Reducing degradation and age of items in imperfect repair modeling

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Abstract. We develop new models for imperfect repair and the corresponding generalized renewal processes for stochastic description of repairable items that fail when their degradation reaches the specified deterministic or random threshold. The discussion is based on the recently suggested notion of a random virtual age and is applied to monotone processes of degradation with independent increments. Imperfect repair reduces degradation of an item on failure to some intermediate level. However, for the nonhomogeneous processes, the corresponding age reduction which sets back the 'clock' of the process is also performed. Some relevant stochastic comparisons are obtained. It is shown that the cycles of the corresponding generalized imperfect renewal process are stochastically decreasing/increasing depending on the monotonicity properties of the failure rate that describes the random failure threshold of an item.

***Keywords*:** imperfect repair; virtual age; renewal processes; remaining lifetime; stochastic comparisons; gamma process

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1. Introduction

Perfect repair/maintenance brings an item to *as good as new* state. It is a common assumption in reliability modeling of repairable systems. However, strictly speaking, there is no perfect repair in real life, as, e.g., even the stored spare parts are aging and, therefore, are not new when used for repair. Therefore, during the last 30 years, a considerable attention in reliability literature was focused on modeling and applications of imperfect repair. One of the most popular imperfect repair models (age reduction) is based on the notion of *virtual age* (Kijima, 1989; Levitin and Lisniansky (2000), Wang and Pham (2006), Doyen and Gaudoin (2004), Finklestein, (2007, 2008,), Krivtsov (2007), Badia and Berrade (2009), Badia et al. (2002), Castro (2013), Tanwar et al. (2014), Dijoux et al. (2016), Nguyen et al. (2017), Zhao et al. (2019) to name a few). In this paper, we will further develop the new approach to virtual age modeling (i.e., employing the notion of a random virtual age defined in Finkelstein and Cha (2021)) and apply it to items with degradation described by the corresponding increasing stochastic processes. But, first, let us recall the classical 'univariate' imperfect repair setting when imperfect repair reduces the age at failure of an item.

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Let T be a lifetime of an item/system described by the Cdf $F_T(t)$ ($\bar{F}_T(t) \equiv 1 - F_T(t)$). The conventional remaining lifetime for an item that was incepted into operation at $t = 0$ and did not fail in $[0, y)$, T_y is described by the following survival function

$$P(T_y > t) = \bar{F}(t \mid y) = \frac{P(T - y > t)}{P(T > y)} = \frac{\bar{F}_T(t + y)}{\bar{F}_T(y)}. \tag{1}$$

Assume that the failure rate $\lambda(t)$ that corresponds to $F_T(t)$ exists and is strictly increasing. This specifically means that an item is aging in the IFR sense, which is the most popular in reliability application notions of aging (Barlow and Proschan (1975). If $\lambda(t)$ is increasing, then its value at t uniquely defines the chronological age of an item, i.e., $\lambda^{-1}(\lambda(t)) = t$. An item fails at $t^* \in (0, \infty)$, which is a realization of T and the repair action is then initiated. The perfect repair reduces an item's age to 0, whereas the minimal repair (Barlow and Proschan, 1975) does not change the age and the distribution of the remaining lifetime.

Assume that repair reduces the age of an item at failure to some intermediate level y, i.e., $0 < y < t^*$ (age reduction), thus (1) can be written as

$$\bar{F}(t \mid y) = \frac{\bar{F}_T(t + y)}{\bar{F}_T(y)} = \exp\left\{-\int_0^t \lambda(y + x)dx\right\}. \tag{2}$$

Relationship (2) means that the 'shape' of the failure rate after this type of imperfect repair remains the same and the function is just 'shifted' on y. The well-known Kijima's models for imperfect repair processes are based on this assumption and consider linear age reduction at each imperfect repair, i.e., $y = qt^*, 0 < q < 1$ (Kijima, 1989).

The foregoing describes a situation when the only information at hand on the state of an item is the time it has been operating. However, at many instances, the performance of the aging/degrading items can be characterized by degradation parameters. In the univariate case, an item usually fails when the corresponding degradation process reaches some threshold. Imperfect repair reduces this degradation to some intermediate value (Castro and Mercier, 2016, Kahle, 2019). See also, e.g., Ponchet et al (2011), where some preventive maintenance optimization problems for the finite horizon and degradation reduction were considered). An important distinction from the age reduction in (1)-(2) is that this value is 'real' and can be observed, whereas y in (1)-(2), is 'imaginary' being the basis for the useful statistical modeling.

A maintenance model with deterioration described by a gamma process and periodic repairs was considered by Mercier and Castro (2019), where the authors compared reduction of accumulated degradation (since the last repair) and the age reduction in the spirit of Kijima 1 model. It is worth noting, in the framework of the referenced paper, that some maintenance actions can act mostly as age reduction, whereas others can reduce age or both age and degradation (see the example of tamping and cleaning of the ballast of a railway track in Sadeghi et al (2018)).

Our approach to modeling degradation and age reduction and the corresponding stochastic analysis are different from those reported in the literature by other authors. The unified treatment of reduction of age and degradation in the current paper is based on the new notion of a random virtual age (Finkelstein and Cha, 2021). It was introduced for defining the remaining lifetime of an item after imperfect repair when the underlying degradation process is nonhomogeneous. In the current paper, we develop further this approach and modify the introduced notion that allows for a more adequate description of the remaining lifetime. Specifically, the relevant stochastic properties are described in detail. As the degradation process is nonhomogeneous, its 'clock' should be also set back upon imperfect repair. This 'bivariate setting' will be discussed in what follows. Imperfect repair can be applied to repairable items at each consecutive repair. Thus, the sequence of operational times forms the corresponding generalized renewal process. Some specific cases for the homogeneous degradation processes will be considered in the paper as well, whereas a more complex setting with nonhomogeneous degradation processes should be addressed in the future work.

The goal of our paper is in introduction and stochastic description of a new model of imperfect repair. When, for instance, the level of degradation after repair is not observed (it is observed in the current version), the corresponding estimation procedures can be also considered. We plan to consider the relevant inferential issues while developing further our model. This can be done using approaches developed in some recent relevant publications (see, e.g., Salles et al (2020), Kamranfar et al (2021), Etminan et al (2022)).

The paper is organized as follows. In section 2 the basic degradation-based setting is described and the notion of a random virtual age is defined. Section 3 deals with a relevant modification of the random virtual age and the corresponding stochastic properties. Section 4 discusses the setting back property for the stochastic process of degradation. Sections 5 and 6 consider the corresponding generalized renewal processes for the cases of deterministic and random thresholds. Finally concluding remarks are given in Section 7.

2. Random degradation-based virtual age

Assume that the observable (continuously monitored or only at failure) internal deterioration process $\{W_t, t \geq 0\}, W_0 = 0$, describes deterioration of an item/system (see, e.g., Berenguer et al. (2003), Meeker and Escobar (2014), Cha and Finkelstein (2013)). Assume that it has *independent increments* and is characterized by the monotonically increasing sample paths. A failure occurs when the process exceeds a deterministic level w. Then, the lifetime of a system, T can be described by the following survival function

$$P(T > t) \equiv \bar{F}(t, w) = P(W_t \leq w). \tag{3}$$

Note that, for the fixed t, $\bar{F}(t, w)$ is, in fact, the Cdf (as a function of w) of a random variable W_t, whereas it is the survival function of a random variable T for the fixed w. In accordance with (3), denote the Cdf $F(t, w) = 1 - \bar{F}(t, w) = P(W_t > w)$.

