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Modeling criminal careers of different levels of offence

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A B S T R A C T

We set up and analyse a mathematical model, the Serious Crime Model, which describes the interaction of mild and serious offenders and potential criminals. However we get more complete results for a simpler version of this model, the Mild Crime Model, with no serious offenders. For the full Serious Crime Model there are two key parameters R_0^1 and R_0^2 corresponding to the basic reproduction number in the mathematics of infectious diseases, which determine the behaviour of the system. For the Simpler Mild Crime Model there is just one such parameter R_0^1. Both forward and backward bifurcation can occur for this second model with two subcritical non-trivial equilibria possible for $R_0^1 < 1$ in the backwards case. For backwards bifurcation there is another threshold value R_0^* such that the upper non-trivial equilibrium is unstable for $R_0^1 < R_0^*$ and stable for $R_0^1 > R_0^*$. For forwards bifurcation there is a second additional threshold value R_0^{**} such that the stability of the unique non-trivial equilibrium switches from unstable to stable as R_0^1 passes through R_0^{**}. At the end we return to the full Serious Crime Model and discuss the behaviour of this model. The results are meaningful and interesting because in all of the other epidemiological and sociological models of which we are aware, analogous thresholds to R_0^* and R_0^{**} do not exist. For forwards bifurcation the unique non-trivial equilibrium, and for backwards bifurcation with two subcritical endemic equilibria the higher non-trivial equilibrium, are also usually always locally asymptotically stable. So our models exhibit unusual and interesting behaviour.

1. Introduction

There has been a great deal of work done on using mathematical models to describe how infectious diseases spread. However similar techniques are applicable to many other areas, for example the spread of rumours or the spread of technological innovations in a populated area. In this paper we attempt to apply these techniques to another area: the spread

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of criminal activity in a population [1–4,6]. We explore the questions of how criminal offences spread and what levels of criminal offences spread, and what the mechanisms of the transmission of violence across individuals and groups are [7,8,13]. We point out that there is a predisposing factor to violence that would result in an act of violence, and that given the highly contagious nature of the causes and effects of violence it is appropriate to refer to it as a type of infectious disease [19,20,37]. In this way, we assume that some behaviours, including some types of violence, may spread in ways analogous to the contagious spread of infectious diseases, a process that has been characterized as *behavioural contagion* [14,17,21,30,31,34]. Our model is more closely aligned to the idea of contagion than with infectious disease transmission as such. The arguments are based on models that have been developed to describe how infectious diseases spread in populations. Thus, our analysis serves to highlight the similarities between violence and disease and to violence being viewed as a disease process. We begin by surveying existing work in the area.

2. Literature review

There has recently been an increase of interest in using mathematical modelling to predict the spread of criminal behaviour. Useful reviews of this area are provided by Lacey et al. [15] and Sookanan et al. [43]. They point out that a variety of approaches have been taken to model how crime spreads through society. The first of these is agent-based models which simulate agents which can transition to different states [5,18]. They also discuss differential equation models similar to the ones which will be discussed here [28,44]. Then they progress to game-theoretic models, including McCalla et al. [25] who study the effect of individual networks and common values on criminal coalitions in an adversarial evolutionary game, and Short et al. [39] who also use an adversarial evolutionary game. They next look at stochastic simulation models [9,27,42]. A final approach is continuous stochastic models [38,40,41].

As our model uses differential equations we focus on differential equation models which have been used to model criminal activity. Nuno et al. [28] describe a differential equation model, with individual terms resembling the terms in a predator-prey model for owners, X, criminals, Y and security guards, Z. A simpler version of this model is considered initially which assumes that Y plus Z is a constant. A bifurcation study is then performed which shows the onset of bistability. For the complete model a number of bifurcations occur. This model is different than the one we shall consider and not directly comparable with the bifurcation diagrams normally found in epidemic models, which we shall discuss in our paper.

Sookanan et al. [44] describe a model that has more similarity to our model and uses techniques from mathematical epidemiology to study the spread of gang membership through a community by interactions amongst the individual members in a gang and the general population. There are three equilibrium states, of these two contain no individuals in the gang. They adapt the SIR (susceptible-infected-removed) model from mathematical epidemiology. There are four compartments, namely non-susceptible community members, N, community members who are potential gang members, S, committed gang members, G and ex-gang members who are in prison, R. An expression for the basic reproduction number of this model, which consists of a system of four differential equations, is derived. Although the figure shows a similar bifurcation diagram to ones found in mathematical epidemiology when there are two types of infectious individuals, or two different types of susceptible individuals, the stability pattern is similar to the usual type with the lower subcritical endemic equilibrium being unstable and the upper subcritical endemic equilibrium being stable for $R_0 < 1$.

Sookanan et al. [43] review mathematical models of crime. They look at models involving differential equations. They discuss the work of Sah [36] where the idea of peer pressure being put on an individual to commit a crime is introduced. This is a concept we also introduce in our model. Sookanan et al. then discuss the model of Ormerod et al. [29] who introduce a differential equation model with similarities to models found in mathematical epidemiology with three categories: individuals susceptible to crime, criminals, and individuals not susceptible to crime. Ormerod et al. also discuss an extension of this basic model. They fit the model to data and perform a stability analysis numerically from the Routh-Hurwitz conditions for the stability of the unique positive model solution. It is possible for the unique endemic equilibrium to exist and be unstable for $R_0 > 1$ for some parameter values. Sookanan et al. then finally discuss agent-based models.

Lacey and Tsardakas [15] discuss a mathematical model using minor and serious criminal activity, similar to the ideas in our paper. ρ_1 and ρ_2 are respectively the number of minor and serious criminals in an area at time t. The behaviour of criminals is driven by the attractiveness of the area which consists of an intrinsic (given) part $A(t)$ and a dynamic part $B(t)$. They describe a system of three differential equations for $\rho_1,\ \rho_2$ and B. In the case that the attractiveness $A(t)$ is constant there is a unique endemic equilibrium which has positive values for ρ_1, ρ_2 and B which they then solve numerically. After that they introduce stochastic noise into the model, solve it numerically and discuss parameter estimation.

Raimundo et al. [33] consider the implications for crime-control policies of criminal career dynamics again using two epidemiologically-based models. In the first of these, the partially contagious criminality model, the variables are people who are not currently offenders but susceptible to offending, people in prison for the first time, people who have been in prison once who are again susceptible to re-offending, people who are currently in prison for the second or subsequent time, and people who have been in prison twice or more but are still susceptible to re-offending. They consider the first model and a modification of it where the flow of people into offending is a function solely of their contact with the incarcerated. For both models the existence and uniqueness of equilibria and their stability and backwards and forwards bifurcation are examined but the bifurcation diagrams have their normal stability pattern. For forward bifurcation the one and only one criminality-present steady-state is always locally asymptotically stable when it exists, and for backwards bifurcation for $R_0 < 1$ the higher criminality-present steady-state is always locally asymptotically stable and the lower criminality-present steady-state is unstable.

In more recent work Mebratie and Dawed [26] again motivated by mathematical models in epidemiology consider a mathematical model of crime dynamics with a disease awareness program, so that the susceptibles are split into aware and unaware individuals, and aware susceptibles have a decreased rate of disease transmission compared with unaware ones. The population also contains compartments of prisoners, non-criminals and police. An expression for the basic reproduction number is derived. There is always a crime-free equilibrium point and if $R_0 > 1$ there is a unique crime-persistent equilibrium point. The unique crime-persistent equilibrium point is locally asymptotically stable when it exists.

Terefe [45] analyses a mathematical model for the diffusion of violence. There are four compartments: susceptible individuals, violence-exposed individuals, violently infectious individuals, negotiators and reconciled individuals. The results are qualitatively similar to Membratie and Dawed [26]. An expression for the basic reproduction number is derived. An equilibrium point free from violence is always possible and there is a violence-persistence equilibrium point which exists for $R_0 > 1$ and is locally stable when it exists.

Teklu and Terefe [46] investigate the presence of violence and racism (separately and together) in a population using infectious disease dynamical methods. Each of violence and racism are considered similar to chronic infections. In the full model the population classes are susceptible to either violence or racism, infected by violence, infected by racism, co-infected with violence and racism, recovered from racism, recovered from violence, violence and racism co-infected and recovered from co-infected. They also study the violence submodel and the racism submodel. In the violence submodel they find the basic reproduction number, the violence-free steady-state and the violence-persistence steady-state. Similarly for the racism submodel they find the basic reproduction number, the racism-free steady-state and the racism-persistence steady-state. Teklu and Terefe study the stability of the racism-free steady-state but not the racism-persistent steady-state in the racism submodel and the violence-free but not the violence-persistent steady-state in the violence submodel. In the full model the co-existence-free steady-state is the steady-state with no violence and no racism present. For this model they discuss the local stability of the co-existence-free steady-state.

Maturu et al. [24] discuss tactics for managing criminal offences in developing countries using a differential equation model. There are a total of five subpopulations, the unemployed, the susceptible, the exposed to crime, the active criminal population, people in vocational training and lastly the employed population. A mathematical model based on ones found in mathematical epidemiology is formulated and analysed. There is a crime-free equilibrium and an expression is derived for the crime basic reproduction number. There is also a unique crime-persistent equilibrium point. The model can display backward bifurcation as R_0 passes though one with the usual stability patterns of one stable higher and one unstable lower crime-persistence equilibrium. The paper concludes with some simulations.

So in summary we have seen that a variety of mathematical models exist in the literature that examine criminality. However as far as we are aware for all of them that examine the bifurcation structure of the equilibria, for forwards bifurcation the unique crime-persistence equilibrium is locally asymptotically stable when it exists, and for backwards bifurcation the upper crime-persistence equilibrium is locally asymptotically stable and the lower crime-persistence equilibrium is unstable. The model of Ormerod et al. [29] has forward bifurcation with a unique crime-persistent equilibrium for $R_0 > 1$ which they showed analytically can be unstable for some parameter values (although they say that it is not possible to deduce the fact that the crime-persistent equilibrium is unstable from their simulations). However they did not look at the bifurcation structure.

3. Our model

Since prevalence of a given type of criminal activity may change the propensity of an individual to engage in that same behaviour, we assume that exposure to violence does not lead immediately to the expression of violence but is one factor that may lead susceptible individuals to commit either mild or serious offences. The underlying principle is that whenever the various parameters combine to produce a situation where an offender (infective person) co-opts or incites to commit an offence (infects), on average, more than one person during the course of their criminal career (infectivity), then sustained criminal activity (an epidemic) is predicted to occur.

In what follows, the model presented here assumes that not all susceptible individuals exposed to violence go on to commit offences as well as that the number of susceptible individuals exposed to infection is much higher than those actually presenting with a disease. We first present the Serious Crime Model, a mathematical model that describes segments of the criminal activity based on participation in crime, incarceration, and recidivism within a population according to different levels of offence. However, due to the complexity of our Serious Crime Model, which is a challenging system to analyse, we will analyse a simplified model. At present, we seek to gain as much mathematical information as we can from the simplified model presented in this paper. Moreover, we intend to address each of the ongoing challenges of the Serious Crime Model in future work. Finally, over time, the focus of this work turned away from a macro-level model of crime to micro-level analyses of so-called criminal careers and how offending is affected over time [23]. The trajectories of individual participation in crime begin with initiation and continue until desistance. Therefore the simplified model is a theoretical framework that that aims to help the authorities develop schemes to control illegal activity.

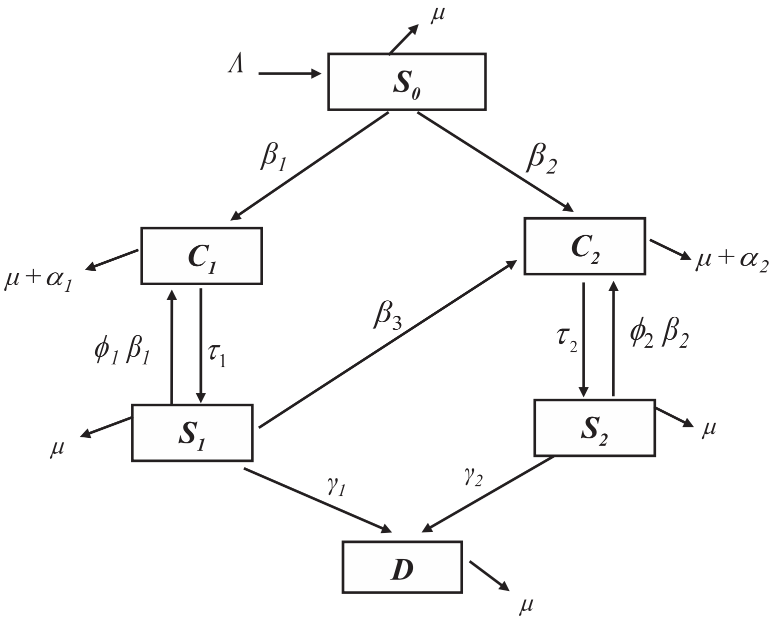

Fig. 1. Diagrammatic representation of the Serious Crime Model (1).

