ are spatially and temporally correlated;
- decimation by D can break temporal correlation and further reduces complexity;
- we can average over consecutive syndrome vectors to increase discrimination;
- $\xi_{n,D}^{(K)}$ is generalised χ^2 distributed if temporal correlation is suppressed [9, 1].

Results I — Statistics

- $M = 6$ sensors, $L = 2$ stationary sources, transfer functions determined by radio propagation model for dense urban environment (polynomial order ≈ 40);
- statistics of output for I_0: no transient versus I_1: transient present:



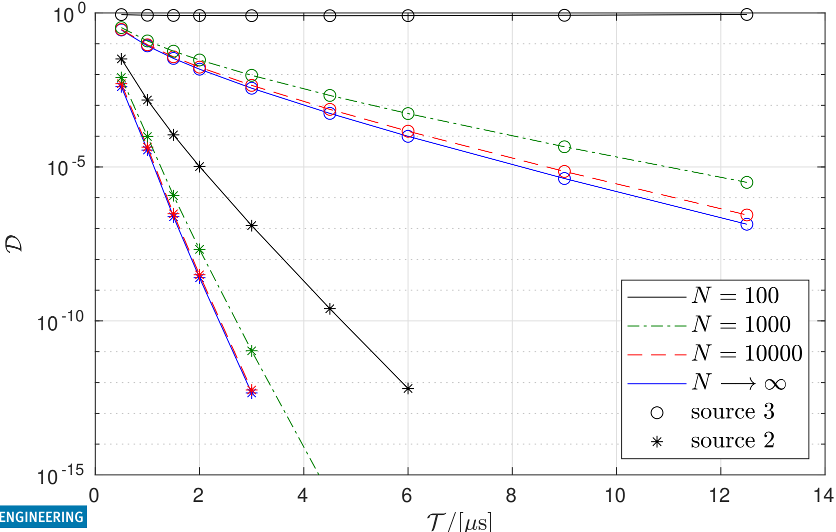
Results II — Sources and Propagation Environment

- Power of contributions for realistic channel scenario:

signal	power
source 1	0.0000 dB
source 2	-4.3494 dB
source 3	-13.2865 dB
noise	-16.2865 dB

Results III — Discrimination vs Decision Time

- Averaging increasingly separates the distributions for I_0 and I_1 — measured as discrimination $\mathcal{D}$: derived from the ROC [5];
- averaging also increases the time to compute $\xi_{n,D}^{(T)} \longrightarrow$ decision time $\mathcal{T}$ (for a 20MHz channel);
- N here is the window over which the space-time covariance is estimated [2, 3, 4].



Summary

- We have proposed a broadband subspace approach to detect the presence of weak transient signals;
- this is based on second order statistics of sensor array data — the space-time covariance matrix — and a polynomial matrix EVD;
- this covariance matrix and its decomposition can be computed off-line;
- a subspace decomposition for the noise-only subspace determines a syndrome vector;
- in the absence of a transient signal, this syndrome only contains noise;
- a transient signal is likely to protrude into the noise-only subspace, and a change in energy can be detected even if the signal is weak;
- discrimination can be traded off against decision time;
- further work: (i) impact of time-varying channels, and (ii) forensic investigation of the transient source once detected.

References I

[1] A. Das and W. S. Geisler.
A method to integrate and classify normal distributions.
arXiv e-prints, page arXiv:2012.14331, Dec. 2020.

[2] C. Delaosa, F. K. Coutts, J. Pestana, and S. Weiss.
Impact of space-time covariance estimation errors on a parahermitian matrix EVD.
In *10th IEEE Workshop on Sensor Array and Multichannel Signal Processing*, pages 1–5, July 2018.

[3] C. Delaosa, J. Pestana, N. J. Goddard, S. Somasundaram, and S. Weiss.
Sample space-time covariance matrix estimation.
In *ICASSP 2019 - 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 8033–8037, May 2019.

[4] C. Delaosa, J. Pestana, N. J. Goddard, S. D. Somasundaram, and S. Weiss.
Support estimation of a sample space-time covariance matrix.
In *Sensor Signal Processing for Defence*, Brighton, UK, March 2019.

[5] J. A. Hanley and B. J. McNeil.
The Meaning and Use of the Area under a Receiver Operating Characteristic (ROC) Curve.
Radiology, 143:26–36, 1982.

[6] J. G. McWhirter, P. D. Baxter, T. Cooper, S. Redif, and J. Foster.
An EVD Algorithm for Para-Hermitian Polynomial Matrices.
IEEE Transactions on Signal Processing, 55(5):2158–2169, May 2007.

[7] S. Redif, J. McWhirter, and S. Weiss.
Design of FIR paraunitary filter banks for subband coding using a polynomial eigenvalue decomposition.
IEEE Transactions on Signal Processing, 59(11):5253–5264, November 2011.

[8] S. Redif, S. Weiss, and J. McWhirter.
Sequential matrix diagonalization algorithms for polynomial EVD of parahermitian matrices.
IEEE Transactions on Signal Processing, 63(1):81–89, January 2015.

References II

[9] H. Solomon and M. A. Stephens.
Distribution of a sum of weighted chi-square variables.
Journal of the American Statistical Association, 72(360):881–885, 1977.

[10] S. Weiss, C. Delaosa, J. Matthews, I. Proudler, and B. Jackson.
Detection of weak transient signals using a broadband subspace approach.
In *International Conference on Sensor Signal Processing for Defence*, 2021.

[11] S. Weiss, J. Pestana, I. Proudler, and F. Coutts.
Corrections to "on the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix".
IEEE Transactions on Signal Processing, 66(23):6325–6327, Dec 2018.

[12] S. Weiss, J. Pestana, and I. K. Proudler.
On the existence and uniqueness of the eigenvalue decomposition of a parahermitian matrix.
IEEE Transactions on Signal Processing, 66(10):2659–2672, May 2018.

[13] S. Weiss, I. Proudler, F. Coutts, and J. Deeks.
Extraction of analytic eigenvectors from a parahermitian matrix.
In *SSPD*, 2020.

[14] S. Weiss, I. K. Proudler, and F. K. Coutts.
Eigenvalue decomposition of a parahermitian matrix: Extraction of analytic eigenvalues.
IEEE Transactions on Signal Processing, 69:722–737, 2021.

[15] S. Weiss, I. K. Proudler, F. K. Coutts, and J. Pestana.
Iterative approximation of analytic eigenvalues of a parahermitian matrix EVD.
In *IEEE International Conference on Acoustics, Speech and Signal Processing*, Brighton, UK, May 2019.