Assume that at some instant of time the degradation $\tilde{w}, 0 < \tilde{w} < w$ is observed. For the homogeneous $\{W_t, t \geq 0\}$, the remaining lifetime is defined by the following survival function

$$\bar{F}(t, w - \tilde{w}) = P(W_t \leq w - \tilde{w}), \tag{4}$$

and it does not depend on the time of observation, whereas for the non-homogeneous process, the remaining lifetime already depends on this time. Indeed, denote the random time to reach the degradation level $\tilde{w}$ by $T_v(\tilde{w})$. It follows from (3) that its Cdf for the fixed $\tilde{w}$ is given by $F(t, \tilde{w}) = 1 - \bar{F}(t, \tilde{w})$. Then the corresponding *remaining lifetime* T_r in this case can be defined by the following survival function that is already the function of both w and $\tilde{w}$ and not only of their difference as in (4)

$$P(T_r > t) \equiv \bar{F}_r(t, w, \tilde{w}) = \int_0^\infty P(W_{x+t} - W_x \leq w - \tilde{w}) f(x, \tilde{w}) dx, \tag{5}$$

where $f(x,\tilde{w}) = \frac{\partial}{\partial x}F(x,\tilde{w})$.

Following Finkelstein and Cha (2021), we will call $T_v(\tilde{w})$ the *random virtual age* (to reach degradation $\tilde{w}$) as opposed to the 'observed' calendar age which is a realization of $T_v(\tilde{w})$. Note that, this is a different notion than that employed in the imperfect repair modelling.

Definition 1.*(Finkelstein and Cha (2021)* The function $F(t,\tilde{w})$ defined by (3) ($w=\tilde{w}$) is the Cdf of the virtual age $T_v(\tilde{w})$, i.e., a random time that is needed for a *statistically identical item* that starts at $t=0$ to accumulate degradation $\tilde{w}$.

One can define an imperfect repair in this framework as reducing deterioration on failure (which is *w*) to $\tilde{w}$, where the remaining lifetime of the repaired item is defined by (5). However, when considering the foregoing definition of the virtual age, one can observe that, as this virtual age is random with unbounded support, it may happen that its realization can exceed the failure time t^*.This, in some sense (at least, realization-wise), contradicts to the essence of any repair that implies that the item should be better after it. We will deal with this consideration in the next section by discussing the modified (conditional) virtual age. On the other hand, for the homogeneous case, the fact that the performance of the item after the imperfect repair is worse than that of a new item obviously holds:

$$\bar{F}(t,w-\tilde{w}) \le \bar{F}(t,w), \tag{6}$$

whereas, for the nonhomogeneous case that needs additional assumption, the forthcoming Theorem 1 holds. But, first, for the presentation sake and further discussions on relevant stochastic comparisons (Müller and Soyan (2002), Belzunce et al. (2015)), we need definitions of the basic stochastic orders and a few supplementary results.

Definition 2 (Shaked and Shanthikumar, 2007). Let X_1 and X_2 be two non-negative continuous (discrete) random variables with the corresponding cdfs $F_{X_1}(t)$ and $F_{X_2}(t)$, respectively. Denote their pdfs (pmfs) and failure rates as $f_{X_1}(t)$, $f_{X_2}(t)$, and $\lambda_{X_1}(t)$, $\lambda_{X_2}(t)$, respectively, if applicable.

(i) If $F_{X_1}(t) \ge F_{X_2}(t)$ for all $t \ge 0$, then X_1 is smaller than X_2 in the usual stochastic order, denoted by $X_1 \le_{st} X_2$ (or $F_{X_1}(\cdot) \le_{st} F_{X_2}(\cdot)$).

(ii) If $\lambda_{X_1}(t) \ge \lambda_{X_2}(t)$ for all $t \ge 0$, then X_1 is smaller than X_2 in the failure rate order, denoted by $X_1 \le_{fr} X_2$ (or $F_{X_1}(\cdot) \le_{fr} F_{X_2}(\cdot)$).

(iii) If $f_{X_1}(t)/f_{X_2}(t)$ decreases in $t \ge 0$ over the union of the supports of X_1 and X_2, then X_1 is smaller than X_2 in the likelihood ratio order, denoted by $X_1 \le_{lr} X_2$ (or $F_{X_1}(\cdot) \le_{lr} F_{X_2}(\cdot)$).

Lemma 1 (Shaked and Shanthikumar, 2007).

(i) If $X_1 \le_{st} X_2$ and $g(\cdot)$ is any increasing (decreasing) function, then $g(X_1) \le_{st} (\ge_{st}) g(X_2)$.

(ii) If X_1 and X_2 are two nonnegative random variables, then the following holds

$$X_1 \le_{lr} X_2 \Rightarrow X_1 \le_{fr} X_2 \Rightarrow X_1 \le_{st} X_2 \Rightarrow E[X_1] \le E[X_2].$$

Theorem 1. *Let* $W_{x+t} - W_x$ *be stochastically increasing in x in the sense of the usual stochastic order for all fixed* $t > 0$. *Then the following inequality holds for the remaining lifetime of the repaired item (where the degradation is reduced to* $\tilde{w} < w$*)*

$$\bar{F}_r(t, w, \tilde{w}) \le \bar{F}(t, w). \tag{7}$$

Proof. The proof is rather straightforward. Indeed, observe that, due to the assumption,

$$\begin{aligned}\bar{F}_r(t, w, \tilde{w}) &= E[P(W_{T_v(\tilde{w})+t} - W_{T_v(\tilde{w})} \le w - \tilde{w})] \\ &\le E[P(W_{T_v(\tilde{w})+t} - W_{T_v(\tilde{w})} \le w)] \le E[P(W_{T_0+t} - W_{T_0} \le w)] = \bar{F}(t, w),\end{aligned}$$

where T_0 is a degenerate random variable taking its value at 0 and the second inequality holds due to the fact that $P(W_{x+t} - W_x \le w)$ is a decreasing function of x and Lemma 1-(i),(ii).

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Note that, in the homogeneous case, obviously, $W_{x+t} - W_x$ does not depend on *x*, whereas the assumption of Theorem 1 means that the process is increasing not only by accumulation of increments but due to the fact that the increments itself are increasing with time.

Example 1. Gamma Process. As stated in the excellent survey by van Noortwijk (2009): "The gamma process is suitable to model gradual damage monotonically accumulating over time in a sequence of tiny increments, such as wear, fatigue, corrosion, crack growth, erosion, consumption, creep, swell, degrading health index, etc". Therefore, given its mathematical tractability, this process is popular in various reliability applications and especially in imperfect maintenance modeling as at many instances some deterioration can be reversed/decreases by maintenance actions.

Let $\{W_t, t \ge 0\}$ be the nonhomogeneous gamma process and denote, for each $x \ge 0$, the pdf of $W(t)$ (the gamma distribution) as

$$g(x,t) = \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x), \quad y \ge 0,$$

where $\lambda > 0$ and $\alpha(t)$ is monotonically increasing in $t \ge 0$, with $\alpha(0) = 0$ (for the homogeneous case $\alpha(t) = \alpha t$). We assume that $\alpha(t)$ is a convex function. Therefore, for each $t \ge 0$, the corresponding cumulative distribution function is defined as (van Norotwijk (2009))

$$\begin{aligned}P(W(t) \le w) = \bar{F}(t, w) &= \int_0^w \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x)\, dx \\ &= 1 - \frac{\Gamma(\alpha(t), \lambda w)}{\Gamma(\alpha(t))},\end{aligned},$$

where $\Gamma(a) = \int_0^\infty z^{a-1} \exp\{-z\} dz$; $\Gamma(a, x) = \int_x^\infty z^{a-1} \exp\{-z\} dz$.