Table 1
Classes of individual in the model.

Variables	Description
S_0	People vulnerable to criminal activity who are not currently involved in illegal acts
C_1	Criminally active mild offenders
S_1	People who have committed a mild offence in the past and are vulnerable to criminality but not currently involved in illegal acts
C_2	Criminally active serious offenders
S_2	People who have committed a serious offence in the past and are vulnerable to criminality but not currently involved in illegal acts
D	Offenders who have completely ceased illegal activity (criminal desisters)

4. The serious crime model formulation

Firstly, we assume that when an individual is charged with a crime, he or she can fall into two categories according to the different levels of offence: mild and serious offences. The difference between the two is evident in definition, severity and sentences. Hence, mild offences are defined as a crime where no injury or force is used on another person and they are often measured in terms of loss to the victim or economic damage. They are most often some type of theft or larceny (bribery, prostitution, tax crimes, fraud, alcohol and drug-related crimes, etc). Serious offences, on the other hand, are considered offences against a person. This means that another person's physical body was harmed during the committing of a crime (robbery, false imprisonment, domestic violence, assault, homicides, sexual abuse, etc).

$N(t)$ is used to denote the entire number of individuals under consideration. These individuals are either susceptible (S), criminally active (C) or desisters (D). All of susceptible individuals, criminally active individuals and model constants are then specified according to offences: mild offences ($i = 1$) and serious offences ($i = 2$). Thus, we will call S_1 a mild ex-offender, i.e., individuals who are ex-offenders with a lifetime history of mild offences who are again susceptible to crime, and S_2 a serious ex-offender, i.e., individuals who are ex-offenders with a lifetime history of serious offences who are again susceptible to crime. Similarly, we will call C_1 criminally active mild offenders, and C_2 criminally active serious offenders.

The diagrammatic representation of the Serious Crime Model is illustrated in Fig. 1. See Table 1 for the meanings of the classes and Table 2 for the meanings of the parameters. Susceptible individuals are generated by recruitment through births and immigration at a rate Λ. Hence, the susceptible population exposed to violence (affected by the disease) will be compartmentalized into categories. Let S_0 be the fraction of susceptible non-offenders who have a criminal propensity, so they are not criminally active but susceptible to crime; S_1 be the fraction of mild ex-offenders and S_2 be the fraction of serious ex-offenders, both again not criminally active but, once more, susceptible to crime. Similarly, criminally active individuals with a lifetime history of serious offences were compared with criminally active individuals having mild offences. We suppose that the criminally active population is divided into two categories, C_1 mild offenders and C_2 serious offenders.

Table 2
Constants of the Model.

Parameters	Description
Λ	Rate of recruitment of susceptible individuals
$\beta_i\ (i=1,2)$	Co-optation rate
β_3	Incitement rate between C_2 and S_1
μ	Natural mortality rate
α_1	Additional mortality rate when in C_1
α_2	Additional mortality rate when in C_2
τ_1	Rate of ceasing criminal activity when in C_1
τ_2	Rate of ceasing criminal activity when in C_2
ϕ_1	Factor of perpetration of violence when in S_1
ϕ_2	Factor of perpetration of violence when in S_2
γ_1	Rate at which individuals in C_1 give up illegal activity
γ_2	Rate at which individuals in C_2 give up illegal activity

The average length of continuous criminal activity, that is, the rate at which inmates move from state C_i to S_i $(i=1,2)$ is given by $1/\tau_i,\ i=1,2$, with $1/\tau_2 < 1/\tau_1$. We also define β_1 and β_2 to be the co-optation rates. These also include individuals who had ceased criminal activity but then came back to crime so that β_1 and β_2 are the rates that individuals perform illegal acts depending on influence by mild and serious criminally active individuals, respectively.

It is also worth noting that in criminal law, incitement is the act of using coercion and other tactics to induce or encourage a person to commit a criminal offence when the potential criminal expresses a desire not to go ahead. The essence of the law of incitement is that a person (the ”inciter”) urges another person or persons (the ”incitee(s)”) to commit a criminal offence [12]. In this way, we assume that if the criminally active individual induces or encourages another offender to commit a serious offence, that otherwise they would not do, the crime has gone from mild to serious offence, not from serious to mild.

Building on the above, we then suppose that mild ex-offenders S_1 can be incited (“reinfected by another virus”) to commit a serious offence by the criminally active serious offenders C_2. In other words, an inciter C_2 urges the incitees S_1 to commit a serious offence. On the other hand, serious ex-offenders S_2, are not encouraged to change their personality and behaviour patterns because they have a tendency to repeat the same offence types in successive crimes as a way of life.

Hence β_3 describes the rate with which serious offenders C_2 contribute to the incitement of the class S_1 to commit serious crimes. The parameter β_3 could be defined as a modification parameter that measures the efficacy of incitement in inducing individuals to commit serious offences. This could be analogous to the antibody-dependence enhancement (ADE) of virus infection. ADE is a disease spreading process causing individuals with their secondary infection to be more infectious (for example serious offenders) than during their first infection (for example mild offenders) by a different disease serotype or strain [35].

In this way and motivated by analogous scenarios where viral production is increased during a secondary infection due to ADE, and violence is increased during the co-optation and recidivism process, we introduce parameters analogous to ADE to increase the probability of an ex-offender committing more offences. Thus, we define ϕ_1 to be the relative increase in the likelihood of the chances of an individual being co-opted to commit a mild crime on contact with a mild offender due to the individual being contacted having a previous history of mild offending (as opposed to no criminal history). Similarly ϕ_2 is the relative increase in the likelihood of the chances of an individual being co-opted to commit a serious crime on contact with a serious offender due to the individual being contacted having a history of serious offending.

We also suppose that crime prevention programs $(\gamma_i,\ i=1,2)$ may change the underlying thinking about engaging in illegal activity [21]. So we assume that such programs make ex-offenders in the S_1 and S_2 classes cease crime and return to the mainstream society, thus lowering recidivism. Hence let D be those individuals who have given up illegal activity either by themselves or due to intervention programs.

Lastly μ is the background per capita death rate, and α_1 and α_2 are the per capita criminality associated extra death rates (prisoners dying due to diseases, for example AIDS-related effects, violence in prison, taking their own life, unintentional harm to themselves, or another cause associated with incarceration). Also as the classes in the model represent populations all the constants of the model are assumed to be non-negative. Additionally the only heterogeneity in the population is due to criminal activity (none, mild or serious) and that within these classes the population is homogeneous (for example we do not consider differences due to age).

We have chosen to study a simple model using only the two classes of criminally active individuals, mild offences (i =1) and serious offences (i =2). In theory the model could be extended to include more levels of criminal offence, however as we wish to focus on qualitative results we keep the model simple by not doing this. Also we suppose that the crime prevention programs occur both in prison and after a prisoner returns to the community. The focus of crime prevention programs is to prevent prisoners or ex-prisoners from engaging in illegal activity and to stop them from coming back to incarceration. It is possible that some prisoners can be involved in illegal acts soon after coming back into the community, or returning to previous illegal activity [47]. This point is of great interest here.

In summary our ordinary differential equation model for the spread of illegal activity in the vulnerable population is

$$\begin{cases} \frac{dS_0}{dt} = \Lambda - \beta_1 S_0 C_1 - \beta_2 S_0 C_2 - \mu S_0 \\ \frac{dC_1}{dt} = \beta_1 S_0 C_1 + \phi_1 \beta_1 S_1 C_1 - (\mu + \alpha_1 + \tau_1) C_1 \\ \frac{dC_2}{dt} = \beta_2 S_0 C_2 + \phi_2 \beta_2 S_2 C_2 + \beta_3 S_1 C_2 - (\mu + \alpha_2 + \tau_2) C_2 \\ \frac{dS_1}{dt} = \tau_1 C_1 - \phi_1 \beta_1 S_1 C_1 - \beta_3 S_1 C_2 - (\gamma_1 + \mu) S_1 \\ \frac{dS_2}{dt} = \tau_2 C_2 - \phi_2 \beta_2 C_2 S_2 - (\gamma_2 + \mu) S_2 \\ \frac{dD}{dt} = \gamma_1 S_1 + \gamma_2 S_2 - \mu D, \end{cases} \tag{1}$$

with generic initial conditions $S_0(0) \geq 0$, $C_1(0) \geq 0$, $C_2(0) \geq 0$, $S_1(0) \geq 0$, $S_2(0) \geq 0$ and $D(0) \geq 0$. By adding the system (1) we find

$$\frac{dN}{dt} = \Lambda - \mu N - \alpha_1 C_1 - \alpha_2 C_2. \tag{2}$$

So if $\alpha_1 = \alpha_2 = 0$ so that there are no extra deaths of criminally active individuals, the population has a constant immigration rate Λ and a constant per capita death rate μ, in other words $\frac{dN}{dt} = \Lambda - \mu N$. There is a single steady state value $N = \Lambda/\mu$ and it is straightforward to show that for any initial value $N(0)$ the population size ultimately approaches Λ/μ. In the case where α_1 and α_2 are not zero it is a consequence of (2) that $\lim_{t\to\infty} N(t) \leq \Lambda/\mu$.

It is straightforward to show that if (1) has a solution which commences in $\mathbb{R}^6_+$ (all variables greater than or equal to zero) it either approaches, comes into, or stays in the subset $\Omega \subset \mathbb{R}^6_+$ given by

$$\Omega = \Big\{ (S_0, C_1, C_2, S_1, S_2, D) \in \mathbb{R}^6_+ : \\ S_0 + C_1 + C_2 + S_1 + S_2 + D \leq \Lambda/\mu \}. \tag{3}$$

If we consider the initial value problem given by equations (1) for solutions originating in Ω it is straightforward to show the existence of solutions and on a maximal interval there is only one solution [10]. As the solutions stay in Ω they are bounded and thus we have that solutions exist analytically and they make biological and sociological sense [11]. So it is enough to consider the system (1) with initial values in Ω. It is important to highlight that throughout this paper, we refer to system (1) as the Serious Crime Model.

4.1. Analytic strategy

Because the Serious Crime Model is very complicated to analyse we will adapt an analytic strategy of simplifying the model so that an analysis of the simplified model will help us to gain some intuition for the dynamical behaviour of the more complex model. This strategy is to decouple the analysis of the equilibrium points of system (1) hence we relax the assumption that the total criminally active population is composed of both mild (C_1) and serious (C_2) offenders by substituting either $C_2 = 0$ or $C_1 = 0$ into system (1).

In this way, we consider that only one level of violence persists in the criminally active population, which yields a simplified system that retains the key features of the Serious Crime Model. Although a less realistic possibility, understanding the dynamical behaviour of this simplified model is a necessary step to check the possibility of occurrence of the phenomenon of bifurcations in the Serious Crime Model. Finally, it is worth noting that this step is necessary to gain some intuition for the understanding of the outstanding challenges of our system (1), mainly the possible ways that bifurcation may occur. Next, we will carry out the brief qualitative analysis of system (1).

5. Brief analysis of the serious crime model

In what follows there are four different possibilities to consider for criminality for the equilibrium points of system (1), which lead to the following criminality equilibrium points.

Path 1. If $C_2 = 0$ and $C_1 = 0$, then system (1) has criminality-free equilibrium $P_0^* = (S_0^*, 0, 0, 0, 0, 0)$ which indicates that the community is free from all forms of criminality.

Path 2. If $C_2 = 0$ and $C_1 \neq 0$, i.e., $\beta_1 S_0 + \phi_1 \beta_1 S_1 - (\mu + \alpha_1 + \tau_1) = 0$, then $P_1^* = (S_0^*, C_1^*, 0, S_1^*, 0, D^*)$ is a mild criminality equilibrium point of system (1), which indicates that the offenders committed mild crimes only. Hence, we can treat system (1) as a Mild Crime Model.

Path 3. If $C_1 = 0$ and $C_2 \neq 0$, i.e., $\beta_2 S_0 + \phi_2 \beta_2 S_2 - (\mu + \alpha_2 + \tau_2) = 0$, then system (1) has $P_2^* = (S_0^*, 0, C_2^*, 0, S_2^*, D^*)$ as a serious criminality equilibrium point, which indicates that the offenders committed serious crimes only. Hence, we can treat system (1) as a Serious Crime Only Model.

Path 4. If $C_1 \neq 0$, i.e., $\beta_1 S_0 + \phi_1 \beta_1 S_1 - (\mu + \alpha_1 + \tau_1) = 0$ and $C_2 \neq 0$, i.e., $\beta_2 S_0 + \phi_2 \beta_2 S_2 - (\mu + \alpha_2 + \tau_2) = 0$, then system (1) has a mild-serious criminality equilibrium point $P_3^* = (S_0^*, C_1^*, C_2^*, S_1^*, S_2^*, D^*)$, which indicates that the offenders committed both mild and serious crimes.

