Relating ordinals in set theory to ordinals in type theory*

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Set theory and dependent type theory are two very different settings in which constructive mathematics can be developed, but not always in comparable ways. Lively discussions on what foundation is "better" are not uncommon. While we do not dare to offer a judgment on this question, we can at least report that the choice of foundation is in a certain sense insignificant for the development of constructive ordinal theory. We consider this an interesting finding since ordinals are fundamental in the foundations of set theory and are used in theoretical computer science in termination arguments [6] and semantics of inductive definitions [1, 5].

Set-theoretic and type-theoretic ordinals coincide In constructive set theory [3], following Powell's seminal work [7], the standard definition of an ordinal is that of a *transitive* set whose elements are again transitive sets. A set x is transitive if for every $y \in x$ and $z \in y$, we have $z \in x$. Note how this definition makes essential use of how the membership predicate $\in$ in set theory is global, by simultaneously referring to $z \in y$ and $z \in x$. In type theory, on the other hand, the statement "if $y : x$ and $z : y$ then $z : x$" is ill-formed, and so ordinals need to be defined differently. In HoTT, an ordinal is defined to be a type equipped with an order relation that is transitive, extensional, and wellfounded [8, §10.3].

A priori, the set-theoretic and the type-theoretic approaches to ordinals are thus quite different. One way to compare them is to interpret one foundation into the other. Aczel [2] gave an interpretation of Constructive ZF set theory into type theory using setoids, which was later refined using a higher inductive type $\mathbb{V}$ in [8, §10.5], referred to as the *cumulative hierarchy*.

The type $\mathbb{V}$ allows us to define a set membership relation $\in$, which makes it possible to consider the type $\mathbb{V}_{\mathrm{ord}}$ of elements of $\mathbb{V}$ that are set-theoretic ordinals. Similarly, we write Ord for the type of all type-theoretic ordinals, i.e., for the type of transitive, extensional, and wellfounded orders. We show that $\mathbb{V}_{\mathrm{ord}}$ and Ord are equivalent, meaning that we can translate between type-theoretic and set-theoretic ordinals.

This translation by itself would not be satisfactory if it was not well-behaved; what makes it valuable is that it preserves the respective order. A fundamental result about type-theoretic ordinals is that the type Ord of (small) ordinals is itself a type-theoretic ordinal when ordered by inclusion of strictly smaller initial segments (also referred to as *bounded simulations*). To complement this, we show that the type $\mathbb{V}_{\mathrm{ord}}$ of set-theoretic ordinals also canonically carries the structure of a type-theoretic ordinal. The isomorphisms that we construct respect these orderings, and our first main result is that Ord and $\mathbb{V}_{\mathrm{ord}}$ are isomorphic as ordinals. Thus, the set-theoretic and type-theoretic approaches to ordinals coincide in HoTT.

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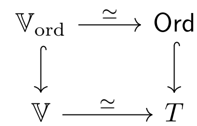
Generalizing from ordinals to sets Since the subtype $\mathbb{V}_{\mathrm{ord}}$ of $\mathbb{V}$ is isomorphic to Ord, a type of ordered structures, it is natural to ask if there is a type of ordered structures that captures *all* of $\mathbb{V}$. That is, we look for a type T of ordered structures such that the diagram on the right commutes. Since $\mathbb{V}$ is $\mathbb{V}_{\mathrm{ord}}$ with transitivity dropped, it is tempting to try to choose T to be Ord without transitivity, i.e., the type of extensional and wellfounded orders. However such an attempt is too naive to work: consider the type-theoretic ordinal α with two elements $0 < 1$, whose corresponding set in $\mathbb{V}_{\mathrm{ord}}$ is the set $2 = \{\emptyset, \{\emptyset\}\}$. The latter is the transitive closure of the non-transitive set $\{\{\emptyset\}\} \subseteq 2$, but the only extensional, wellfounded order whose transitive closure is α is α itself. In other words, there cannot be an order-preserving isomorphism between $\mathbb{V}$ and the type of extensional, wellfounded orders, since there is no corresponding order for the set $\{\{\emptyset\}\}$ — we need additional structure to fully capture this set.



To this end, we introduce the theory of *(covered) marked extensional wellfounded orders (mewos)*, i.e., extensional wellfounded orders with additional structure in the form of a *marking*. The idea is that the carrier of the order also contains elements representing elements of elements of the set, with the marking designating the "top-level" elements: the set $\{\{\emptyset\}\}$ is again represented by the order α with two elements $0 < 1$, but with only element 1 marked. Such a marking is *covering* if any element can be reached from a marked top-level element, i.e., if the order contains no "junk". Since every ordinal can be equipped with the trivial covering by marking all elements, the type Ord of ordinals is a subtype of the type of covered mewos $\mathsf{MEWO}_{\mathsf{cov}}$.

After developing the theory of covered mewos, which requires some extra care compared to ordinals as the orders involved are not necessarily transitive, we are then indeed able to prove that $\mathbb{V}$ and $\mathsf{MEWO}_{\mathsf{cov}}$ are both extensional wellfounded orders and, when equipped with the trivial marking, are equal as covered mewos. Thus, we can take T to be $\mathsf{MEWO}_{\mathsf{cov}}$ in the diagram above.

Formalization and preprint Our results are presented in our recent preprint [4] which is supported by a complete Agda formalization, an HTML rendering of which can be found at `https://tdejong.com/agda-html/st-tt-ordinals/index.html`.

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