For the corresponding increment in (5), we have

$$P(W_{x+t}-W_x \le w-\tilde{w}) = 1-\frac{\Gamma((\alpha(t+x)-\alpha(x)),\lambda(w-\tilde{w}))}{\Gamma(\alpha(t+x)-\alpha(x))}.$$

It is easy to see that for $x_1 < x_2$, the ratio of the pdfs of $W_{x_1+t}-W_{x_1}$ and $W_{x_2+t}-W_{x_2}$ is given by

$$\frac{f_{W_{x_1+t}-W_{x_1}}(u)}{f_{W_{x_2+t}-W_{x_2}}(u)} = \frac{\Gamma(\alpha(x_2+t)-\alpha(x_2))}{\Gamma(\alpha(x_1+t)-\alpha(x_1))}\lambda^{(\alpha(x_1+t)-\alpha(x_1))-(\alpha(x_2+t)-\alpha(x_2))}u^{(\alpha(x_1+t)-\alpha(x_1))-(\alpha(x_2+t)-\alpha(x_2))},$$

which is decreasing in *u*, as $\alpha(x_1+t)-\alpha(x_1) < \alpha(x_2+t)-\alpha(x_2)$ due to the assumption that $\alpha(t)$ is convex. This implies that $W_{x_1+t}-W_{x_1} \le_{lr} W_{x_2+t}-W_{x_2}$, and, accordingly, $W_{x_1+t}-W_{x_1} \le_{st} W_{x_2+t}-W_{x_2}$. Thus, the assumption of Theorem 1 holds and therefore, inequality (7) also holds for the gamma processes with convex $\alpha(t)$. Note also that $\alpha(t)$ defines the rate of jumps in the process (whereas λ governs its size).

3. Modified virtual age

In the setting discussed in the previous section, it can formally happen that a realization of the random virtual age after the imperfect repair of an item is larger than the time at failure t^*. This is because $T_v(\tilde{w})$ is independent from the previous time-wise operational history. It can be considered as unrealistic at certain instances, as one should expect that the age of an item after the imperfect repair should also be smaller in realizations (note that often stochastic dominance defined in the previous section is still a practically admissible assumption). This situation can be resolved by considering the adjusted (dependent on realization at failure) virtual age $T_v(\tilde{w},t^*)$ with the pdf

$$g(x,\tilde{w},t^*) = \begin{cases} \dfrac{f(x,\tilde{w})}{P(W_{t^*} > \tilde{w})}, 0 \le x < t^* \\ 0, x \ge t^* \end{cases}, \tag{8}$$

where t^* is the observation of the failure time. The cdf for this *conditional* virtual age is

$$G(x,\tilde{w},t^*) = \begin{cases} \dfrac{\int_0^x f(y,\tilde{w})dy}{P(W_{t^*} > \tilde{w})}, 0 \le x < t^* \\ 1, x \ge t^* \end{cases} \tag{9}$$

If t^* is not observed, the unconditional distribution of the new “adjusted virtual age” that takes into account the fact that it cannot be larger than the random time of failure is

$$G_u(x,\tilde{w}) = \int_0^\infty G(x,\tilde{w},t^*) f(t^*,w)dt^*, \tag{10}$$

where $f(t^*,w)$ is our initial pdf for the time of reaching the threshold w (was denoted as $f(x,w)$).

Remark 1. The virtual age suggested in Definition 1 along with the degradation level after imperfect repair, i.e., the pair $\{T_v(\tilde{w}), \tilde{w}\}$ uniquely defines (for each realization of $T_v(\tilde{w})$) the state of an item after repair (nonhomogeneous process of degradation). Thus, the time of failure t^* is out of the picture, which, in our opinion, is an advantage when modelling the process of imperfect repairs. The foregoing adjustment, defined in (8)-(10), on one hand, eliminates the described drawback of the model (which is often quite acceptable in practice, in our view), but brings the time of failure into the play, on the other hand. Thus, the two versions of a random virtual age have their pros and contras, however, both of them, in our opinion can, be justified in practice in contrast to traditional age reduction modelling in (1)-(2) and its further variants. However, it should be noted that the proposed notions apply to a more specific class of distributions based on the corresponding degradation modelling. Another supplementary option based on deterministic virtual age of a different nature is described in the next section.

Thus, in accordance with the model (8)-(9), assume that we observe both $(\tilde{w}, t^*)$. Denote the corresponding remaining lifetime by $T_r(\tilde{w}, t^*)$. Then the survival function that describes this random variable is given by

$$P(T_r(\tilde{w}, t^*) > t) \equiv \bar{F}_r(t, w, \tilde{w}; t^*) = \int_0^\infty P(W_{x+t} - W_x \le w - \tilde{w}) g(x, \tilde{w}, t^*) dx .$$

Following the similar procedure as that in the proof of Theorem 1, it can be easily shown that for any t^* and $\tilde{w} < w$

$$\bar{F}_r(t, w, \tilde{w}; t^*) \le \bar{F}(t, w) .$$

In addition, we have the following comparison results that are proved for a general case of the nonhomogeneous $\{W_t, t \ge 0\}, W_0 = 0$ and, obviously, hold for the homogeneous case (note that, as usual, “increasing” means “non-decreasing”).

Theorem 2. *Let $W_{x+t} - W_x$ be stochastically increasing in x in the sense of the usual stochastic order for all fixed $t > 0$.*

(i)If $t_1^* < t_2^*$, then $T_r(\tilde{w}, t_1^*) \ge_{st} T_r(\tilde{w}, t_2^*)$, for all fixed $\tilde{w}$.

(ii)If $\tilde{w}_1 < \tilde{w}_2$ and $F(\cdot, \tilde{w}_1) \le_{lr} F(\cdot, \tilde{w}_2)$, then $T_r(\tilde{w}_1, t^*) \ge_{st} T_r(\tilde{w}_2, t^*)$, for all fixed t^*.

Proof.

(i) Observe that $P(T_r(\tilde{w}, t^*) > t)$ is the expectation of $P(W_{x+t} - W_x \le w - \tilde{w})$ with respect to the distribution $g(x, \tilde{w}, t^*)$. For $t_1^* < t_2^*$,

$$\frac{g(x, \tilde{w}, t_1^*)}{g(x, \tilde{w}, t_2^*)} = \frac{P(W_{t_2^*} > \tilde{w})}{P(W_{t_1^*} > \tilde{w})}, \text{ for } 0 \le x < t_1^* ,$$

and

$$\frac{g(x,\tilde{w},t_1*)}{g(x,\tilde{w},t_2*)}=0\,,\ t_1*\le x<t_2*\,.$$

Therefore, $\frac{g(x,\tilde{w},t_1*)}{g(x,\tilde{w},t_2*)}$ decreases in x over the union of the supports of the two distributions, which means that $G(\cdot,\tilde{w},t_1*)\le_{lr} G(\cdot,\tilde{w},t_2*)$. Due to Lemma 1-(ii), this again implies $G(\cdot,\tilde{w},t_1*)\le_{st} G(\cdot,\tilde{w},t_2*)$. On the other hand, due to the assumption that $W_{x+t}-W_x$ is stochastically increasing in x in the sense of the usual stochastic order, $P(W_{x+t}-W_x\le w-\tilde{w})$ is decreasing function of x. Then, due to Lemma 1-(i),(ii),

$$P(T_r(\tilde{w},t_1*)>t)=\int_0^\infty P(W_{x+t}-W_x\le w-\tilde{w})g(x,\tilde{w},t_1*)dx$$

$$\ge\int_0^\infty P(W_{x+t}-W_x\le w-\tilde{w})g(x,\tilde{w},t_2*)dx=P(T_r(\tilde{w},t_2*)>t)\,,$$

which completes the proof.