Having established the above paths, we begin by analysing the stability of criminality-free equilibrium (Path 1). In the following we also compute for system (1) an explicit expression for the thresholds separating the steady state with no crime and the steady state where crime is endemic between the criminality-free equilibrium and the criminality-endemic equilibria, analogous to the basic reproduction number for infectious diseases R_0 [22,32,48]. We shall see that these thresholds are the same as the threshold of each simplified system.

Some of the key parameters of the system are the rate of co-optation of non-criminals and the rate of incitement of non-criminals to serious criminal activity (β_1, β_2 and β_3) as well as the relative increases ϕ_1 (or ϕ_2) of the chances of committing a mild (or serious) crime on contact with a mild (or serious) offender due to the individual being contacted having a previous history of mild (or serious) offending. We shall investigate how these parameters affect the thresholds and the proportion of criminally active individuals in the system.

5.1. Local stability of criminality-free equilibrium and reproduction numbers

From Path 1 it follows then that equations (1) have a steady state with no crime present given by $P_0^* = (\Lambda/\mu, 0, 0, 0, 0, 0)$. To determine its stability, we look at the Jacobian of equations (1) at P_0^*. So the steady-state P_0^* with no crime present is locally asymptotically stable, if $R_0^i < 1$ with $i = 1, 2$, where R_0^1 and R_0^2 are given by

$$R_0^1 = \frac{\beta_1}{(\mu+\alpha_1+\tau_1)} \frac{\Lambda}{\mu} \tag{4}$$

and

$$R_0^2 = \frac{\beta_2}{(\mu+\alpha_2+\tau_2)} \frac{\Lambda}{\mu}. \tag{5}$$

These are called the Criminality Reproduction Numbers (CRNs). Similarly to epidemic models [22] the CRN R_0^1 (or R_0^2), represents the "average expected number of new offenders originated by a single offender in class C_1 (or C_2), whilst in a criminal career". In practice, one criminally active individual C_1 (or C_2) gets into contact with ex-offenders S_1 (or S_2) and successfully induces R_0^1 (or R_0^2) persons to commit crime. In other words, R_0^1 and R_0^2 are the expected numbers of susceptibles who perform illegal acts due to association with one offender in the class C_1 and C_2, respectively.

Thus, equations (4) and (5) are the thresholds that distinguish the path in which all solutions converge to the criminality-free equilibrium P_0^* from the path in which all solutions converge to P_1^* or P_2^*, as well as P_3^*. In what follows, an important result is then established.

Lemma 1. *Provided $R_0^1 < 1$ and $R_0^2 < 1$, the criminality-free equilibrium P_0^* of the model (1) is locally asymptotically stable, otherwise it is unstable.*

Following terminology of the basic reproduction number for infectious diseases, Lemma 1 implies that is it possible to eliminate the criminality (or disease) from the community when $R_0^1 < 1$ and $R_0^2 < 1$, if the initial size of the criminally active individuals (or infectious individuals) of model (1) are in the basin of attraction of the criminality-free equilibrium P_0^*.

It should be noted however that the equations (4) and (5) for the CRN do not include the factor ϕ_i, $(i = 1, 2)$ of the relative increase in the chances of committing a crime due to the previous offending history of the individuals being contacted or the incitement rate between mild ex-offenders and serious offenders given by β_3, despite the fact that these terms should contribute significantly to the emergence of new offending individuals (C_1 and C_2). Hence, this already suggests that R_0^1 and R_0^2 alone are unable to quantify some key features of the dynamics of the criminality into the community, and is in fact the first sign that bifurcations might be involved. In general, in a dynamical system, when a parameter is varied, then the differential system may change. It can happen that a slight variation in a parameter can have significant impact on the solution: an equilibrium can become unstable and a periodic solution may appear or a new stable equilibrium may appear making the previous equilibrium unstable.

In our simplified criminality models, we will see that the parameters ϕ_i, $(i = 1, 2)$ also play an important role and they are responsible for the presence of qualitative changes in dynamical behavior of each system.

Besides that, when parameter values cross the threshold (bifurcation value), the solution structure changes qualitatively and the simplified system undergoes bifurcation. However, due to the complexity of our simplified systems, the solutions do not have a concise, explicit form, and we can prove the existence and uniqueness of the equilibrium points performing the dynamics only numerically. In the following, we will focus on the pathway when there is no serious offender (Path 2) which leads to a Mild Crime Model. Moreover, we investigate the possibility of backward bifurcation, especially where the criminality-free equilibrium co-exists with two criminality equilibria [32].

Remembering that since we use either $C_2 = 0$ or $C_1 = 0$ in system (1), both simplified criminality models become symmetric and they have, therefore, similar dynamical behaviour.

In what follows, due to the symmetry, the results obtained for the Mild Crime Model can be then translated into the Serious Crime Only Model (Path 3). Finally, the full Serious Crime Model is more difficult to analyse, but a better understanding

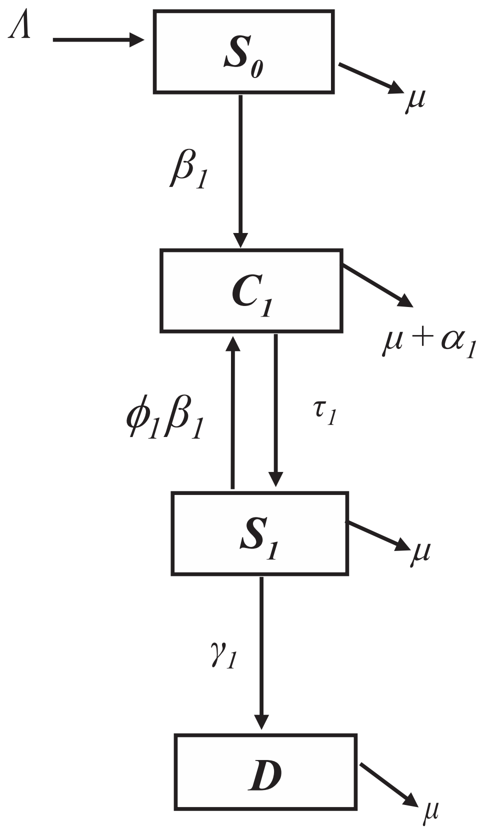

Fig. 2. The flow diagram for the Mild Crime Model (6).

of the dynamic of the simplified models, will provide tools for a better understanding and knowledge of the dynamical behavior of the full Serious Criminality System (Path 4).

6. Analysis of the mild crime model

Firstly, for the sake of simplicity, here and throughout this paper we will use the same notation of variables S_0, C_1, S_1 and D of system (1) in the simplified model. Moreover, the sociological classes given in Table 1 and the model constants given in Table 2 still hold for the Mild Crime Model.

Hence, from Path 2, by taking $C_2 = 0$ into system (1), the simpler more intuitive approach is to examine the Mild Crime Model, given by

$$\begin{cases} \frac{dS_0}{dt} = \Lambda - \beta_1 S_0 C_1 - \mu S_0 \\ \frac{dC_1}{dt} = \beta_1 S_0 C_1 + \phi_1 \beta_1 S_1 C_1 - (\mu + \alpha_1 + \tau_1) C_1 \\ \frac{dS_1}{dt} = \tau_1 C_1 - \phi_1 \beta_1 S_1 C_1 - (\gamma_1 + \mu) S_1 \\ \frac{dD}{dt} = \gamma_1 S_1 - \mu D, \end{cases} \tag{6}$$

with generic initial conditions $S_0(0) \geq 0$, $C_1(0) \geq 0$, $S_1(0) \geq 0$ and $D(0) \geq 0$.

It can be seen that model (6) is much simpler than system (1), but an important (and still complex) model because it gives an understanding of the bifurcation that is likely to occur in the system (1), and provides conclusions that coincide with Path 3. The diagrammatic representation of the Mild Crime Model given by system (6) is shown in Fig. 2 and the sociological classes given in Table 1 and the model constants given in Table 2 still hold. From system (6), evaluated at an equilibrium from the second equation one gets either $C_1 = 0$ or $\beta_1 S_0 + \phi_1 \beta_1 S_1 - (\mu + \alpha_1 + \tau_1) = 0$. For $C_1 = 0$, then model (6) has the criminality-free equilibrium given by $P_0 = (S_0, C_1, S_1, D) = (\Lambda/\mu, 0, 0, 0)$ which indicates that the community is free from criminality. To examine whether the criminality-free steady-state is stable or not the Jacobian matrix of (6) is examined at P_0. However, note that simplified system (6) has the same Criminality Reproduction Number as system (1) which is given by equation (4). In what follows, as we did previously, an important result can be then established.

Lemma 2. *Provided $R_0^1 < 1$, the criminality-free equilibrium P_0 of the model (6) is locally asymptotically stable, otherwise it is unstable.*

We have the following theorem:

Theorem 1. *The equations (6) possess:*

(i) *One and only one steady state greater than zero P_1 if $b_0 < 0$ (i.e., $R_0^1 > 1$);*
(ii) *One and only one steady state greater than zero P_1 if $b_0 = 0$ (i.e., $R_0^1 = 1$) and $b_1 < 0$;*
(iii) *One and only one steady state greater than zero P_1 if $b_0 > 0$ (i.e., $R_0^1 < 1$), $b_1 < 0$ and $b_1^2 - 4b_2b_0 = 0$;*
(iv) *Two positive equilibria, P_1, if $b_0 > 0$ (i.e., $R_0^1 < 1$), $b_1 < 0$ and $b_1^2 - 4b_2b_0 > 0$ with $\phi_1 > \phi_1^{**2}$;*
(v) *No positive equilibrium, otherwise.*

Proof. See Appendix A. □

It should be mentioned that we choose to examine bifurcation as β_1 varies. Moreover as for $\phi_1 > \phi_1^{**2}$ model (6) has two positive equilibria then we can suspect that there is a threshold criterion for parameter β_1 where model (6) could undergo backwards bifurcation. However to find out whether backwards bifurcation can occur in the system of differential equations (6) we must introduce a second subordinate critical value. This will be denoted R_0^{thr}. Because the system of differential equations (6) is too complicated to find the exact formula of the value of R_0^{thr} by hand, its value can be found by computational methods.

We next examine the behaviour of the feasible (greater than zero) steady states P_1^- and P_1^+ of system (6) in terms of both parameters ϕ_1 and β_1.

Theorem 2. *The stability of a positive equilibrium P_1 (P_1^- or P_1^+) is determined by the roots of the third degree characteristic equation $F_3(\lambda) = 0$ where $F_3(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Here*

$$\begin{aligned}
a_3 &= 1,\\
a_2 &= \beta_1 C_1(1+\phi_1) + 2\mu + \gamma_1,\\
a_1 &= \mu(\mu+\gamma_1) + \beta_1 C_1\left[k_1 + 2\mu + \gamma_1 - \phi_1(1-\phi_1)\left[R_0^1 k_4 - \frac{1}{\eta_1}k_1(R_0^1 - 1)\right]\right] + \phi_1\beta_1^2 C_1^2,\\
a_0 &= \beta_1 C_1[2\phi_1\beta_1 C_1(\mu+\alpha_1) + (\mu+\gamma_1)(\mu+\alpha_1) + \gamma_1\tau_1(1-\phi_1) + \phi_1\mu k_1(1-R_0^1)],
\end{aligned} \tag{7}$$

with $k_4 = \frac{(\mu+\alpha_1)C_1}{\mu(\phi_1-\phi_1^c)}$.

Proof. See Appendix B. □

To verify the stability of the equilibrium point P_1 we look at the Routh-Hurwitz stability criterion which establishes that if $a_0 > 0$, $a_1 > 0$, $a_2 > 0$ and $a_1a_2 - a_0 > 0$ hold then $J(P_1)$ given by (B.1) has eigenvalues with real part less than zero and so the steady state is locally asymptotically stable [16,33]. Because all of the model constants are greater than or equal to zero and $C_1 > 0$, using (7) we deduce that a_2 always exceeds zero. However, the conditions $a_0 > 0$, $a_1 > 0$ and $a_1a_2 - a_0 > 0$ are complex, thereby making it impossible to verify these conditions analytically. Therefore, the stability analysis will be performed only numerically.

After extensive numerical simulations we conjecture that, although necessary, the Routh-Hurwitz stability criterion is not sufficient to guarantee the stability of the positive criminality endemic equilibrium P_1^+ of system (6). Moreover, the positive criminality endemic equilibrium P_1^- of system (6) is always unstable.

We would emphasise that the purpose of these simulations is to understand the theoretical behaviour of the model and we do not claim to be using real parameter values in our study. Note also that we follow Raimundo et al. [32] and take the total population recruitment rate Λ to be equal to the per capita death rate μ. At first sight this may appear not to be a realistic choice as it implies that the criminality-free (disease-free) equilibrium point has population size $\Lambda/\mu = 1$. However we would like to note that by changing the units in which the population is measured this restriction is not as limiting as it may appear. If we take a model with general population recruitment rate Λ and per capita death rate μ, then the size of the crime-free (or the disease-free) equilibrium population is Λ/μ. If we change the units of the population and take Λ/μ as the unit of population size then with this unit we have $\Lambda = \mu$. However were we to do this then we would also need to change the units of the β's (β_1, β_2 and β_3) so that they also use this unit. But the point is that the restriction $\Lambda = \mu$ is not as limiting as it may first appear.

