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Fast iterative solvers for geomechanics in a commercial FE code

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ABSTRACT

There is a pressing need to improve the feasibility of three-dimensional finite element (FE) methods applied to many problems in civil engineering. This is particularly the case for static analyses in geotechnical engineering: ideally, models would be 3D, follow the actual geometry, use non-linear material formulations and allow simulation of construction sequences, and all of this with a reasonable degree of accuracy. One major obstacle to improvements in this regard is the difficulty in solving of the set of (linearised) algebraic equations which arises from a typical discretisation approach. Very large systems become cumbersome for direct techniques to solve economically. This paper describes the incorporation of iterative (rather than direct) solution techniques, developed through University research, into commercial FE software for geotechnics.

1 Background

Finite element modelling is widely used in civil engineering and particularly in geotechnics where typical problems include the prediction of movements due to construction operations such as tunnelling and retaining wall erection. Accurate predictions require highly refined models in 3D and the use of (often) complex nonlinear elasto-plastic material models for soils, which take account of the stress histories of all points in the soil domain. These requirements lead to large FE models (i.e. many degrees of freedom) and many load steps, due to the nonlinearity. Highly efficient storage schemes and optimized direct solvers (e.g. [1]) have improved the feasibility of solving very large systems, however, the memory requirements of direct solvers will eventually place a limit on what is possible using the computing resources typically available to industry, e.g. a desktop workstation and a maximum runtime of 12 hours. Potentially, iterative solvers provide a solution to this problem since they require a fraction of the storage of a direct solver for a large system, however convergence then becomes the major issue.

Nonlinear FE analysis leads to the solution of a series of linear systems each

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{1}$$

involving the structure stiffness matrix $\mathbf{K}$ and load vector $\mathbf{f}$ which must be solved for the nodal displacements $\mathbf{u}$. These equations are usually solved with well-established incremental or iterative solution techniques (such as the modified Euler or Newton-Raphson methods). The choice of iterative method depends partly on the nature of the linear system. For symmetric systems the conjugate gradient (CG) method can be used while for unsymmetric systems the Generalized Minimal Residual (GMRES) or Biconjugate Gradient Stabilized (BiCGSTAB) methods can be used [2]. For all of these Krylov methods, convergence depends crucially on the eigenvalue spectrum of the coefficient matrix in the linear system. Convergence can be improved by preconditioning: theoretically, this is equivalent to replacing $\mathbf{K}$ by a preconditioned matrix $\mathbf{P}^{-1}\mathbf{K}$ whose eigenvalue spectrum facilitates faster iterative convergence. Considerable research has been carried out in recent years to find inexpensive ways to generate suitable preconditioners $\mathbf{P}$ for a variety of problems with different types of $\mathbf{K}$.

Recent collaboration between Augarde and Ramage has led to the development of preconditioned iterative approaches for systems arising from FE models incorporating geotechnical materials [3]. In particular, a new element-based version of a class of matrix reduction techniques was developed which proved to be extremely effective for unstructured elasto-plastic footing problems when tested on a sequence of model problems [4]. The approach takes each element stiffness matrix $\mathbf{K}_e$ and applies two reductions as follows (using the terminology of [5]). Firstly a D-reduction of $\mathbf{K}_e$ is obtained by neglecting any connections between degrees of freedom of different types (i.e. links between x and y degrees of freedom in 2D) with the resulting matrix known as the *separate displacement component* of $\mathbf{K}_e$. Secondly a *C-reduction* of $\mathbf{K}_e$ is carried out by lumping any positive off-diagonal entries in a row of $\mathbf{K}_e$ onto the diagonal. The resulting matrix is factorised using the approach described in [6] and a preconditioning matrix produced. Full details of this preconditioner (named DC-EMF) are given in [4].

This paper describes the incorporation and initial testing of iterative solvers (including DC-EMF) into commercial software through collaboration between academics in mathematics and engineering and a commercial software house (OASYS Limited (`http://www.oasys-software.com`); OASYS is part of Ove Arup & Partners Ltd (`http://www.arup.com`)). The project is funded through the Collaborative Training Account of Strathclyde University. The collaboration provides both a strong "testbed" for these solvers on problems not usually available to academics, and access to state of the art research by industry very soon after initial development in academia.