(ii) For $\tilde{w}_1<\tilde{w}_2$,

$$P(T_r(\tilde{w}_1,t*)>t)=\int_0^\infty P(W_{x+t}-W_x\le w-\tilde{w}_1)g(x,\tilde{w}_1,t*)dx$$

$$\ge\int_0^\infty P(W_{x+t}-W_x\le w-\tilde{w}_2)g(x,\tilde{w}_1,t*)dx\,.$$

Due to the condition that $F(\cdot,\tilde{w}_1)\le_{lr} F(\cdot,\tilde{w}_2)$,

$$\frac{g(x,\tilde{w}_1,t*)}{g(x,\tilde{w}_2,t*)}=\frac{P(W_{t*}>\tilde{w}_2)}{P(W_{t*}>\tilde{w}_1)}\frac{f(x,w_1)}{f(x,w_2)}$$

decreases in x over the union of the supports of the two distributions. This implies that $G(\cdot,\tilde{w}_1,t*)\le_{lr} G(\cdot,\tilde{w}_2,t*)$, which again means that $G(\cdot,\tilde{w}_1,t*)\le_{st} G(\cdot,\tilde{w}_2,t*)$. Then, due to Lemma 1-(i),(ii),

$$\int_0^\infty P(W_{x+t}-W_x\le w-\tilde{w}_2)g(x,\tilde{w}_1,t*)dx$$

$$\ge\int_0^\infty P(W_{x+t}-W_x\le w-\tilde{w}_2)g(x,\tilde{w}_2,t*)=P(T_r(\tilde{w}_2,t^*)>t)\,,\text{ for all }\ t\,.$$

By combining this inequality with the one for $(T_r(\tilde{w}_1,t*)>t)$ above we have

$$P(T_r(\tilde{w}_1,t^*)>t)\ge P(T_r(\tilde{w}_2,t^*)>t)\,,\text{ for all }\ t\,,$$

which completes he proof.

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Remark 2. At first sight the result in (i) seems counterintuitive. However, it can be loosely explained by conditioning in (8)-(9), as larger values of t^* result in stochastically larger values of time that correspond to degradation $\tilde{w}$ after the imperfect repair. On the other hand, the result in (ii) is intuitively natural: the larger level of degradation after imperfect repair implies the stochastically smaller remaining lifetime. However, for proving it, we need the stronger (likelihood ratio) ordering, for the corresponding times to reach the degradation levels, as the usual stochastic ordering that holds 'automatically' in this case is not sufficient.

Now, in accordance with (10), suppose that we observe only $\tilde{w}$. In this case, the remaining lifetime is denoted by $T_r(\tilde{w})$ and the corresponding survival function is given by

$$P(T_r(\tilde{w}) > t) = \int_0^\infty P(W_{x+t} - W_x \le w - \tilde{w}) g_u(x, \tilde{w}) dx ,$$

where $g_u(x,\tilde{w}) = \int_0^\infty g(x,\tilde{w},t^*) f(t^*,w) dt^*$. We have the following comparison result that is similar to (ii) of the previous theorem.

Theorem 3. *Let $W_{x+t} - W_x$ be stochastically increasing in x in the sense of the usual stochastic order for all fixed $t > 0$. If $\tilde{w}_1 < \tilde{w}_2$ and $F(\cdot,\tilde{w}_1) \le_{lr} F(\cdot,\tilde{w}_2)$, then $T_r(\tilde{w}_1) \ge_{st} T_r(\tilde{w}_2)$.*

Proof.

By the same procedures as those described in the proof of Theorem 2, for $\tilde{w}_1 < \tilde{w}_2$, we have $G(\cdot,\tilde{w}_1,t^*) \le_{st} G(\cdot,\tilde{w}_2,t^*)$, which means $G(x,\tilde{w}_1,t^*) \ge G(x,\tilde{w}_2,t^*)$, for all x. Then,

$$G_u(x,\tilde{w}_1) = \int_0^\infty G(x,\tilde{w}_1,t^*) f(t^*,w) dt^* \ge \int_0^\infty G(x,\tilde{w}_2,t^*) f(t^*,w) dt^* = G_u(x,\tilde{w}_2) ,$$

which implies that $G_u(\cdot,\tilde{w}_1) \le_{st} G_u(\cdot,\tilde{w}_2)$. Then, due to Lemma 1-(i),(ii),

$$\begin{aligned} P(T_r(\tilde{w}_1) > t) &= \int_0^\infty P(W_{x+t} - W_x \le w - \tilde{w}_1) g_u(x,\tilde{w}_1) dx \\ &\ge \int_0^\infty P(W_{x+t} - W_x \le w - \tilde{w}_2) g_u(x,\tilde{w}_1) dx \\ &\ge \int_0^\infty P(W_{x+t} - W_x \le w - \tilde{w}_2) g_u(x,\tilde{w}_2) dx = P(T_r(\tilde{w}_2) > t) , \end{aligned}$$

which completes he proof.

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4. Setting back the clock for the degradation process

In the imperfect repair model defined by relationship (2), the chronological time y is, in a way, a proxy for degradation (when the failure rate is increasing). Thus, we reduce the time at failure to some intermediate level during the imperfect repair and, therefore, we reduce the corresponding degradation as well. The setting described in our paper is different, as we observe degradation on one hand and the time of observation, on the other. Thus, the same degradation can be accumulated during different intervals of time in accordance with different realizations of the degradation process. This was reflected via the foregoing original (not modified) notion of a random virtual age. The important assumption for this notion is that it is defined for the statistically identical item, whereas the imperfect repair is performed on a *specific* item and this specific item continues operation after the repair. It was mentioned that this assumption is natural and not restrictive (at least far less restrictive than assumptions in the classical model (2)). Besides, our additional reasoning in (8)-(10) ensures that after the repair the virtual age, in realizations, is smaller than the age at failure (before imperfect repair) as well. Thus, on the one hand, we have the observed value of degradation after the imperfect repair, $\tilde{w}$, whereas on the other hand, we have its own time (clock) of the stochastic process of degradation, $\{W_t, t \geq 0\}$, which is also reduced.

Example 2. Poisson process. As another illustrative example, in addition to the gamma process considered above (inverse Gaussian is also a candidate in the suggested framework), we can think about the Poisson counting processes, which can describe the corresponding jump process. Then the threshold w and the reduced degradation $\tilde{w}$ after the imperfect repair are integers. The corresponding increment for integer d for the nonhomogeneous Poisson process (NHPP) is defined as

$$P(W_{x+t} - W_x < d) = \exp\{-R(x,t)\}\sum_{i=0}^{d-1}\frac{(R(x,t))^i}{i!};\ R(x,t) = \int_x^{x+t} r(u)du\,,$$

where $r(t)$ is the rate of the NHPP. For the homogeneous Poisson process (HPP) with rate r, obviously,

$$P(W_{x+t} - W_x < d) = \exp\{-rt\}\sum_{i=0}^{d-1}\frac{(rt)^i}{i!};\ \forall x > 0\,,$$

which can describe the reliability of the 1 out of d cold standby system (in the time interval $[0,t)$) with the i.i.d., exponentially distributed components and the failure rate r. A system fails when d components fail. The imperfect repair replaces only $\tilde{d}$ ($\tilde{d} < d$) components.

Remark 3. In view of other possible applications, consider the NHPP of shocks affecting a system. Each shock results in the fixed amount of degradation, thus the number of experienced shocks defines the level of degradation of a system. One can think, e.g., about the antenna of a radio telescope that consists of numerous sections or any other sectional device, or any device consisting of i.i.d. components of the same 'additive functionality'. Assume that each shock results in a failure/destruction of one component with probability p and is harmless to a system with the complementary probability 1-p. Thus, the process of effective (causing damage) shocks in accordance with the thinning procedure is again NHPP with rate $p\lambda(t)$, whereas a system fails when the total number of failed components reaches d. The origin of shocks can be different depending on operational properties, e.g., voltage surges, wind gusts, mechanical obstacles, etc.

It should be also noted that, for the NHPP, the imperfect repair along with reducing degradation to a deterministic level reduces the time at failure (in accordance with Definition 1) to a random one. We believe that it is the adequate way to deal with the 'bivariate' degradation-age reduction. However, a question can arise:

What can be a reasonable, deterministic alternative to the random virtual age defined above that takes into account these two aspects(time upon failure and degradation) of imperfect repair in some aggregated way?