Fig. 3 and Fig. 4 show the plotting of the Routh-Hurwitz conditions for $a_0 > 0$, $a_1 > 0$ and $a_1a_2 - a_0 > 0$ at P_1^+ as R_0^1 increases. To illustrate we chose some values for ϕ_1, namely $\phi_1 = 2.0$ and $\phi_1 = 4.0$ (see Fig. 3) and $\phi_1 = 12.311$ and $\phi_1 = 61.411$ (see Fig. 4). The solid curve represents where the conditions $a_0 > 0$, $a_1 > 0$ and $a_1a_2 - a_0 > 0$ hold. The dashed curve represents where those conditions do not hold. Note that P_1^+ will be locally asymptotically stable if $a_0 > 0$, $a_1 > 0$, $a_2 > 0$ and $a_1a_2 - a_0 > 0$ are simultaneously satisfied.

Now that we have examined when the steady states of the system of differential equations (6) can be stable, as a next step we want to check if system (6) undergoes backward and forward bifurcations. This is a subject of major importance. We then explore the system behaviour when $R_0^1 < 1$ and $R_0^1 > 1$.

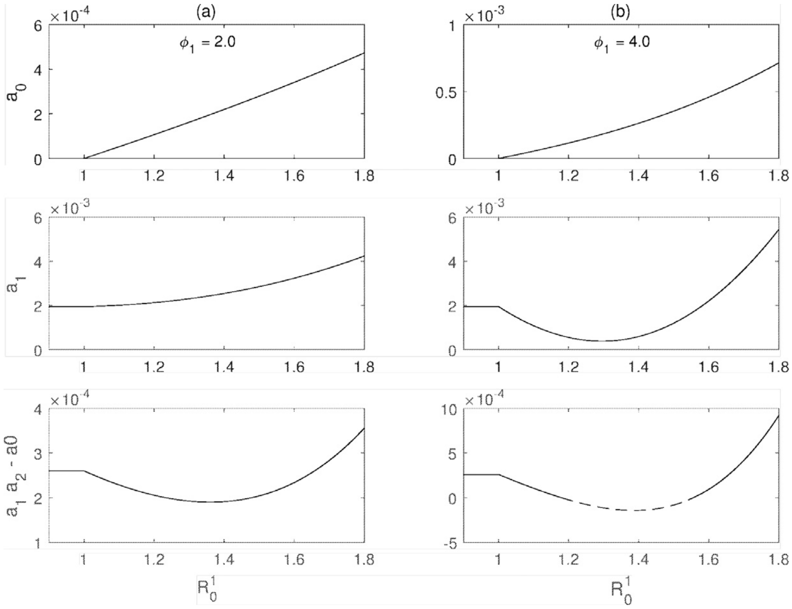

Fig. 3. Plot of the Routh-Hurwitz stability conditions for the 3rd degree polynomial (B.2) for $\phi_1 = 2.0$ and $\phi_1 = 4.0$ as R_0^1 increases. The solid curve represents where the Routh-Hurwitz stability criterion holds, while the dashed curve represents where the criterion does not hold. Parameter values used are as given in Table 3.

Table 3
Constants used in system (1).

Constant	Value
Λ	0.0167 (*per annum*)
β_1	*To be altered* (*per annum*)
β_2	*To be altered* (*per annum*)
β_3	0.15 (*per annum*)
μ	0.0167 (*per annum*)
α_1	0.05 (*per annum*)
α_2	0.1 (*per annum*)
τ_1	0.2 (*per annum*)
τ_2	0.067 (*per annum*)
ϕ_1	*To be altered* (*per annum*)
ϕ_2	*To be altered* (*per annum*)
γ_1	0.1 (*per annum*)
γ_2	0.08 (*per annum*)

Next we will see that falling back into criminality is not solely a function of the starting value of the class C_1 to tell us whether or not backwards bifurcation will occur, there also exist other critical values for the occurrence of both bifurcations. It is important to stress that due to the complexity, these critical values will be calculated only computationally. Doing this our numerical computational results demonstrate the existence of threshold values where model (6) undergoes backward and forward bifurcations.

7. Numerical results

Despite the innocuous form of the equations of system (6), it may be impossible to find exact solutions, P_1^+ and P_1^-. Hence, we resort to numerical simulations to gain some insight into the behaviour of the model (6) as well as the serious model (1). The differential equations are integrated using MATLAB's ODE45 integrator. For the simulations presented here, we use a sample collection of parameters that are similar to those used by Raimundo et al. [32]. These do not necessarily represent realistic values, as at this point our main goal is to examine the behaviour rather than the solutions of systems.

As stated earlier a system that exhibits the *usual* backward bifurcation has two positive equilibria given by the locally asymptotically stable equilibrium (corresponding to the higher solution of our system, C_1^+) and the unstable equilibrium (corresponding to the smaller solution of our system, C_1^-) which coexist with the equilibrium with no crime present when R_0^1 is immediately less than one. This characteristic, however, may not always be the case for system (6) as shown in Fig. 5.

In Fig. 5 the profile of the proportion of mild offender incarcerated and criminally active populations C_1^- and C_1^+ are plotted as functions of R_0^1 with some values of the parameter ϕ_1. Furthermore, C_1^- and C_1^+, which in turn correspond to

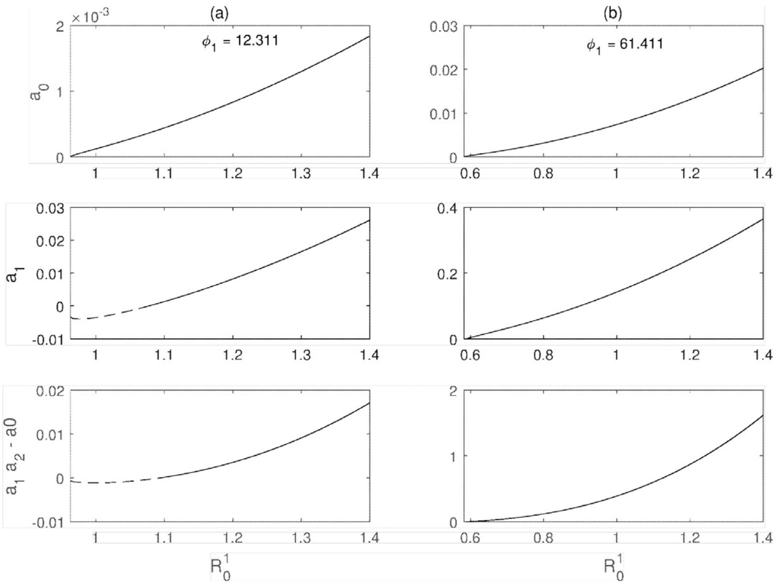

Fig. 4. Plot of the Routh-Hurwitz stability conditions for the 3rd degree polynomial (B.2) for (a) ϕ_1=12.311 and (b) $\phi_1 = 61.411$ as R_0^1 increases. The solid curve represents where the Routh-Hurwitz stability criterion holds, while the dashed curve represents where the criterion does not hold. Parameter values used are as given in Table 3.

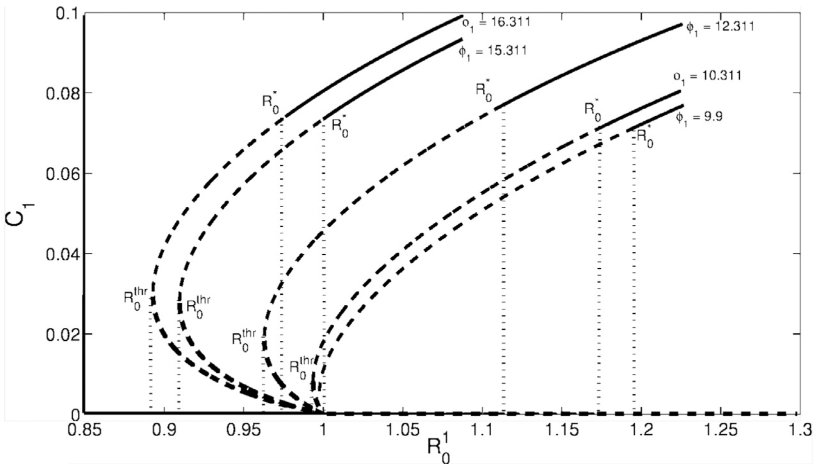

Fig. 5. Qualitative illustration of *special* backward bifurcation for model (6) where β_1 is chosen as a bifurcation parameter. For $\phi_1 = 9.9$, $R_0^* = 1.19$ and $R_0^{thr} = 0.997572$. For $\phi_1 = 10.311$, $R_0^* = 1.17$ and $R_0^{thr} = 0.993615$. For $\phi_1 = 12.311$, $R_0^* = 1.11$ and $R_0^{thr} = 0.962917$. For $\phi_1 = 15.311$, $R_0^* = 1.00195$ and $R_0^{thr} = 0.9096323$. For $\phi_1 = 16.311$, $R_0^* = 0.976363$ and $R_0^{thr} = 0.8928309$. The dashed curve represents the instability and the solid curve represents the stability of P_1^+ (greater, thin line), P_1^- (smaller, thick line) and P_0. Table 3 lists the numerical evaluations used for the model parameters.

the smaller and higher positive criminality equilibria of system (6) P_1^- and P_1^+, respectively, coexist with the criminality-free equilibrium P_0.

Nevertheless, despite system (6) satisfying one of the classical requirements of the occurrence of backward bifurcation, it is important to note from Figure 5 that in addition to the bifurcation threshold, R_0^{thr}, there is another critical value, namely $R_0^1 = R_0^*$, where the system (6) changes its behavior, that is, its stability. Thus, $R_0^{thr} < R_0^1 < 1$ is no longer sufficient to guarantee the phenomenon of backward bifurcation with a stable and an unstable steady state at each of these R_0^1 values.

For example, for $\phi_1 \geq 9.9$, it can be observed from Fig. 5 that the system of differential equations (6) demonstrates backwards bifurcation. To see this, first note that P_1^+ is stable for $R_0^1 > R_0^* > 1$. Note again that only for $\phi_1 > 15.311$, is P_1^+ stable for $R_0^* < 1$. However, if $R_0^{thr} < R_0^1 < R_0^*$, then P_1^+ is unstable, and these characteristics are not indicative of *usual* backward bifurcation (in other words one stable and one unstable equilibrium for $R_0^1 < 1$ when two endemic equilibria exist). We conjecture therefore that in such a case system (6) requires another critical value to have a backward bifurcation, namely $\phi_1 = \phi_1^{back}$.

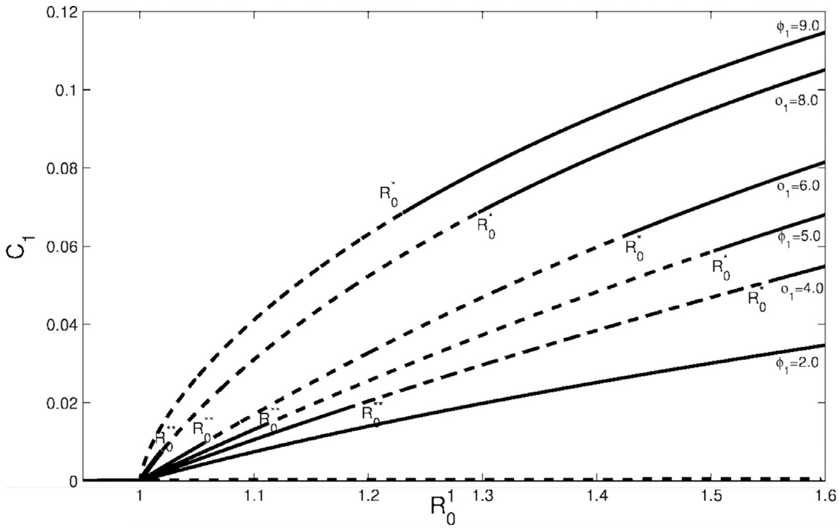

Fig. 6. For 2.0 $< \phi_1 \leq 9.0$, model (6) undergoes *special* forward bifurcation. P_1^+ is locally asymptotically stable for $R_0^1 > R_0^*$ and $R_0^1 < R_0^{**}$ and unstable for $R_0^{**} < R_0^1 < R_0^*$. For $\phi_1 = 2.0$ model (6) undergoes *usual* forward bifurcation and P_0 is locally asymptotically stable if $R_0^1 < 1$. For $R_0^1 > 1$, P_0 loses its stability and P_1^+ becomes locally asymptotically stable. The dashed curve represents the instability and the solid curve represents the stability of P_1^+ and P_0. Parameter values used are as given in Table 3.

Table 4
Usual **backward** bifurcation, $\phi_1 = 61.411$ ($\phi_1 > \phi_1^{back}$).

