The ordinals in set theory and type theory are the same

Tom de Jong[1], Nicolai Kraus[1], Fredrik Nordvall Forsberg[2], and Chuangjie Xu[3]

[1] University of Nottingham, Nottingham, UK
{tom.dejong, nicolai.kraus}@nottingham.ac.uk
[2] University of Strathclyde, Glasgow, UK
fredrik.nordvall-forsberg@strath.ac.uk
[3] SonarSource GmbH, Bochum, Germany
chuangjie.xu@sonarsource.com

Set theory or type theory; which one is "better" for constructive mathematics? While we do not dare to offer an answer to this question, we can at least report that when it comes to constructive ordinal theory, the choice between these two foundations is insignificant: the set-theoretic and type-theoretic ordinals coincide. We consider this an interesting finding since ordinals are fundamental in the foundations of set theory and are used in theoretical computer science in termination arguments [7] and semantics of inductive definitions [1, 5].

Comparing ordinals in set theory and (homotopy) type theory In constructive set theory [3], following Powell's seminal work [9], the standard definition of an ordinal is that of a *transitive* set whose elements are again transitive sets. A set x is transitive if for every $y \in x$ and $z \in y$, we have $z \in x$. Note how this definition makes essential use of how the membership predicate $\in$ in set theory is global, by referring to both $z \in y$ and $z \in x$. In type theory, on the other hand, the statement "if $y : x$ and $z : y$ then $z : x$" is ill-formed, and so ordinals need to be defined differently. In the homotopy type theory book [10], an ordinal is defined to be a type equipped with a proposition-valued order relation that is transitive, extensional, and wellfounded [10, §10.3]. Extensionality implies that the underlying type of an ordinal is a set [6].

A priori, the set-theoretic and the type-theoretic approaches to ordinals are thus quite different. One way to compare them is to interpret one foundation into the other. Aczel [2] gave an interpretation of Constructive ZF set theory into type theory using setoids, which was later refined using a higher inductive type $\mathbb{V}$ [10, §10.5], referred to as the *cumulative hierarchy*. Using the set membership relation $\in$ on the cumulative hierarchy, we can construct the subtype $\mathbb{V}_{\mathrm{ord}}$ of elements of $\mathbb{V}$ that are set-theoretic ordinals. Similarly, we write Ord for the type of all type-theoretic ordinals, i.e., for the type of transitive, extensional, and wellfounded order relations. A fundamental result about type-theoretic ordinals is that, using univalence, the type Ord of (small) ordinals is itself a type-theoretic ordinal when ordered by inclusion of strictly smaller initial segments (also referred to as *bounded simulations*), and we show that the type $\mathbb{V}_{\mathrm{ord}}$ of set-theoretic ordinals also canonically carries the structure of a type-theoretic ordinal.

Next, we show that $\mathbb{V}_{\mathrm{ord}}$ and Ord are equivalent, meaning that we can translate between type-theoretic and set-theoretic ordinals. Furthermore, the isomorphism that we construct respects the order structure of $\mathbb{V}_{\mathrm{ord}}$ and Ord, which means that Ord and $\mathbb{V}_{\mathrm{ord}}$ are isomorphic as (large) ordinals. Thus, the set-theoretic and type-theoretic approaches to ordinals coincide in homotopy type theory:

Theorem 1 ([4, Theorem 33] ✿). *The ordinals* Ord *and* $\mathbb{V}_{\mathrm{ord}}$ *are isomorphic (as type-theoretic ordinals). Hence, by univalence, they are equal.* □

Generalising from ordinals to sets Since the subtype $\mathbb{V}_{\mathrm{ord}}$ of $\mathbb{V}$ is isomorphic to Ord, a type of ordered structures, it is natural to ask if there is a type of ordered structures that captures *all* of $\mathbb{V}$. That is, we look for a type T of ordered structures such that Diagram 1 below commutes.

Since $\mathbb{V}$ is $\mathbb{V}_{\text{ord}}$ with transitivity dropped, it is tempting to try to choose T to be Ord without transitivity, i.e., the type of extensional and wellfounded relations. However such an attempt cannot work for cardinality reasons: for example, the set-theoretic ordinal $2 = \{\emptyset, \{\emptyset\}\}$ corresponds to the type-theoretic ordinal α with elements $0 < 1$, but there are more subsets of 2 than subrelations of α. Instead we need additional structure to capture the elements of elements (of elements ...) of sets. To this end, we introduce the type $\mathsf{MEWO}_{\mathsf{cov}}$ of *(covered) marked extensional wellfounded order relations (mewos)*, i.e., extensional wellfounded relations with additional structure in the form of a *marking*.[1] The idea is that the carrier of the order also contains "deeper" elements of elements of the set, with the marking designating the "top-level" elements. Such a marking is *covering* if any element can be reached from a marked top-level element, i.e., if the order contains no "junk". Since every ordinal can be equipped with the trivial covering by marking all elements, the type Ord of ordinals is a subtype of the type of covered mewos. Taking $T = \mathsf{MEWO}_{\mathsf{cov}}$, this gives the inclusion $\mathsf{Ord} \hookrightarrow T$ in Diagram 1.

$$\begin{array}{ccc} \mathbb{V}_{\text{ord}} & \xrightarrow{\simeq} & \mathsf{Ord} \\ \big\downarrow & & \big\downarrow \\ \mathbb{V} & \xrightarrow{\simeq} & T \end{array} \qquad (1)$$

To show also $\mathbb{V} \simeq \mathsf{MEWO}_{\mathsf{cov}}$, we develop the theory of covered mewos: the type of covered mewos is itself a covered mewo, with order $<$ given by an appropriately modified notion of bounded simulation (to take the lack of transitivity into account), and covered mewos are closed under both singletons and least upper bounds of arbitrary (small) families of covered mewos. We can then show that indeed $T = \mathsf{MEWO}_{\mathsf{cov}}$ fulfils the requirements of Diagram 1:

Theorem 2 ([4, Theorem 76]). *The structures* $(\mathbb{V}, \in)$ *and* $(\mathsf{MEWO}_{\mathsf{cov}}, <)$ *are equal as covered mewos.* □

Full Paper and Formalisation More details are available in our paper at LICS this year [4]. We have also formalised all our results in Agda. An HTML rendering can be found at the URL `https://tdejong.com/agda-html/st-tt-ordinals/index.html`.

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