2 Results

The aim of this section is to compare the performance of the recently developed element-based preconditioning procedure in [4] against other iterative solution methods and the existing direct solver in the FE software SAFE. Due to space limitations, full details of the solvers and preconditioners are not included here. All calculations used versions of SAFE with different routines called for different solvers. A number of example linear elastic geotechnical test problems were analysed varying the material properties (i.e. ν, Poisson's ratio), mesh refinement and type of solver. The role of Poisson's ratio in the convergence of iterative solvers has been shown to be of major significance even compared to onset of yielding in elasto-plastic analyses [4]. The solution techniques used were as follows:

1. Element based CG solver with no preconditioner. Element-based methods do not require the storage of any *assembled* matrices. One of the main attractions of these methods is that they allow relatively straightforward parallel implementation of iterative solvers.
2. Element-based, with diagonal preconditioner. $\mathbf{P} = \mathrm{diag}(\mathbf{K}) = \sum_{\mathrm{e}=1}^{\mathrm{E}} \mathbf{C}_{\mathrm{e}}^{\mathrm{T}} \,\mathrm{diag}(\mathbf{K}_{\mathrm{e}})\, \mathbf{C}_{\mathrm{e}}$ where $\mathbf{C}_e$ is the Boolean matrix linking entries in the element stiffness matrix $\mathbf{K}_e$ with the global stiffness matrix, and E is the number of elements.
3. Element based with "element-by-element" preconditioner following [7]. $\mathbf{P} = \mathbf{D}^{1/2}\left[\prod_{e=1}^{E}\mathbf{L}_e\right]\left[\prod_{e=1}^{E}\mathbf{D}_e\right]\left[\prod_{e=E}^{1}\mathbf{L}_e^T\right]\mathbf{D}^{1/2}$ where $\mathbf{L}_e$ and $\mathbf{D}_e$ are lower triangular and diagonal matrices found from a factorization of a regularized version of $\mathbf{K}_e$.
4. Assembled with SSOR preconditioning.
5. Assembled with Incomplete Cholesky preconditioning.
6. Assembled with Incomplete Cholesky preconditioning including thresholding and dropping.

7. Oasys SAFE's existing direct solver.

8. Element-based with DC-EMF preconditioning following [4].

The simple rigid footing problem is studied (geometry as shown in Fig. 1) where only one-half of the domain (shown shaded) is modelled due to symmetry. Firstly, structured meshes of $N \times N$ 8-noded

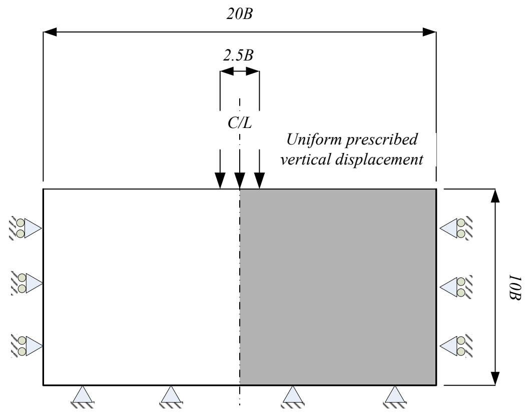
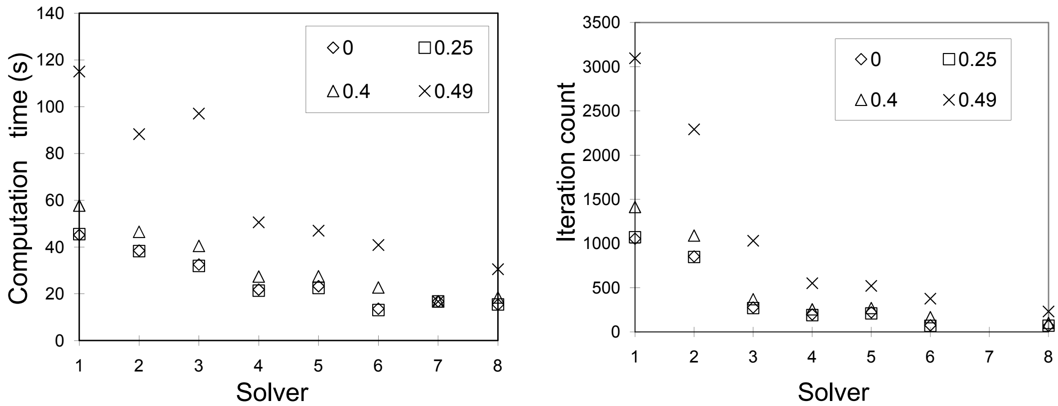