β_1 (per year)	R_0^1	C_1^-	C_1^+	stable
0.155221	0.582	0	0	P_0
0.199221	0.747	0.0067	0.122	P_1^+ or P_0
0.259221	0.972	0.0003	0.156	P_1^+ or P_0
0.2667	1	0	0.159	P_1^+
0.32692	1.226	0	0.178	P_1^+

Another crucial question is to investigate if system (6) also undergoes foward bifurcation. In such a case, we will see that the classical requirement $R_0^1 > 1$, where P_0 loses stability and P_1^+ becomes stable, is necessary but it is not sufficient.

For example, referring to Fig. 6, it is also apparent that system (6) undergoes the phenomenon of forward bifurcation. As particular examples we illustrate forward bifurcation for various values of ϕ_1 between $\phi_1 \cong 2.0$ and $\phi_1 \cong 9.33$. However, also note that in addition there are other critical values, namely $R_0^1 = R_0^*$ and $R_0^1 = R_0^{**}$ where the system (6) changes its stability. Hence, $R_0^1 > 1$ is no longer sufficient to guarantee either the phenomenon of *usual* forward bifurcation or the stability of P_1^+. As before, we also conjecture therefore that in such a case system (6) requires another critical value to have a foward bifurcation, namely $\phi_1 = \phi_1^{forw}$.

Finally, it is important to stress that when the aforementioned classical requirements are satisfied, *usual* backward and forward bifurcations are expected to occur, but only for some values of parameter ϕ_1. Therefore, we now state the following.

Lemma 3. *The model (6) exhibits,*

(i) *Usual backward bifurcation whenever* $\phi_1 > \phi_1^{back}$.
(ii) *Usual forward bifurcation whenever* $\phi_1 < \phi_1^{forw}$.
(iii) *Otherwise, system (6) does not exhibit the usual stability bifurcation pattern.*

Table 4 indicates the existence of two positive real solutions for equation (A.2), namely C_1^+ and C_1^-, when $R_0^1 < 1$. Translating it into the equilibrium values of system (6) this corresponds to two equilibria, confirming that system (6) undergoes *usual* backward bifurcation with one criminality equilibrium P_1^+ (which corresponds to the higher equilibrium C_1^+), another criminality equilibrium P_1^- (which corresponds to the smaller equilibrium C_1^-) and the criminality-free equilibrium P_0. In such a case, whenever $\phi_1 > \phi_1^{back}$, P_1^- is unstable while the stability either of P_1^+ or P_0 will depend on the initial condition of system (6). We chose $\phi_1 = 61.411$ because this phenomenon occurs for very large values of the parameter ϕ_1. In particular, in a neighborhood of $\phi_1 = 61.411$ is where the phenomenon of *usual* backward bifurcation starts to take place. Unfortunately, the critical value ϕ_1^{back} cannot be determined analytically, so we carried out this task using numerical simulations.

Similarly, the phenomenon of forward bifurcation is shown in Table 5. In such a case the criminality-free equilibrium P_0 is locally asymptotically stable for $R_0^1 \leq 1$. If R_0^1 increases a little the equilibrium value of C_1^+ will also increase. If the initial value of C_1 sits close to the region of attraction of the positive endemic equilibrium P_1^+ and R_0^1 passes through the critical

Table 5
Usual **forward** bifurcation, $\phi_1 = 1.5$ ($\phi_1 < \phi_1^{forw}$).

β_1 (per year)	R_0^1	C_1^-	C_1^+	stable
0.266698	0.9999925	0	0	P_0
0.266699	0.9999963	0	0	P_0
0.2667	1.0	0	0	P_0
0.266701	1.0000037	0	0.00000028	P_1^+
0.266702	1.0000075	0	0.00000056	P_1^+

Table 6
Special **backward** bifurcation, $\phi_1 = 9.4$ ($\phi_1 < \phi_1^{back}$).

β_1 (per year)	R_0^1	C_1^-	C_1^+	stable
0.266685	0.999944	0	0	P_0
0.266687	0.999951	0.00052	0.001097	P_1^+ or P_0
0.266696	0.999985	0.00012	0.001508	P_1^+ or P_0
0.2667	1.0	0	0.001627	P_1^+
0.266725	1.000094	0	0.002157	P_1^+
0.28	1.049869	0	0.030648	limit cycle
0.33	1.237345	0	0.073818	P_1^+ (stable focus)

point $R_0^1 = 1$ the limiting value of the trajectory will suddenly jump from being close to $C_1 = 0$ to being close to C_1^+ (which corresponds to P_0 being unstable). Hence for $R_0^1 > 1$ the criminality-free equilibrium P_0 loses its stability and the criminality endemic equilibrium P_1^+ becomes stable. In particular, as we mentioned earlier the phenomenon of usual forward bifurcation starts to take place at $\phi_1 = 2.0$ (see Fig. 6).

Finally it is interesting to compare these critical values specified above. If $\phi_1 < \phi_1^{forw}$ (*usual* forward bifurcation) and $R_0^1 < 1$ we expect C_1, the level of criminality in the population, to ultimately die out. For $\phi_1 > \phi_1^{back}$ (*usual* backward bifurcation) and $R_0^{thr} < R_0^1 < 1$ the limiting value of the amount of criminality C_1 can either be zero or a higher value according to the initial conditions of the system (6). Moreover, it should be noticed that R_0^{thr} decreases as ϕ_1 increases. Also for R_0^1 fixed smaller levels of endemic criminality are observed for smaller values of ϕ_1 and conversely a greater crime prevention effort would be needed for larger values of ϕ_1.

We are interested in the points where $\phi_1 > \phi_1^{forw}$ and $\phi_1 < \phi_1^{back}$, which lead to the appearance of an unusual phenomenon of stability patterns in the bifurcation. Moreover, keeping in mind the existence of R_0^* and R_0^{**}, throughout this paper we will refer to this phenomenon as the *special* forward and *special* backward bifurcations. As seen in Fig. 5, when $R_0^1 < 1$, the model (6) has two positive criminality endemic equilibria and only one when $R_0^1 > 1$ which is the signature of a backward bifurcation. In addition, after extensive numerical simulations it can be seen that model (6) undergoes *special* backwards bifurcation for values of ϕ_1 in the region $\phi_1^0 \le \phi_1 \le \phi_1^{back}$ where $\phi_1^0 \approx 9.33$ and $\phi_1^{back} \approx 61.411$ (see Table 6 and Table 4). Similarly, as seen in Fig. 6, it can also be seen that model (6) undergoes *special* forward bifurcation for $\phi_1^{forw} \le \phi_1 \le \phi_1^0$ where $\phi_1^{forw} \approx 2.0$ (see Table 7 and Table 5). These restrictions are needed so that we can show that the behaviour of the differential equations (6), including whether or not the steady states are stable, depend not only on the starting values of the differential equations but also on R_0^1 and ϕ_1. Moreover, we conjecture therefore that R_0^*, R_0^{**}, ϕ_1^{forw} and ϕ_1^{back} play an important role because it would appear that they are also responsible for the presence of the *special* backward and forward bifurcations.

Fig. 7 exhibits the *special* backward phenomenon for $\phi_1 = 16.311$ as β_1 increases. As an example, for $\beta_1 = 0.2602$ (per year), where $R_0^{thr} < R_0^1 < R_0^*$, both positive criminality endemic equilibria P_1^- and P_1^+ are unstable. The Jacobian matrix (B.1) at P_1^+ has one real negative eigenvalue plus two complex conjugate eigenvalues with positive real part while at P_1^- the eigenvalues are real and one of them is positive. In contrast, for $\beta_1 = 0.2609$ (per year), where $R_0^* < R_0^1 < 1$, the Jacobian matrix (B.1) at P_1^+ has one negative real eigenvalue plus two complex conjugate eigenvalues with negative real part and system (6) converges to a stable limit cycle. In such a case the positive criminality endemic equilibrium P_1^- is always unstable (see Fig. 5).

It should be mentioned that we considered the same initial condition to the system (6) for both cases (a) and (b) of Fig. 7 to show that R_0^* and ϕ_1 are also responsible for the presence of the *special* backward bifurcation. Moreover, independently of the initial condition of system (6), P_0 is always locally asymptotically stable if $R_0^{thr} < R_0^1 < R_0^*$ and P_1^+ is always locally asymptotically stable if $R_0^1 > R_0^* > 1$. On the other hand, if $R_0^* < R_0^1 < 1$, then P_0 and P_1^+ will both be locally asymptotically stable and the limiting behaviour of the system (6) will depend on its initial condition.

However, the phenomenon of *special* backward bifurcation changes when ϕ_1 decreases and gets closer to $\phi_1^0 \approx 9.33$ where the system (6) changes from *special* backward to *special* forward bifurcation.

Up to now we have looked at finding the equilibrium points and examining how the paths of (6) behave in the locality of the equilibrium points. This yields clues as to the potential behaviour of the other paths, particularly if they approach the equilibrium points sufficiently closely. An additional factor which may effect how the paths behave is if one of them traces

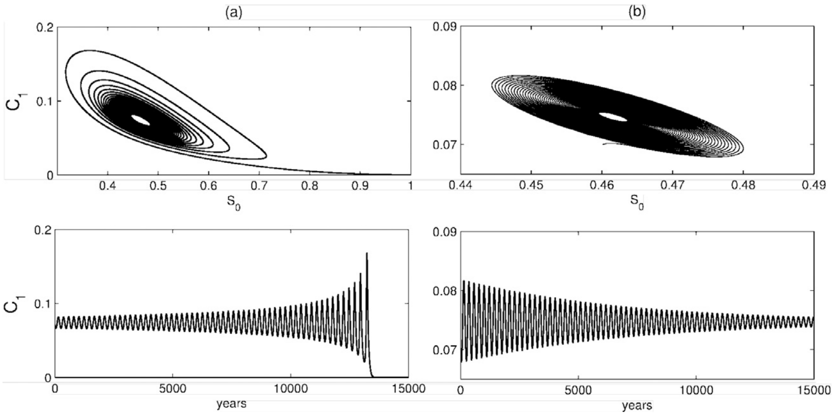

Fig. 7. *Special* backward bifurcation for $\phi_1 = 16.311$ ($\phi_1 < \phi_1^{back}$), $R_0^* = 0.976363$ and $R_0^{thr} = 0.8928309$. In both cases, (a) and (b), the initial conditions are the same. (a) $\beta_1 = 0.2602$ (per year), $R_0^1 = 0.975628$, $C_1 = 0$, P_0 is asymptotically stable. (b) $\beta_1 = 0.2609$ (per year), $R_0^1 = 0.978253$, system (6) converges to a stable limit cycle. Parameter values used are as given in Table 3.

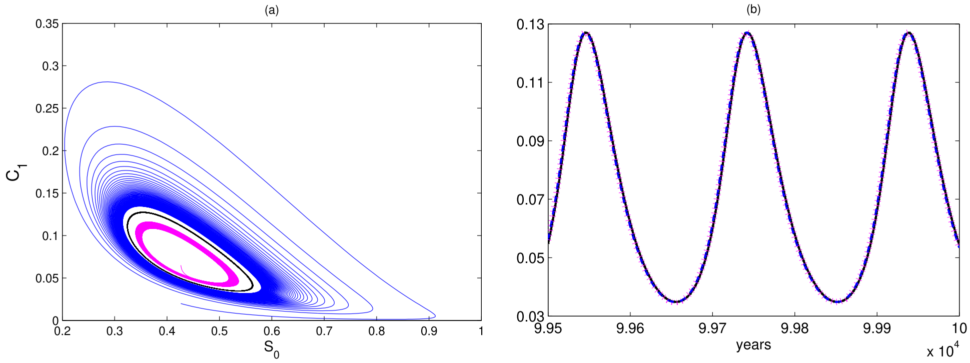

Fig. 8. *Special* backward bifurcation when $\phi_1 = 10.311$ and $R_0^1 = 1.176$. (a) Inner limit cycle with two complex eigenvalues with positive real part (the trajectory starting nearest the centre, the magenta color in the online version); outer limit cycle with two complex eigenvalues with negative real part (the trajectory starting on the outside, the blue color in the online version); stable limit cycle (the trajectory dividing these two cases, the black color in the online version). (b) Both inner and outer spirals approach the closed orbit. Parameter values used are as given in Table 3. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

out a closed path, let K denote such a path. If such a closed path exists, the nearby paths should exhibit similar behaviour to K. The paths close to K can either spiral in towards K, or spiral further out from K, alternatively they could be closed curves with a certain period. In the event that K is an isolated closed path it is called a limit cycle. The most significant type of limit cycle is the stable limit cycle where all paths close to K approach K.