To answer this question, first recall the age correspondence principle (Finkelstein and Cha (2013), Meeker and Escobar (2014)). Consider two items with stochastically ordered lifetimes, i.e.,

$$\bar{F}_2(t) \le \bar{F}_1(t),\ t \in (0,\infty). \tag{11}$$

This inequality implies the following equation

$$F_1(t) = F_2(D(t)), \quad D(0) = 0, t \in (0,\infty), \tag{12}$$

where $D(t) \le t$ is an increasing function that defines time that the second less reliable item to operate in order to have the same survival probability as the first one that operates during time t.

Relationship (12) can be viewed as an equation for obtaining the function $D(t)$, as

$$\exp\left\{-\int_0^{D(t)} \lambda_2(u)du\right\} = \exp\left\{-\int_0^t \lambda_1(u)du\right\}$$

$$\Rightarrow \int_0^{D(t)} \lambda_2(u)du = \int_0^t \lambda_1(u)du, \tag{13}$$

where $\lambda_1(u)$ and $\lambda_2(u)$ are the corresponding failure rates under the assumption that densities exist. Note that $\int_0^t \lambda_2(u)du \ge \int_0^t \lambda_1(u)du,$ for all $t \ge 0$, due to (11) and thus there exists $D(t) \le t$ satisfying (12) and (13). In this way, we can make the age correspondence for different items with ordered lifetimes. This principle is widely used in accelerated life testing (Nelson (1990)) and can be also interpreted as a general Accelerated Life Model (ALM) (see also Cox and Oakes (1984), Meeker and Escobar (2014), Finkelstein (2008), to name a few) with a time-dependent scale-transformation function $D(t) \le t$.

We can use this reasoning for our setting as well. When the imperfect repair reduces wear from w to $\tilde{w}$, it 'creates' an item with a stochastically smaller lifetime and, therefore, an age at failure t^* (on reaching w) corresponds to the smaller age $v(t^*, \tilde{w}, w) \equiv D(t^*) \le t^*$, on reaching degradation $\tilde{w}$. Thus, distinct from the random virtual age in Definition 1, we have the following definition for the deterministic virtual age that is formulated in terms of the age correspondence principle (13). In this way, in fact, we average the random virtual ages also complying with the property discussed in (8)-(10).

Definition 3. Let the failure of a new item in the described failure model occur at time $t^* \in (0,\infty)$ and the imperfect repair reduces degradation at failure w to some intermediate level $\tilde{w}, 0 < \tilde{w} < w$. Determine $D(t)$ by Eq. (13), where $\lambda_1(t)$ is the failure rate of an item with the failure threshold w and $\lambda_2(t)$ is that of an item with the failure threshold $\tilde{w}$. Then the function $v(t) = D(t)$ defines the Virtual Age of the underlying stochastic process after the repair (for t=t*).

Example 3. In this example, we illustrate how $D(t^*)$ can be obtained when the degradation process follows a gamma process. As in Example 1, let $\{W_t, t \geq 0\}$ be the nonhomogeneous gamma process and the pdf of $W(t)$ is given by

$$g(x,t) = \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x), \quad y \geq 0,$$

where $\lambda > 0$ and $\alpha(t)$ is monotonically increasing in $t \geq 0$, with $\alpha(0) = 0$. Suppose that the imperfect repair reduces degradation level w at the failure time t^* to some intermediate level $\tilde{w}, 0 < \tilde{w} < w$. In this case,

$$\bar{F}_2(t) = \int_0^{\tilde{w}} \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x) dx \leq \int_0^{w} \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x) dx = \bar{F}_1(t),$$

and

$$\int_0^t \lambda_2(u) du = -\ln\left(\int_0^{\tilde{w}} \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x) dx\right)$$

$$\geq -\ln\left(\int_0^{w} \frac{1}{\Gamma(\alpha(t))} \lambda^{\alpha(t)} x^{\alpha(t)-1} \exp(-\lambda x) dx\right) = \int_0^t \lambda_1(u) du,$$

for all $t \geq 0$. When $t = 0$, $\int_0^t \lambda_2(u) du = 0$, and $\Lambda_2(s) = \int_0^s \lambda_2(u) du$ is a strictly increasing function of s. Thus, for fixed t^*, there exists $0 < s^* < t^*$ such that $\int_0^{s^*} \lambda_2(u) du = \int_0^{t^*} \lambda_1(u) du$, from which $D(t^*) = s^*$. Thus, $D(t^*)$ can be obtained by increasing the function $\Lambda_2(s) = \int_0^s \lambda_2(u) du$ until it attains $\int_0^{t^*} \lambda_1(u) du$.

Example 4. Assume, as in Example 2, that degradation is modelled by the Poisson process with rate λ. Let the threshold level be d (integer), and after failure at time t^* and subsequent imperfect repair it has been reduced to $\tilde{d} < d$. In this case,

$$\bar{F}_2(t) = \exp\{-\lambda t\} \sum_{i=0}^{\tilde{d}-1} \frac{(\lambda t)^i}{i!} \leq \exp\{-\lambda t\} \sum_{i=0}^{d-1} \frac{(\lambda t)^i}{i!} = \bar{F}_1(t)$$

and

$$\int_0^t \lambda_2(u) du = -\ln\left(\exp\{-\lambda t\} \sum_{i=0}^{\tilde{d}-1} \frac{(\lambda t)^i}{i!}\right) \geq -\ln\left(\exp\{-\lambda t\} \sum_{i=0}^{d-1} \frac{(\lambda t)^i}{i!}\right) = \int_0^t \lambda_1(u) du,$$

for all $t \geq 0$. Then, by similar procedure as described in Example 3, $D(t^*)$ can be found.

Thus, the pair $(v(t^*), \tilde{w})$, defines the described 'deterministic' model (similar to the random model $(T_v(\tilde{w}), \tilde{w})$). It should be noted that, as in (1)-(2), the described approached is based on realizations of the failure time. It is also clear that, similar to the random model, the notion of virtual age is irrelevant (for defining the remaining lifetime) for the homogeneous degradation processes. Therefore, in the next section, we consider the nonhomogeneous $\{W_t, t \geq 0\}$.

5. Process of imperfect repairs (deterministic threshold)

Recall briefly the process of imperfect repairs governed by the Kijima 2 model. Let an item start operating at $t = 0$. The first cycle duration is described by the Cdf $F(t)$ with the corresponding failure rate $\lambda(t)$. Let the first failure (and the instantaneous imperfect repair) occur at $X_1 = x_1$. Assume that the imperfect repair decreases the age of an item to $qx_1, 0 \leq q \leq 1$. Thus, the second cycle of the point process starts with the virtual age $v_1 = qx_1$ and the cycle duration X_2 is distributed as $F(t \mid v_1)$ with the failure rate $\lambda(t + v_1),\ t \geq 0$ as defined in (1)-(2). Therefore, the virtual age after the second repair is $v_2 = q(qx_1 + x_2) = q^2 x_1 + qx_2, \ldots$, and after the nth repair

$$v_n = q^n x_1 + q^{n-1} x_2 + \ldots + qx_n = \sum_{i=0}^{n-1} q^{n-i} x_{i+1} \tag{14}$$

This is also referred to as the ARA∞ model in Doyen and Gaudoin (2004). Relationship (14) is written for realization in an explanatory way. The stochastic properties (including the limiting ones) of the sequence of random virtual ages and of the cycles were studied, e.g., in Finkelstein (2007) and Liu et al (2020).