Figure 1: Geometry of the footing problem

quadrilateral elements with uniform material properties were used. Figure 2 shows computation times and iteration counts for a mesh with $N = 64$, (i.e. 4096 elements; 12545 nodes). (Iteration counts are not included for solver 7 as this is the direct solver). With lower mesh densities the direct solver wins as expected confirming that direct solvers are the preferred choice for 2D analyses.

The iteration count (and hence computation time) increase as Poisson's ratio increases as the linear system to be solved becomes less positive definite; the off-diagonal terms become larger relative to the diagonal terms. Figure 2 shows that the DC-EMF solver is a competitive iterative solver as compared to the other iteartive solvers tested. Figure 3(a) compares the direct solver with the DC-EMF solver for this problem for two values of Poisson's ratio in which r_t is the ratio of computation time for the DC-EMF solver against the direct solver. Figure 3(a) indicates that as problem size increases the DC-EMF solver starts to beat the direct solver for this problem.

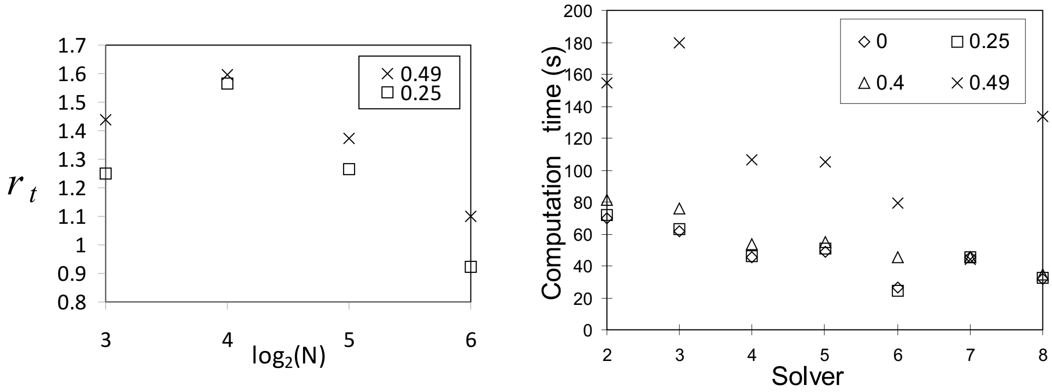


(a) Computation time

(b) Iteration count

Figure 2: Results for the uniform material problem, $N = 64$

When the mesh is unstructured (containing 6435 elements and 19652 nodes), for the same problem (maintaining uniform material properties) the results are excellent for DC-EMF with $\nu = 0, 0.25, 0.4$ but less promising for $\nu = 0.49$. Figure 3(b) shows that for $\nu = 0.49$ the DC-EMF solver suddenly performs much worse than other iterative solvers and the direct solver.



(a) Comparison of DC-EMF solver and direct solver for varying problem size

(b) Results for unstructured mesh

Figure 3:

3 Discussion

The initial results presented above appear to show that iterative solution methods can begin to challenge direct solvers for large FE problems although care must be taken in their use as minor changes to the model can affect convergence. The maximum size of problem studied here is, however, relatively small compared to a reasonable mesh in 3D, therefore these conclusions may be even more significant for 3D, where direct solvers will definitely struggle, and makes the development of robust iterative solvers more of a priority. It is rare for academics to have the opportunity to implement their ideas rapidly into commercial software and all parties hope this collaboration can continue to the benefit of all.

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