In this way, for illustrative purposes only, Fig. 8 shows these periodic solutions of system (6) which correspond to a closed orbit in the phase plane for $\phi_1 = 10.311$, $\beta_1 = 0.313599578$ (per year) and $R_0^1 = 1.1758514$. The value of β_1 was chosen in the neighborhood of the critical value where the real part of the eigenvalues changes sign when it passes through zero, that is, where the pairs of complex conjugate eigenvalues of the Jacobian matrix associated to system (6) change from positive to negative real part. In this case, the criminality-free equililibrium is always unstable. Fig. 8a shows the stable limit cycle (the trajectory dividing the other two cases, the black color in the online version), the nearby inner (the trajectory starting nearest the centre, the magenta color in the online version), and outer (the trajectory starting on the outside, the blue color in the online version) curves spiral towards this cycle on both sides. Fig. 8b shows how the inner and outer spirals approach the closed orbit. In such a case, both equilibria P_1^+ and P_0 are unstable. Along the same lines, and to gain more insights for the underlying dynamics of the phenomenon of *special* forward bifurcation, a qualitative illustration is given by Fig. 9 according Table 7.

We are interested in the points where the stability of the equilibria change. Although it was shown numerically that subcritical endemic equilibria can appear at $R_0^1 = R_0^{thr}$ and forward or backward bifurcation at $R_0^1 = 1$, this phenomenon has a different interpretation because there are other critical values for R_0^1, namely R_0^{**} and R_0^*, and ϕ_1, namely ϕ_1^{forw} and ϕ_1^{back}.

Although, numerous researchers have attempted to include the recidivism in the spread of criminality, none of these has explored its role in the formation of *special* forward and backward bifurcations as we do. Backward bifurcation still has major implications for infectious diseases, since control programs based on reducing R_0^1 below unity may be ineffective given the disease might be able to persist indefinitely under such conditions, as a result of reinfection (this is still a controversy).

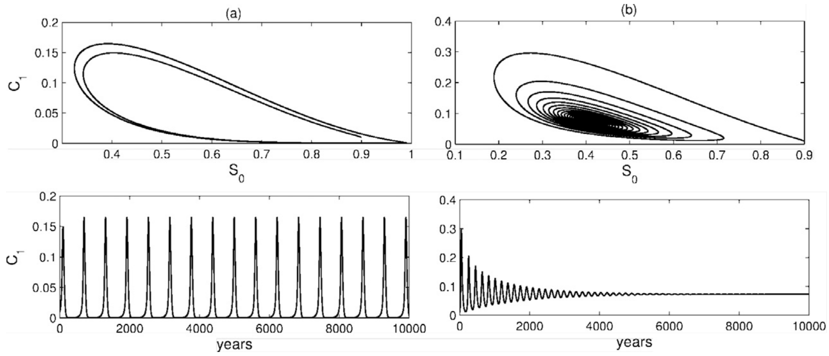

Fig. 9. *Special* forward bifurcation when $\phi_1 = 9.3$. (a) $\beta_1 = 0.28$ (per year), system (6) has two complex eigenvalues with positive real part and converges to a stable limit cycle; (b) $\beta_1 = 0.33$ (per year), system (6) has two complex eigenvalues with negative real part and converges to a stable focus. In both cases, (a) and (b), the initial conditions are the same. Parameter values used are as given in Table 3 and Table 7.

Table 7
Special **forward** bifurcation $\phi_1 = 9.3$ ($\phi_1 > \phi_1^{forw}$).

β_1 (per year)	R_0^1	C_1^-	C_1^+	stable
0.266685	0.999944	0	0	P_0
0.266687	0.999951	0	0	P_0
0.266696	0.999985	0	0	P_0
0.2667	1.0	0	0	P_0
0.266725	1.000094	0	0.000889	P_1^+
0.28	1.049869	0	0.029564	limit cycle
0.33	1.237345	0	0.07281	P_1^+ (stable focus)

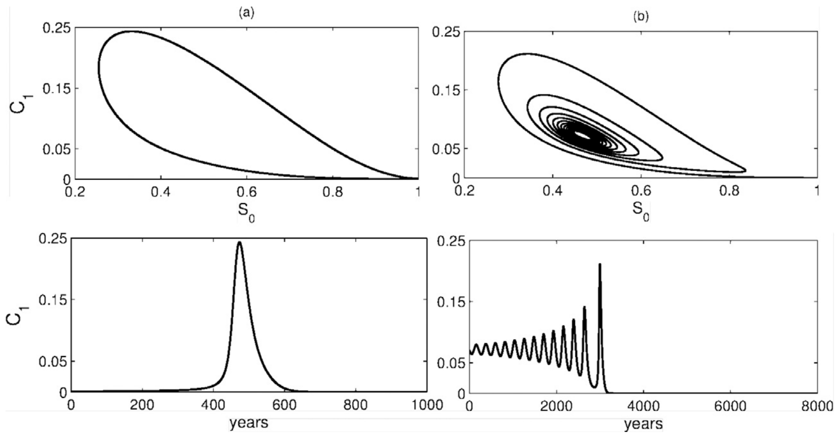

Fig. 10. *Special* backward bifurcation for system (6) when $\phi_1 = 15.311$, $\beta_1 = 0.2649942$ (per year), $R_0^{thr} = 0.909633$ and $R_0^1 = 0.993604$. (a) The initial condition is close to P_0. (b) The initial condition is close to P_1^+. System (6) has a unique locally asymptotically stable criminality-free equilibrium P_0. Parameter values used are as given in Table 3.

However, in our model we also have the parameter ϕ_1 which can work as a recidivism prevention. Certainly this non-intuitive possibility will have to be taken into account if programs are designed to prevent vulnerable individuals from relapsing into crime. In the cases that we have just considered the crime prevention programs will be a function of the starting values of the various subpopulations (including C_1), $\phi_1 > \phi_1^{forw}$ and $\phi_1 < \phi_1^{back}$ to indicate the possibility of both *special* forward and *special* backward bifurcations.

Thus, we can use our analysis to evaluate the relative effectiveness of various intervention strategies with respect to parameter ϕ_1, and determine the necessary level of incitement to reduce or eradicate the criminal offences.

For example if β_1 and ϕ_1 are fixed such that $R_0^{thr} < R_0^1 < 1$ and $\phi_1 > \phi_1^{**2}$ it is still possible to eradicate the recidivism of C_1 if the initial condition sits close to the attraction region of the positive equilibrium P_1^+ or the criminality-free equilibrium P_0.

Fig. 10 shows the effort to avoid the recidivism prevalence and provides an example of this attraction region for an arbitrary choice of the parameters β_1 and ϕ_1, namely $\beta_1 = 0.2649942$ (per year) and $\phi_1 = 15.311$. In Fig. 10a. the initial condition $y_0 = (S_0, C_1, S_1, D) = (0.999, 0.001, 0, 0)$ is close to P_0, while in Fig. 10b. the initial condition $y_0 = (S_0, C_1, S_1, D) =$

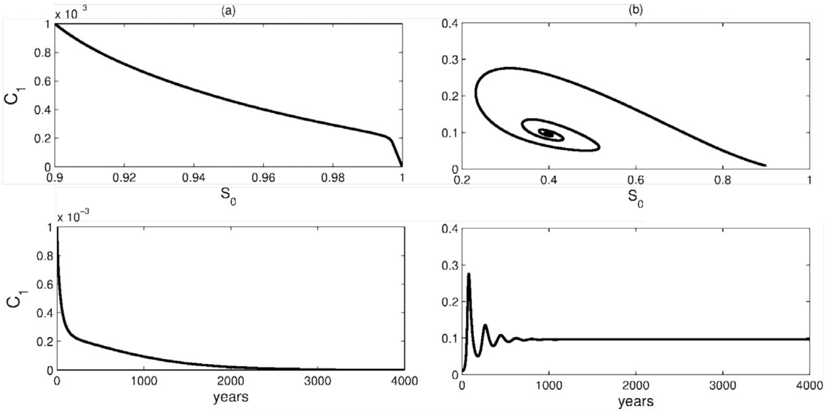

Fig. 11. *Special* backward bifurcation for system (6) when $\phi_1 = 19.311$, $\beta_1 = 0.2649942$ (per year), $R_0^{thr} = 0.84717$ and $R_0^1 = 0.993604$. The initial condition of system (6) is close to P_0. (a) $y_0 = (S_0, C_1, S_1, D) = (0.9, 0.001, 0, 0)$ (b) $y_0 = (S_0, C_1, S_1, D) = (0.9, 0.01, 0, 0)$. Parameter values used are as given in Table 3.

$(0.467786, 0.07, 0.035181, 0.210663)$ is close to P_1^+. At P_0, all eigenvalues of the Jacobian matrix associated to system (6) are real negative, at P_1^- we have one real positive eigenvalue, at P_1^+ we have pairs of complex conjugate eigenvalues with positive real part. Hence, system (6) undergoes a *special* backward bifurcation where two unstable positive criminality endemic equilibria annihilate each other leaving the criminality-free equilibrium as the only locally asymptotically stable equilibrium independently of the initial conditions of system (6).

In summary we report the following implications that arise from previous analysis:

(i) For $\phi_1 = 15.311$ and $\beta_1 \leq 0.24259791$ (per year), i.e., $R_0^1 < 0.90963$. In this case $\phi_1 < \phi_1^{**2}$, $b_1 < 0$, $b_1^2 - 4b_2b_0 < 0$ and $b_0 > 0$, such that the equation (A.2) has pairs of positive complex conjugate solutions, that is, $C_1 = 0$ is a unique solution for system (6). Hence, model (6) has a unique locally asymptotically stable criminality-free equilibrium P_0.

(ii) For $\phi_1 = 15.311$ and $0.24259791 < \beta_1 < 0.2667$ (per year), i.e., $0.90963 < R_0^1 < 1$. In this case, $\phi_1 > \phi_1^{**2}$, $b_1 < 0$ and $b_1^2 - 4b_2b_0 > 0$ and $b_0 > 0$, such that the equation (A.2) has two positive real solutions C_1^- and C_1^+. Thus, system (6) has two positive criminality equilibria, namely P_1^- (smaller) and P_1^+ (higher). Since at P_1^+, two eigenvalues are complex conjugate eigenvalues with positive real part, while at P_1^- one eigenvalue has a positive real part, both equilibria are unstable. Hence, model (6) has a unique locally asymptotically stable criminality-free equilibrium P_0.

Hence, for $\phi_1 = 15.311$ and 0.24259791 (per year)$< \beta_1 < 0.2667$ (per year) system (6) undergoes a *special* backward bifurcation where the criminality-free equilibrium is a unique locally asymptotically stable independently of the initial conditions of system (6). As an example, see Fig. 10.

In contrast, with increasing values of ϕ_1 such that $\phi_1 >> \phi_1^{**2}$ system (6) still has two positive criminality equilibria, namely P_1^- (smaller) and P_1^+ (higher), but the eigenvalues of the Jacobian matrix associated to system (6) change from real to complex conjugate. Moreover, the real part of the complex conjugate eigenvalues changes from positive to negative values. An example, for $\phi_1 = 19.311$, as shown in Fig. 11:

(i) At P_1^-, when 0.226003 (per year) $< \beta_1 < 0.2667$ (per year) one eigenvalue is always real and positive, so P_1^- is unstable.

(ii) At P_1^+, when 0.226003 (per year) $< \beta_1 < 0.226191$ (per year) we have two real positive eigenvalues and when 0.226192 (per year) $< \beta_1 < 0.243046$ (per year) we have complex conjugate eigenvalues with positive real part. In both cases, P_0 is always stable, independently of the initial condition of system (6) (such as the situation shown in Fig. 10).

(iii) Finally, for 0.243047 (per year) $< \beta_1 < 0.2667$ (per year) the complex eigenvalues have negative real part. In such a case, either P_1^+ or P_0 is stable (see Fig. 11).

Fig. 11 shows not only that the stability of P_1^+ depends on the initial condition, as predicted by *special* backward bifurcation, but also how small changes in the initial condition of C_1 will impact the behavior of system (6) determining whether society converges to a low crime level or a high crime level.

In Fig. 11b. P_1^+ is a stable focus (pairs of complex conjugate eigenvalues with negative real part). Although, $R_0^{thr} < R_0^1 < 1$, the situation is more dramatic if the value of parameter ϕ_1 increases because the effort to avoid the recidivism prevalence cannot be efficient, unless the initial condition causes system (6) to converge to P_0, as shown in Fig. 11a.

Finally it is useful to examine some general points for $R_0^1 = 1$ and $R_0^1 > 1$. If $R_0^1 = 1$, then if $b_0 = 0$, (A.2) either has exactly one solution greater than zero (if $b_1 < 0$) or no positive root (if $b_1 > 0$). So for $R_0^1 = 1$ and $b_1 < 0$ the model (6) has exactly one non-zero steady state with illegal activity present, given by P_1^+ (case ii). For $b_1 > 0$ the model (6) has a criminality-free equilibrium P_0.