Let us consider now the model introduced in the previous section. The first cycle in our process of (instantaneous) imperfect repairs with duration X_1 is, in accordance with (3), described by the following survival function

$$P(T_1 > t) = \bar{F}_1(t, w) \equiv \bar{F}(t, w) = P(W_t \leq w). \tag{15}$$

Let the first failure (and the instantaneous imperfect repair) occur at $X_1 = x_1$ (previously the notation for the time of the first failure was $x_1 = t^*$). Obtain, in accordance with (12), the virtual age $v_1(x_1) = D_1(x_1)$ (for the process of degradation) after the first imperfect repair that brings degradation to $\tilde{w}$ from equation

$$P(W_t \leq w) = P(W_{v_1(t)} \leq \tilde{w}) \tag{16}$$

after letting $t = x_1$, i.e.,

$$P(W_{x_1} \leq w) = P(W_{v_1(x_1)} \leq \tilde{w}) \tag{17}$$

Thus, $v_1(x_1)$ is the 'deterministic' virtual age at the beginning of the *second cycle*. As in the case of the random virtual age, it obviously exists for the homogeneous degradation process as well, but, bears no influence on the further trajectory of the imperfectly repairable item.

Therefore, we consider the nonhomogeneous process, $\{W_t, t \geq 0\}$. What happens next. An item starts the second cycle having the initial degradation $\tilde{w}$ and the initial age of the degradation process $v_1(x_1)$ (it was 0 at the start of the first cycle). Duration of the second cycle is defined by the corresponding remaining lifetime to failure (the latter occurs, as on the first cycle, at the threshold w). Thus, instead of $\{W_t, t \geq 0\}, W_0 = 0$ on the first cycle, we consider the shifted process on the second cycle $\{W_{v_1(x_1)+t}, t \geq 0\}$, and the corresponding survival function of its duration T_2, due to the property of independent increments, is defined as

$$P(T_2 > t) = P(W_{v_1(x_1)+t} - W_{v_1(x_1)} \leq w - \tilde{w}). \tag{18}$$

The second imperfect repair reduces degradation to the larger level than the first one, thus $\tilde{w}_2 \in (\tilde{w}, w)$ and the new increment of virtual age $v_2(x_2)$ should be obtained from the equation

$$P(W_{v_1(x_1)+x_2} - W_{v_1(x_1)} \leq w - \tilde{w}) = P(W_{v_1(x_1)+v_2(x_2)} - W_{v_1(x_1)} \leq \tilde{w}_2 - \tilde{w}), \tag{19}$$

where x_2 is a realization of a lifetime on a second cycle. For obtaining (19), we use (17), where the right-hand side is the probability that the accumulated wear in (0, $v_1(x_1)$) that started from "0" at time 0 would be smaller than $\tilde{w} - 0$. Now, at $v_1(x_1)$, we have already initial wear $\tilde{w}$. Then, applying the same arguments, the right-hand side should be the probability that the accumulated wear in ($v_1(x_1)$, $v_1(x_1) + v_2(x_2)$) which started from "$\tilde{w}$" at time $v_1(x_1)$ would be smaller than $\tilde{w}_2 - \tilde{w}$.

Observe that $w - \tilde{w} > \tilde{w}_2 - \tilde{w}$, and therefore, $v_2(x_2) < x_2$ and the virtual age just after the second imperfect repair is given by $v_1(x_1) + v_2(x_2)$. Then, the third cycle can be described by

$$P(T_3 > t) = P(W_{v_1(x_1)+v_2(x_2)+t} - W_{v_1(x_1)+v_2(x_2)} \leq w - \tilde{w}_2)$$

and, similarly, the n-th cycle by

$$P(T_n > t) = P(W_{v_1(x_1)+v_2(x_2)+\ldots+v_{n-1}(x_{n-1})+t} - W_{v_1(x_1)+v_2(x_2)+\ldots+v_{n-1}(x_{n-1})} \leq w - \tilde{w}_{n-1}), \tag{20}$$

where, $\tilde{w}_{n-1} \in (\tilde{w}_{n-2}, w)$, whereas the virtual age for the degradation process at the start of the (n+1)-th cycle, $v_1(x_1) + v_2(x_2) + \cdots + v_{n-1}(x_{n-1}) + v_n(x_n)$ should be obtained from the equation

$$P(T_n > x_n) = P(W_{v_1(x_1)+v_2(x_2)+\ldots+v_{n-1}(x_{n-1})+v_n(x_n)} - W_{v_1(x_1)+v_2(x_2)+\ldots+v_{n-1}(x_{n-1})} \leq \tilde{w}_n - \tilde{w}_{n-1}) \tag{21}$$

where, $\tilde{w}_n \in (\tilde{w}_{n-1}, w)$ and letting $t = x_n$. In this case, $v_n(x_n) < x_n$. Clearly,

$$v_1(x_1) < v_1(x_1) + v_2(x_2) < v_1(x_1) + v_2(x_2) + v_3(x_3) < \cdots \tag{22}$$

Therefore, if $W_{x+t} - W_x$ is stochastically increasing in x in the sense of the usual stochastic order the sequence of cycles durations is stochastically decreasing, i.e.,

$$P(T_1 > t) > P(T_2 > t) > P(T_3 > t).... \tag{23}$$

Note that there can be different practically sound models for the imperfect repair levels $\tilde{w}_i, i = 1, 2, \ldots$. For instance, the level of wear after each imperfect repair can increase in a geometric-type way, i.e.,

$$\tilde{w}_1 \equiv \tilde{w} = (1-\rho)w, \tilde{w}_2 = (1-\rho^2)w, \ldots; w - \tilde{w}_i = \rho^i w; \ 0 < \rho < 1.$$

6. Process of imperfect repairs (random threshold)

Consider now a random failure threshold L described by the Cdf $G(w)$ and the pdf $g(w)$. Assume also that the degradation process is homogeneous. The nonhomogeneous case is much more cumbersome and will be discussed elsewhere.

The time to failure of an item is described by the following survival function

$$\bar{F}(t) = \int_0^\infty \bar{F}(t, w) g(w) dw = \int_0^\infty P(W_t \leq w) g(w) dw = P(W_t \leq L). \tag{24}$$

Let the first failure (and the instantaneous imperfect repair) occur for realization of threshold $L \equiv L_1$ denoted by w_1. Assume that imperfect repair reduces this degradation proportionally to the value $qw_1, 0 \leq q \leq 1$. Distinct from the age reduction described previously, this operation now can be practically justified as, at many instances, this is the clear 'physical action'. The first cycle's duration T_1 is described by the Cdf $F_1(t) = F(t, w_1)$. The second cycle starts with an item that has degradation qw_1. We assume that the repair *does not change the distribution of the threshold* $G(w)$ (similar to (1)) and the corresponding remaining degradation to this threshold, similar to (1), is described by:

$$G(w \mid qw_1) = \frac{G(w + qw_1) - G(qw_1)}{\bar{G}(qw_1)}; \ g(w \mid qw_1) = \frac{g(w + qw_1)}{\bar{G}(qw_1)}. \tag{25}$$

Thus, the duration of the second cycle T_2 is defined by the following conditional survival function (SF) (conditional distribution of T_2)

$$\bar{F}_2(t, w_1) = \int_0^\infty \bar{F}(t, w) g(w \mid qw_1) dw. \tag{26}$$

Let realization of a threshold L_2 on the second cycle be w_2 and the imperfect repair decrease the accumulated degradation to $q(qw_1 + w_2)$. Then the duration of the third cycle T_3 is given by the following conditional SF (conditional distribution of T_3)

$$\bar{F}_3(t, w_1, w_2) = \int_0^\infty \bar{F}(t, w) g(w \mid q(qw_1 + w_2)) dw, \tag{27}$$

The structure of degradations qw_1, $q(qw_1 + w_2)$ is similar to virtual ages in the Kijima-2 model described in the beginning of the previous section, i.e., qx_1, $q(qx_1 + x_2)$. However, there is nothing 'virtual' now, as degradation is the observed quantity. Note also that in realizations, these values are not always ordered as, e.g., $qw_1 < q(qw_1 + w_2)$ is not true for all realizations. Thus, the n-th cycle (after the $(n-1)$-th imperfect repair) can be defined, using relationship similar to (14), for the corresponding realizations as the following conditional distribution of T_n

$$\bar{F}_n(t, w_1, w_2, ..., w_{n-1}) = \int_0^\infty \bar{F}(t, w) g(w \mid \sum_{i=1}^{n-1} q^{n-i} w_i) dw . \quad (28)$$

As realizations of $qw_1, q^2 w_1 + qw_2, q^3 w_1 + q^2 w_2 + qw_3, ...$, are not necessarily ordered, the corresponding *conditional distributions* of $\{L_i\}, i = 1, 2, ...,$ are not necessarily stochastically ordered. This implies that the conditional distributions of $\{T_i\}, i = 1, 2, ...$ defined in (24), (26), (27), (28) are also not stochastically ordered.