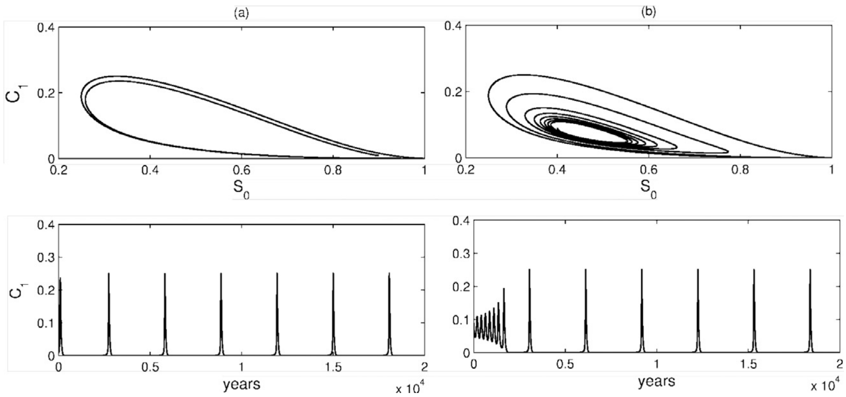

Fig. 12. For model (6): $\phi_1 = 15.311$, $\beta_1 = 0.2667$ (per year) where $R_0^1 = 1.0$. (a) The initial condition is close to P_0: $y_0 = (S_0, C_1, S_1, D) = (0.9, 0.001, 0, 0)$. (b) The initial condition is close to P_1^+: $y_0 = (S_0, C_1, S_1, D) = (0.4, 0.09, 0.03, 0.1)$. Parameter values used are as given in Table 3.

For example, Fig. 12 shows this scenario by setting $\phi_1 = 15.311$, when $\beta_1 = 0.2667$ (per year), that is, when $R_0^1 = 1$ the system (6) has the positive equilibrium P_1^+ and the criminality-free equilibrium P_0. At P_1^+ the Jacobian matrix associated to system (6) has a pair of positive complex conjugate eigenvalues, and the attractor becomes a limit cycle. At the crime-free equilibrium, the eigenvalues of the Jacobian matrix associated to system (6) are real and negative, but one of them is zero. In this case, from equation (A.5), one gets $\phi_1^{**1} = \phi_1^{**2}$.

As mentioned earlier, because of a complex scenario found in system (1), the implementation of an analytic strategy was a visualization to guide us towards a more thorough understanding of the Serious Crime Model. Hence, by using either $C_2 = 0$ or $C_1 = 0$ in system (1), the simplified crime models become symmetric, so the analysis of the Mild Crime Model can be extended to the Serious Crime Model.

Although the analytical strategy has been implemented as a tool to facilitate the analysis of the Serious Crime Model, and the simplified form of the Mild Crime Model has presented a very complex dynamic behaviour, such a strategy was still very beneficial, corroborating the need of a future work to understand in depth the dynamics of the full Serious Crime Model. The challenges that we will have in the analysis of the Serious Crime Model will be complex. We could exemplify by mentioning what happens to the changes in the dynamic behaviour of the system (1) with respect to thresholds R_0^1, R_0^2 and R_0^{thr}.

For example, considering the system (1) with the initial conditions in the attraction region of each equilibrium point by numerical simulations we have:

(a) If $R_0^1 < 1$, $R_0^2 < 1$ then the crime-free equilibrium point P_0^* is stable,
(b) If $R_0^1 > 1$, $R_0^2 < 1$ then point P_1^* is stable if ϕ_1 and β_1 are in the region of stability of P_1^*,
(c) If $R_0^1 < 1$, $R_0^2 > 1$ then the point P_2^* is stable if ϕ_2 and β_2 are in the region of stability of P_2^*,
(d) If $R_0^1 > 1$, $R_0^2 > 1$ then the point P_3^* is stable if ϕ_1, ϕ_2, β_1 and β_2 are in the region of stability of P_3^*.

Note that even if conditions (b), (c) and (d) are satisfied this does not necessarily imply that the equilibrium points P_1^* or P_2^* will be stable because depending on the initial conditions we may be in their region of instability. The case will be a little more difficult for P_3^* because in this case, in addition to the adequate initial conditions for system (1), we must also have the values ϕ_1, ϕ_2, β_1 and β_2 within the stability region of P_3^*.

To understand cases (a), (b), (c) and (d), look at Figs. 5 concerning the Mild Crime Model. Note that if the *usual* bifurcation occurs then P_1^* would be stable for $R_0^{thr} < R_0^1 < 1$ (backward bifurcation) and for $R_0^1 > 1$ (forward bifurcation). But as detailed earlier this does not happen because the convergence of system (6) to P_1^* depends on many factors, mainly on the values of ϕ_1. However for case (d) we would have to study the stability of the point P_3^* which is a very complex analysis. Analogous to cases (b) and (c) even if the conditions $R_0^1 > 1$, $R_0^2 > 1$ were satisfied this would not imply the stability of P_3^*.

Bifurcation phenomena also occur in the Serious Crime Model but with a much greater complexity because the dynamic behaviour of the system (1) also depends on the values of ϕ_1, ϕ_2, β_1 and β_2 and the initial conditions which must be in the attraction regions of each of the points P_1^*, P_2^* and P_3^*.

Note for example that $R_0^1 > 1$ and $R_0^2 < 1$ should be a necessary and sufficient condition for the point P_1^* to be stable, but due to the occurrence of the *special* bifurcation a deeper analysis concerning the thresholds of the parameter ϕ_1 and the initial condition of the system (6) is needed. Therefore due to the complexity of system (1), we will present the simplest numerical case where the parameter values are the same for the Mild and Serious components of the full combined Mild-Serious Crime Model. Thus we will make $\beta_1 = \beta_2$ where $R_0^1 = R_0^2 > 1$, β_3 is small, $\alpha_1 = \alpha_2$, $\tau_1 = \tau_2$, $\gamma_1 = \gamma_2$, $\phi_1 = \phi_2$ and the initial conditions vary according to the regions of attraction of each of the equilibrium points.

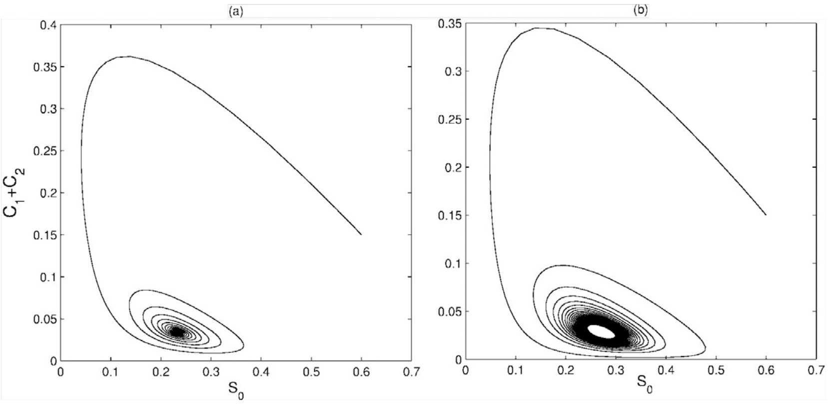

Fig. 13. For $\phi_1 = \phi_2$, $\beta_1 = \beta_2$ where $R_0^1 = R_0^2 > 1$. (a) Initial condition $y_0 = (0.6, 0.05, 0.1, 0.1, 0.1, 0.05)$: system (1) converges to $P_2^* = (0.23, 0, 0.034, 0, 0.11, 0.52)$ which is a stable focus. (b) Initial condition $y_0 = (0.6, 0.1, 0.05, 0.1, 0.1, 0.05)$: $P_1^* = (0.28, 0.022, 0, 0.08, 0, 0.54)$ is unstable and system (1) converges to a limit cycle.

Fig. 13 shows that for $\phi_1 = \phi_2$ and $\beta_1 = \beta_2$, with the initial conditions (a) $y_0 = (0.6, 0.05, 0.05, 0.1, 0.1, 0.1)$ and $y_0 = (0.6, 0.05, 0.1, 0.1, 0.1, 0.05)$ we have the convergence of the system (1) to the equilibrium point P_2^* which is a stable focus; (b) if the initial condition is $y_0 = (0.6, 0.1, 0.05, 0.1, 0.1, 0.05)$ we have the convergence of the system (1) to a limit cycle.

8. Conclusion

In this manuscript we have discussed differential equation models which describe how criminal behaviour potentially spreads amongst groups of individuals. The mathematical techniques used in this paper are similar to those used in mathematical models of how diseases spread.

We were interested in analysing a model that divided potential criminal individuals into mild offenders and serious offenders. Here we introduced the Serious Crime Model (1). We introduced terms ϕ_1 and ϕ_2 corresponding to the relative increase in the likelihood of an individual being induced to perform a mild (or serious) offence due to the person who was contacted having a record of mild (or serious) offending. We then performed a brief analysis of the Serious Crime Model and identified four equilibria, the crime-free equilibrium point P_0^*, the mild criminality-only equilibrium point P_1^*, the serious criminality-only equilibrium point P_2^* and the mild and serious criminality equilibrium point P_3^*. We identified two key parameters R_0^1 and R_0^2 which uniquely identify the qualitative behaviour of the system. These correspond to the basic reproductive number in mathematical epidemiology. However the Serious Crime Model was too complicated to analyse in this paper.

We therefore turned our attention to a simplified version of the Serious Crime Model, the Mild Crime Model (6) with only mild offenders. We examined analytically conditions for this model to have zero, one or two non-trivial equilibria. Here there is one parameter R_0^1 which describes the behaviour of the system. We find that subcritical bifurcation can occur with two non-trivial equilibria possible for $R_0^1 < 1$. We examined these conditions in terms of the co-option parameter ϕ_1. For $R_0^1 > 1$ the Mild Crime Model has a unique non-trivial equilibrium. This is also true for $R_0^1 = 1$ if $\phi_1 > k_1\phi_1^c/\tau_1$. For $R_0 < 1$, if ϕ_1^{**2} denotes the largest real root of (A.5) then system (6) will have two non-trivial equilibria P_1^+ and P_1^- for $\phi_1 > \phi_1^{**2}$. In the case where $\phi_1 = \phi_1^{**2}$ there is just one non-trivial equilibrium of model (6). For $\phi_1 < \phi_1^{**2}$ there are no non-trivial equilibria of model (6).

We then looked at the local stability behaviour of the steady states, both analytically and numerically. Both forward and backward bifurcation were possible but some unusual and interesting stability patterns could occur. For backward bifurcation it was possible to have both non-trivial equilibria unstable for $R_0^1 < 1$. There is a critical value of R_0^1, R_0^* such that the non-trivial equilibrium P_1^+ with the larger C_1 value is unstable for $R_0^1 < R_0^*$ and stable for $R_0^1 > R_0^*$. R_0^* may be less than or greater than one. The non-trivial equilibrium P_1^- with the smaller C_1 values appears always to be unstable when it exists.

For forwards bifurcation there is a second critical value R_0^{**} so that the unique non-trivial equilibrium P_1^+ is locally asymptotically stable for $R_0^1 < R_0^{**}$, unstable for $R_0^{**} < R_0^1 < R_0^*$ and locally asymptotically stable again for $R_0^1 > R_0^*$.

In terms of ϕ_1 there were critical values ϕ_1^{forw} and ϕ_1^{back} with $\phi_1^{forw} < \phi_1^{back}$ and the system undergoes usual forwards bifurcation with the normal local stability behaviour of the non-trivial steady states for $\phi_1 < \phi_1^{forw}$, and usual backwards bifurcation with the normal local stability behaviour of the non-trivial steady states if $\phi_1 > \phi_1^{back}$. There is a critical value ϕ_1^0 between ϕ_1^{forw} and ϕ_1^{back} such that model (6) undergoes the *special* backwards bifurcation with unusual stability pattern described above if $\phi_1^0 < \phi_1 \le \phi_1^{back}$ and the *special* forwards bifurcation with unusual stability pattern described above if $\phi_1^{forw} \le \phi_1 < \phi_1^0$.

We then returned to the full Serious Crime Model but the potential behaviour was much more complex. We did some limited simulations for this model with $\beta_1 = \beta_2, \alpha_1 = \alpha_2, \tau_1 = \tau_2, \gamma_1 = \gamma_2$ and $\phi_1 = \phi_2$. But the behaviour of this full model is very complex and further analysis and numerical simulation of this model is needed.

So we have developed a novel mathematical model for how individuals are induced into crime. There are a variety of mathematical models for how criminality of various types spreads through a population using various techniques. However in most models of which we are aware (in mathematical criminology or epidemiology) when there is forward bifurcation the unique persistence equilibrium is always locally asymptotically stable when it exists and when there is backwards bifurcation the lower persistence equilibrium is unstable for $R_0 < 1$ and the upper persistence equilibrium is always stable. Our results are novel in both cases because for forward bifurcation the persistence equilibrium switches from instability along the bifurcation curve to stability as does the upper persistence equilibrium in the case of backwards bifurcation. This type of switching along the bifurcation curve is interesting and unusual and we are not are aware of it having been shown before.

Data availabiilty

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration of competing interest

The authors have no relevant financial or non-financial interests to disclose.

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Appendix A

Proof of Theorem 1.