However, even though realizations $qw_1, q^2 w_1 + qw_2, q^3 w_1 + q^2 w_2 + qw_3, ...$ are not necessarily ordered, the *unconditional distributions* for the corresponding random variables are stochastically ordered, as will be shown below. In the following, denote by Z_n the initial degradation value at the start of the n-th cycle, $n = 1, 2, ...$, $Z_1 = 0$. Then $Z_n = q^{n-1} L_1 + q^{n-2} L_2 + \cdots + q\, L_{n-1}$. In Finkelstein (2007), the comparison results on virtual ages at the starts of the cycles in the imperfect repair model described by (14) have been obtained. As the construction of the initial degradation at each cycle is the same as the construction of initial virtual ages in the imperfect repair model described by (14), the following result is due to Finkelstein (2007).

Theorem 4. *Suppose that the distribution of a threshold $G(x)$ is absolutely continuous with the hazard rate $\lambda_L(x)$ and it does not change after imperfect repairs defined by (1) Then, we have the following results.*

(i) $Z_1 \le_{st} Z_2 \le_{st} Z_3 \le_{st} ...$

(ii) If additionally, $\lambda_L(x)$ is increasing(decreasing), then the wear increments at each cycle are stochastically decreasing (increasing):

$L_1 \ge_{st} L_2 \ge_{st} L_3 \ge_{st} ... (L_1 \le_{st} L_2 \le_{st} L_3 \le_{st} ...)$

Denote by $g_n(w)$ the pdf of L_n, $n = 1, 2, ...$. Then,

$$P(T_n > t) \equiv \bar{F}_n(t) = \int_0^\infty \bar{F}(t, w) g_n(w) dw ,$$

where $\bar{F}(t, w)$ is an increasing function of w. Then, from Lemma 1-(i),(ii) and Theorem 4, we have the following result on stochastic comparison of cycle durations T_n, $n = 1, 2, ...$ for increasing (decreasing) $\lambda_W(x)$:

$$T_1 \ge_{st} T_2 \ge_{st} T_3 \ge_{st} ... (T_1 \le_{st} T_2 \le_{st} T_3 \le_{st} ...). \quad (29)$$

Thus, under the assumptions of Theorem 4, if e.g., the failure rate that corresponds to the distribution of a failure threshold is increasing, degradation on each cycle and the cycle duration are stochastically decreasing with its number.

The alternative technical proof of this result for the decreasing $\lambda_L(x)$ is deferred to the Appendix.

7. Concluding remarks

Generalizing and extending our previous work (Finkelstein and Cha (2021)), we develop further a basic approach to modelling imperfect repair that differs from the conventional virtual age models reported in the literature. We consider monotone homogeneous and nonhomogeneous processes of degradation with independent increments.

An imperfect repair decreases degradation at failure of an item to some intermediate level. For the homogeneous case, this level completely defines the future performance, however, in the nonhomogeneous case, the corresponding age reduction, which is called "setting back the clock for the degradation process", is performed.

We employ and develop further the new notion of virtual age by conditioning on the event that its value cannot exceed the age of an item upon failure. We obtain some relevant stochastic comparisons for this conditional virtual age as well as for the corresponding remaining lifetimes with different thresholds and different underlying processes of degradation. The homogeneous and non-homogeneous gamma processes are chosen for illustration. Another practically sound option would be the inverse Gaussian process.

The generalized renewal processes are studied based on the suggested imperfect repair modelling. By analogy with Finkelstein (2007) and also using the alternative new proof, under certain assumptions, we show that the cycles of this process are stochastically decreasing (deterministic threshold) and stochastically decreasing or increasing (depending on the monotonicity properties of the failure rate that describes the distribution of the failure threshold).

The future research can focus on various generalizations of the developed methodology. For instance, deterioration processes with dependent increments can be considered. On the applications side, relevant imperfect preventive maintenance models could be also discussed. For instance, Theorem 4 describes aging in the process of imperfect repairs. Therefore, this result can be useful for optimal maintenance modeling when perfect repair is executed after the k-th imperfect repair.

Appendix

We first need some definitions and preliminary lemmas on multivariate stochastic orders.

Definition A1. Let $\mathbf{X}=(X_1,X_2,...,X_n)$ and $\mathbf{Y}=(Y_1,Y_2,...,Y_n)$ be two n dimensional random vectors such that

$$P(\mathbf{X}\in U)\le P(\mathbf{Y}\in U),\text{ for all upper sets } U\subseteq \mathbf{R}^n.$$

Then $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the usual stochastic order, denoted by $\mathbf{X}\le_{st}\mathbf{Y}$.

Lemma A1. (Shaked and Shanthikumar, 2007). *Let* $\mathbf{X}=(X_1,X_2,...,X_n)$ *and* $\mathbf{Y}=(Y_1,Y_2,...,Y_n)$ *be two* n *dimensional random vectors. If*

$$X_1 \le_{st} Y_1$$

and, for $i=2,3,...,n$*,*

$$(X_i \mid X_1=x_1,X_2=x_2,...,X_{i-1}=x_{i-1}) \le_{st} (Y_i \mid Y_1=y_1,Y_2=y_2,...,Y_{i-1}=y_{i-1}) \text{ whenever } x_j \le y_j,\ \ j=1,2,...,i-1,$$

then $\mathbf{X} \le_{st} \mathbf{Y}$.

Lemma A2. (Shaked and Shanthikumar, 2007). *Let* $\mathbf{X}=(X_1,X_2,...,X_n)$ *and* $\mathbf{Y}=(Y_1,Y_2,...,Y_n)$ *be two* n *dimensional random vectors. If* $\mathbf{X} \le_{st} \mathbf{Y}$ *and* $\mathbf{g}:\ \mathbf{R}^n \to \mathbf{R}^k$ *is any* k *dimensional increasing (decreasing) function, for any integer* k*, then the* k *dimensional vectors* $\mathbf{g}(\mathbf{X})$ *and* $\mathbf{g}(\mathbf{Y})$ *satisfy* $\mathbf{g}(\mathbf{X}) \le_{st} (\ge_{st}) \mathbf{g}(\mathbf{Y})$.

Theorem A1. *Suppose that the distribution* $G(x)$ *is DFR, i.e.,* $\lambda_L(x)$ *is decreasing in* x*. Then,*

$$L_1 \le_{st} L_2 \le_{st} L_3 \le_{st} \dots .$$

Proof.

First, we show $L_1 <_{st} L_2$. Observe that

$$P(L_1 > w) = \bar{G}(w) = \exp\{-\int_0^w \lambda_L(x)dx\},$$

whereas

$$P(L_2 > w \mid L_1 = w_1) = \bar{G}(w \mid qw_1) = \exp\{-\int_0^w \lambda_L(qw_1 + x)dx\},$$

and $P(L_2 > w) = E[P(L_2 > w \mid L_1)]$. When $\lambda_L(x)$ is decreasing,

$$P(L_1 > w) \le P(L_2 > w \mid L_1 = w_1), \text{ for all realization of } w_1 > 0, \text{ for all } w > 0. \quad \text{(A1)}$$

This implies that $P(L_1 > w) \le E[P(L_2 > w \mid L_1)] = P(L_2 > w)$, for all $w > 0$.

Observe that

$$P(L_3 > w \mid L_1 = w_1, L_2 = w_2) = \bar{G}(w \mid q(qw_1 + w_2)) = \exp\{-\int_0^w \lambda_L(q(qw_1 + w_2) + x)dx\}.$$

Thus, $P(L_3 > w \mid L_1 = w_1, L_2) = \exp\{-\int_0^w \lambda_L(q^2 w_1 + qL_2 + x)dx\} \ge \exp\{-\int_0^w \lambda_L(qL_2 + x)dx\}$, and

$$P(L_3 > w \mid L_1 = w_1) = E_{(L_2 \mid L_1 = w_1)}[P(L_3 > w \mid L_1 = w_1, L_2)] \ge E_{(L_2 \mid L_1 = w_1)}[\exp\{-\int_0^w \lambda_L(qL_2 + x)dx\}].$$

On the other hand, $P(L_2 > w) = E[\exp\{-\int_0^w \lambda_L(qL_1 + x)dx\}]$. As $L_1 \le_{st} (L_2 \mid L_1 = w_1)$, for all realization of $w_1 > 0$, due to Lemma 1-(i),(ii),

$$P(L_3 > w \mid L_1 = w_1) \ge E_{(L_2|L_1=w_1)}[\exp\{-\int_0^w \lambda_L(qL_2 + x)dx\}]$$

$$\ge E[\exp\{-\int_0^w \lambda_L(qL_1 + x)dx\}] = P(L_2 > w)\text{, for all realization of } w_1 > 0\text{, for all } w > 0,$$

which implies that $P(L_3 > w) \ge P(L_2 > w)$, for all $w > 0$.

In general, for $n = 3,4,...,$

$$P(L_n > w \mid L_1 = w_1, L_2 = w_2, ..., L_{n-1} = w_{n-1}) = \exp\{-\int_0^w \lambda_L(q^{n-1}w_1 + q^{n-2}w_2 + ... + qw_{n-1} + x)dx\}$$

and

$$P(L_{n+1} > w \mid L_1 = w_1, L_2 = w_2, ..., L_n = w_n) = \exp\{-\int_0^w \lambda_L(q^n w_1 + q^{n-1}w_2 + ... + qw_n + x)dx\}.$$

Thus,

$$P(L_n > w) = E_{(L_1, L_2 ..., L_{n-1})}[\exp\{-\int_0^w \lambda_L(q^{n-1}L_1 + q^{n-2}L_2 + ... + qL_{n-1} + x)dx\}]$$

and

$$P(L_{n+1} > w \mid L_1 = w_1) = E_{(L_2, L_3 ..., L_n | L_1 = w_1)}[P(L_{n+1} > w \mid L_1 = w_1, L_2, L_3, ..., L_n)]$$

$$\ge E_{(L_2, L_3 ..., L_n | L_1 = w_1)}[\exp\{-\int_0^w \lambda_L(q^{n-1}L_2 + ... + q^2 L_{n-1} + qL_n + x)dx\}].$$

Now we stochastically compare $(L_1, L_2, ..., L_{n-1})$ with $(L_2, L_3, ..., L_n \mid L_1 = w_1)$. From (A1),

$$L_1 \le_{st} (L_2 \mid L_1 = w_1).$$

Also,

$$P(L_i > w \mid L_1 = w_1, L_2 = w_2, ..., L_{i-1} = w_{i-1})$$

$$= \exp\{-\int_0^w \lambda_L(q^{i-1}w_1 + q^{i-2}w_2 + ... + qw_{i-1} + x)dx\},$$

whereas

$$P(V_i > w \mid L_1 = w_1, V_1 = v_1, V_2 = v_2, ..., V_{i-1} = v_{i-1})$$

$$= \exp\{-\int_0^w \lambda_L(q^i w_1 + q^{i-1}v_1 + ... + qv_{i-1} + x)dx\},$$

where $V_j = L_{j+1}$, $j = 1,2,...$. Then, we have

$$\exp\{-\int_0^w \lambda_L(q^{i-1}w_1 + q^{i-2}w_2 + ... + qw_{i-1} + x)dx\} \le \exp\{-\int_0^w \lambda_L(q^i w_1 + q^{i-1}v_1 + ... + qv_{i-1} + x)dx\},$$

for all $w > 0$, whenever $w_j \le v_j$, $j = 1,2,...,i-1$, which implies that

$$(L_i \mid L_1 = w_1, L_2 = w_2, ..., L_{i-1} = w_{i-1}) \le_{st} (V_i \mid L_1 = w_1, V_1 = v_1, V_2 = v_2, ..., V_{i-1} = v_{i-1}), \quad \text{(A2)}$$

whenever $w_j \le v_j$, $j = 1,2,...,i-1$. Inequality (A2) holds for all $i = 2,...,n-1$. Thus, by Lemma A1, $(L_1, L_2, ..., L_{n-1}) \le_{st} (V_1, V_2, ..., V_{n-1} \mid L_1 = w_1)$ and thus

$$(L_1, L_2, ..., L_{n-1}) \leq_{st} (L_2, L_3, ..., L_n \mid L_1 = w_1). \tag{A3}$$

Observe that

$$h(w_1, w_2, ..., w_{n-1}) = \exp\{-\int_0^w \lambda_L (q^{n-1} w_1 + q^{n-2} w_2 + ... + q w_{n-1} + x) dx\}$$

is an increasing function of $(w_1, w_2, ..., w_{n-1})$. Then, from (A3) and Lemma A3,

$$P(L_n > w) = E_{(L_1, L_2 ..., L_{n-1})}[\exp\{-\int_0^w \lambda_L (q^{n-1} L_1 + q^{n-2} L_2 + ... + q L_{n-1} + x) dx\}]$$

$$\leq E_{(L_2, L_3 ..., L_n \mid L_1 = w_1)}[\exp\{-\int_0^w \lambda_L (q^{n-1} L_2 + ... + q^2 L_{n-1} + q L_n + x) dx\}] \leq P(L_{n+1} > w \mid L_1 = w_1),$$

for all realization of $w_1 > 0$, for all $w > 0$. This, again implies that

$$P(L_n > w) \leq P(L_{n+1} > w), \text{ for all } w > 0.$$

Therefore, we have shown that $P(L_n > w) \leq P(L_{n+1} > w)$, for all $n = 1, 2, ...$

■

Theorem A2. *Suppose that the distribution $G(x)$ is DFR, i.e., $\lambda_L(x)$ is decreasing in x. Then,*

$$Z_1 \leq_{st} Z_2 \leq_{st} Z_3 \leq_{st} \dots .$$

Proof.

It is obvious that $Z_1 \leq_{st} Z_2$ is obvious. In the proof of Theorem A1, we have shown that

$$(L_1, L_2, \cdots, L_n) \leq_{st} (L_2, L_3, \cdots, L_{n+1} \mid L_1 = w_1).$$

Furthermore,

$$h(w_1, w_2, ..., w_{n-1}) = q^{n-1} w_1 + q^{n-2} w_2 + \cdots + q\ w_{n-1}$$

is an increasing function of $(w_1, w_2, ..., w_{n-1})$. Then, from Lemma A2,

$$q^{n-1} L_1 + q^{n-2} L_2 + \cdots + q\ L_{n-1} \leq_{st} (q^{n-1} L_2 + q^{n-2} L_3 + \cdots + q L_n \mid L_1 = w_1)$$

$$\leq_{st} (q^n w_1 + q^{n-1} L_2 + q^{n-2} L_3 + \cdots + q L_n \mid L_1 = w_1),$$

which implies that $Z_n \leq_{st} (Z_{n+1} \mid L_1 = w_1)$, for all realizations of w_1. Therefore, we have

$$Z_n \leq_{st} Z_{n+1},\ n = 1, 2, ...$$

■

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