We have seen that the equilibrium solutions of system (6) imply that either $C = 0$ or $\beta_1 S_0 + \phi_1\beta_1 S_1 - (\mu + \alpha_1 + \tau_1) = 0$ and that $C = 0$ corresponds to the criminality equilibrium P_0. In contrast, for $\beta_1 S_0 + \phi_1\beta_1 S_1 - (\mu + \alpha_1 + \tau_1) = 0$, system (6) has a mild criminality equilibrium $P_1 = (S_0, C_1, S_1, D)$, with

$$\begin{cases} S_0 = \frac{\Lambda}{\mu R_0^1} - \phi_1 S_1 \\ S_1 = \frac{1}{\eta_1}\left[\frac{\Lambda}{\mu}\left(1 - \frac{1}{R_0^1}\right) - \frac{(\mu+\alpha_1)C_1}{\mu}\right] \\ D = \frac{\gamma_1}{\mu} S_1, \end{cases} \tag{A.1}$$

where $\eta_1 = \phi_1^c - \phi_1$, with $\phi_1^c = 1 + \frac{\gamma_1}{\mu}$.

First note that $S_0 > 0$ if and only if $S_1 < \frac{\Lambda}{\phi_1 \mu R_0^1}$ and $D > 0$ if and only if $S_1 > 0$. Moreover, assuming $\eta_1 > 0$ (or $\phi_1 < \phi_1^c$) then $S_1 > 0$ if and only if $C_1 > \frac{\Lambda}{\mu+\alpha_1}\left(1 - \frac{1}{R_0^1}\right)$, when $R_0^1 > 1$. In contrast for $\eta_1 < 0$ (or $\phi_1 > \phi_1^c$) then $S_1 > 0$ if and only if $C_1 > \frac{\Lambda}{\mu+\alpha_1}\left(1 - \frac{1}{R_0^1}\right)$, if $\phi_1 = \phi_1^c$ then $C_1 = \frac{\Lambda}{\mu+\alpha_1}\left(1 - \frac{1}{R_0^1}\right)$.

By replacing the expression for S_1 given by (A.1), into the third equation of system (6) an equation for $C_1 > 0$ is obtained as

$$b_2 (C_1)^2 + b_1 C_1 + b_0 = 0, \tag{A.2}$$

where

$$\begin{aligned} b_2 &= \phi_1\beta_1(1 + \frac{\alpha_1}{\mu}), \\ b_1 &= \phi_1^c(\mu + \alpha_1) + \tau_1(\phi_1^c - \phi_1) + \phi_1 k_1 (1 - R_0^1), \\ b_0 &= \frac{\Lambda(\mu + \gamma_1)}{\mu R_0^1}(1 - R_0^1), \end{aligned} \tag{A.3}$$

and $k_1 = (\mu + \alpha_1 + \tau_1)$.

The quadratic equation (A.2) can be analyzed for the possibility of multiple endemic equilibria when $R_0^1 < 1$. Note that the endemic equilibria of system (6) can be obtained by solving for C_1 from (A.2), and substituting the positive values of C_1 into the expressions in (A.1).

In this scenario, let us now analyze the signs of coefficients (A.3) to determine when the quadratic (A.2) has a unique solution greater than zero or two such solutions and translate our findings in terms of sociologically meaningful (i.e. greater than zero) endemic equilibria of (6).

We are searching for whether two or more steady states of (6) are possible in order to investigate the possibilities of bifurcation. Moreover, it is especially interesting to emphasize that the strategy of our analyses is to understand the criminal career and the role of both the co-optation rate (β_1) and the factor ϕ_1 corresponding to the multiplicative increase in the co-optation rate if the individual who is contacted has a history of offending. Hence, to check the possibilities of bifurcation we carry out this approach by determining two other critical values, namely $\phi_1 = \phi_1^*$ and $\phi_1 = \phi_1^{**}$ which are calculated by the analysis of the coefficients (A.3) and and when (A.2) has real roots which are greater than zero. This idea is explored more deeply below.

Because all the constants of the system of equations (6) are taken to be non-negative it is a consequence of (A.3) that b_2 is always greater than zero; $b_0 < 0$ for $R_0^1 > 1$ and $b_0 > 0$ for $R_0^1 < 1$. If $R_0^1 > 1$ and we substitute $C = C^* = \frac{\Lambda}{\mu+\alpha_1}\left(1 - \frac{1}{R_0^1}\right)$ into equation (A.2) then we get

$$b_2(C^*)^2 + b_1 C^* + b_0 = \tau_1(\phi_1^c - \phi_1)\frac{R_0^1 - 1}{R_0^1}\frac{\Lambda}{\mu+\alpha_1},$$

so if $\phi_1 < \phi_1^c$ then $C^* > C_1$, if $\phi_1 = \phi_1^c$ then $C_1 = C^*$ and if $\phi_1 > \phi_1^c$ then $C^* < C_1$. So, by the comments after Lemma 2 $S_1 > 0$, thus P_1 is biologically feasible. Thus, system (6) has a unique positive equilibrium, P_1, for $b_0 < 0$, i.e, when $R_0^1 > 1$. Moreover, system (6) also has one and only one steady state which is greater than zero, P_1, if $b_0 = 0$, i.e., when $R_0^1 = 1$ and $b_1 < 0$, i.e., when $\phi_1 > \frac{k_1}{\tau_1}\phi_1^c > 1$.

Let us now examine where (A.2) has two real roots strictly greater than zero when $b_0 > 0$, (i.e., $R_0^1 < 1$), $b_1 < 0$ and $b_1^2 - 4b_2 b_0 > 0$. Therefore, supposing that (A.2) has two real roots strictly greater than zero, write C_1^- and C_1^+ to be respectively the lower and the bigger value of C_1, which in turn correspond to the smaller and higher positive criminality equilibria of system (6), namely P_1^- and P_1^+, respectively.

From (A.3), we know $b_0 > 0$ for $R_0^1 < 1$. Also, if $\phi_1 < \phi_1^c$, then $b_1 > 0$. Therefore, $b_1 < 0$ only makes sense when $\phi_1 > \phi_1^c$ which implies that if $R_0^1 < 1$ then $S_1 > 0$. In what follows, $b_1 < 0$ if and only if $\phi_1 > \phi_1^*$, where

$$\phi_1^* = \frac{\phi_1^c}{(R_0^1 - k_3)}, \tag{A.4}$$

with $R_0^1 > k_3 = \frac{\mu+\alpha_1}{k_1}$ (i.e., $\frac{\beta_1\Lambda}{\mu} > \mu + \alpha_1$) and $\frac{\mu+\alpha_1}{k_1} < 1$.

Although, the other critical value, namely $\phi_1 = \phi_1^{**}$, does not have an explicit form due to the high complexity of coefficients (A.3), the inequality $b_1^2 - 4b_2 b_0 > 0$ can be expressed as $b_1^2 - 4b_2 b_0 = d_2(\phi_1)^2 + d_1\phi_1 + d_0 > 0$. Hence ϕ_1^{**} can be found numerically by solving the following quadratic equation

$$d_2(\phi_1)^2 + d_1\phi_1 + d_0 = 0, \tag{A.5}$$

where

$$\begin{aligned} d_2 &= \left[(\mu+\alpha_1) - \frac{\beta_1\Lambda}{\mu}\right]^2, \\ d_1 &= 2\phi_1^c\left[(\mu+\alpha_1)k_1(R_0^1 - 1) - \tau_1\frac{\beta_1\Lambda}{\mu}\right], \\ d_0 &= (\phi_1^c k_1)^2. \end{aligned} \tag{A.6}$$

Since all model parameters are assumed non negative, $\frac{\beta_1\Lambda}{\mu} > (\mu+\alpha_1)$ and $R_0^1 < 1$, it follows from (A.6) that $d_2 > 0$ and $d_0 > 0$. Moreover

$$\begin{aligned} d_1^2 - 4d_0 d_2 &= 4\phi_1^{c2}\left\{\left[(\mu+\alpha_1)k_1(R_0^1 - 1) - \tau_1\beta_1\frac{\Lambda}{\mu}\right]^2 - k_1^2\left[\mu+\alpha_1 - \beta_1\frac{\Lambda}{\mu}\right]^2\right\}, \\ &= 4\phi_1^{c2}\left\{\left[\tau_1\beta_1\frac{\Lambda}{\mu} - (\mu+\alpha_1)k_1(R_0^1 - 1)\right]^2 - k_1^2\left[\beta_1\frac{\Lambda}{\mu} - (\mu+\alpha_1)\right]^2\right\}. \end{aligned}$$

It is straightforward to show that

$$\tau_1\beta_1\frac{\Lambda}{\mu} - (\mu+\alpha_1)k_1(R_0^1 - 1) > k_1\left[\beta_1\frac{\Lambda}{\mu} - (\mu+\alpha_1)\right] > 0.$$

Hence $d_1^2 - 4d_0 d_2 > 0$ and as also $d_1 < 0$ then the quadratic equation (A.5) always has two strictly positive real roots. To simplify our notation, let ϕ_1^{**1} and ϕ_1^{**2} be the positive real roots of equation (A.5), with $\phi_1^{**1} < \phi_1^{**2}$. Hence $b_1^2 - 4b_2 b_0 > 0$ and thus (A.2) has two real solutions strictly greater than zero when either $\phi_1 < \phi_1^{**1}$ or $\phi_1 > \phi_1^{**2}$.

However, due to complexity of the coefficients (A.6), the positive roots ϕ_1^{**1} and ϕ_1^{**2} can only be calculated numerically. Note that when $\phi_1 = \phi_1^*$

$$d_2(\phi_1^*)^2 + d_1\phi_1^* + d_0 = \frac{4(\phi_1^c)^2}{(R_0^1 - k_3)^2}\left[\beta_1\frac{\Lambda}{\mu} - (\mu+\alpha_1)\right](\mu+\alpha_1)(R_0^1 - 1) < 0.$$

Hence $d_2(\phi_1^*)^2 + d_1\phi_1^* + d_0 < 0$ so $\phi_1^{**1} < \phi_1^* < \phi_1^{**2}$. Moreover our numerical simulations confirmed this result. From now on, therefore, we assume that $\phi_1^{**1} < \phi_1^* < \phi_1^{**2}$. Furthermore, since to provide two positive roots C_1 for the quadratic equation (A.2) we need $\phi_1 > \phi_1^*$, the solution of equation (A.5) is then given by $\phi_1 > \phi_1^{**2}$.

Next we turn to the implications of this result for equation (A.5). Firstly, note that if $b_0 > 0$, i.e., $R_0^1 < 1$, the equation (A.2) has pairs of complex conjugate values for C_1 with positive real parts whenever $\phi_1^* < \phi_1 < \phi_1^{**2}$. In this case, $b_1 > 0$ and $b_1^2 - 4b_2b_0 < 0$. Hence, model (6) has only the criminality-free equilibrium P_0 which is locally asymptotically stable. In contrast if $\phi_1 > \phi_1^{**2}$, then the equation (A.2) has two strictly positive real roots. In this case $b_1 < 0$ and $b_1^2 - 4b_2b_0 > 0$. Hence for $\phi_1 > \phi_1^{**2}$ model (6) has two positive equilibria, namely P_1^+ and P_1^-. If $\phi_1 = \phi_1^{**2}$ then these two positive equilibria co-incide, $b_1^2 = 4b_2b_0$ and there is a unique positive equilibrium. This completes the proof of Theorem 1.

Appendix B

Proof of Theorem 2.

The stability of P_1 is governed by the following Jacobian matrix

$$J(P_1) = \begin{bmatrix} J_{11} & -\beta_1 S_0 & 0 & 0 \\ \beta_1 C_1 & J_{22} & \phi_1\beta_1 C_1 & 0 \\ 0 & \tau_1 - \phi_1\beta_1 S_1 & J_{33} & 0 \\ 0 & 0 & \gamma_1 & -\mu \end{bmatrix} \tag{B.1}$$

where $J_{11} = -\beta_1 C_1 - \mu$; $J_{22} = \beta_1 S_0 + \phi_1\beta_1 S_1 - k_1$; $J_{33} = -\phi_1\beta_1 C_1 - (\mu+\gamma_1)$.

The Jacobian matrix (B.1) gives explicitly one eigenvalue, namely $\lambda_1 = -\mu < 0$, and the remaining eigenvalues are found by the corresponding third degree characteristic equation $F_3(\lambda) = 0$, where

$$\begin{aligned} F_3(\lambda) =& (J_{33} - \lambda)(J_{11} - \lambda)(J_{22} - \lambda) + \beta_1^2 S_0 C_1 (J_{33} - \lambda) \\ & - (\phi_1\beta_1 C_1)(\tau_1 - \phi_1\beta_1 S_1)(J_{11} - \lambda). \end{aligned} \tag{B.2}$$

To consider the characteristic polynomial (B.2) in terms of the Criminality Reproduction Number, R_0^1, we rewrite the expression (B.2) as a third degree polynomial, in its following closed-form $F_3(\lambda)$ as defined in the statement of Theorem 2. This completes the proof of Theorem 2.

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