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On Decidability of Recursive Weighted Logics

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Abstract In this paper we develop and study two recursive weighted logics (RWLs) $\mathcal{L}^w$ and $\mathcal{L}^t$, which are multimodal logics that express qualitative and quantitative properties of labelled weighted transition systems (LWSs). LWSs are transition systems describing systems with quantitative aspects. They have labels with both actions and real-valued quantities representing the costs of transitions with respect to various resources. RWLs use first-order variables to measure local costs. The main syntactic operators are similar to the ones of timed logics for real-time systems. $\mathcal{L}^w$ has operators that constrain the value of resource-variables at the current state. $\mathcal{L}^t$ extends $\mathcal{L}^w$ by having quantitative constraints on the transition modalities as well. This extension makes sure that $\mathcal{L}^t$ is adequate, i.e., the semantic equivalence induced by $\mathcal{L}^t$ coincides with the weighted bisimilarity of LWSs. In addition, our logic is endowed with simultaneous recursive equations, which allow encoding of properties of infinite behaviours. We prove that unlike in the case of the timed logics, the satisfiability problems for RWLs are decidable. The proofs use a variant of the region construction technique used in literature with timed automata, which we adapt to the specific settings of RWLs. For $\mathcal{L}^t$ we also propose an attractive alternative proof which makes use of the algorithm for $\mathcal{L}^w$.

Keywords labelled weighted transition system · multimodal logic · maximal fixed point computation · weighted constraints · satisfiability · model construction

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1 Introduction

Formal specification, verification and analysis of behavioural properties of software-based systems has emerged as a useful method for validating a design before the implementation of the system has started. However, in the setting of embedded software systems – or cyber-physical systems – the correctness is intimately linked to the resource constraints of the execution platform as well as quantitative aspects of the physical environment to be controlled. Hence, specification and verification should not only consider functional properties (correctness, predictability, etc) but also non-functional properties such as those related to resource constraints (time, energy, bandwidth, etc). To deal with the growing complexity of embedded systems – in size as well as in features – there is a pressing need for tools that provide computer-assistance to the verification and analysis, with the area of model checking providing a number of such fully automated tools. Within model checking various state-machine based modelling formalisms have surfaced, which allow for such quantitative aspects to be expressed, especially time constraints, with the well-established notion of timed automata [Alur and Dill, 1990] being ideal for modelling such aspects. Time is however not the only quantity that is relevant for embedded systems; another important quantitative aspect is energy, which may be consumed or – for certain systems – harvested. Here, the extension to weighted timed automata [Behrmann et al, 2001; Alur et al, 2001] allows for such constraints to be modelled and efficiently analysed.

In this paper, we put forward the notion of labelled weighted transition systems (LWS) as the semantic foundation for models of systems with quantitative aspects. More precisely, an LWS is a labelled transition system that has a number of different resources, which allow us to model the quantitative consumptions of resources. Its transitions are labelled at the same time with both actions and real values, representing the costs of the corresponding transitions in terms of resources. Note that with respect to *weighted automata* [Droste et al, 2009], our notion of weighted transition system can also be infinite and infinitely branching; and most probably it can be further generalized for a general semiring instead of finite reals. In order to use a variant of the region construction technique developed for timed automata in [Alur and Dill, 1990; Alur et al, 1990] and to involve the reset operations of event related resource-variables, we only consider non-negative labels in this paper. The work might be further extended to include negative weights, but this requires an extensive analysis of the semantics and of the model theory for this logic, which we intend to address in future works.

In order to specify and reason about not only the qualitative but also about quantitative properties of the systems, several quantitative interpretations of temporal logics for games, general quantitative transition systems and real-time systems are studied [De Alfaro et al, 2003, 2004, 2005; Henzinger et al, 2005]. In this paper, we consider two recursive weighted logics (RWLs) – $\mathcal{L}^w$ and $\mathcal{L}^t$ – semantically to be interpreted over LWSs. The RWLs are extensions of the *weighted modal logic* [Larsen and Mardare, 2014] with maximal fixed points. The maximal fixed points – which are defined by simultaneous recursive equations [Larsen, 1990; Cleaveland et al, 1992; Cleaveland and Steffen, 1993] in this paper – allow encoding of properties of infinite behaviours including safety and cost-bounded liveness. They specify the weakest properties satisfied by the recursive variables. Similar results can be proven for the logic also involving the minimal fixed points under the restriction of alternation-free [Larsen et al, 2015], but the proof method presented here cannot be adapted to the case of minimal fixed points, neither the efficient algorithms that we discuss in what follows. Nevertheless, the logic including only the maximal fix points is already very interesting as it can encode a large class of important properties, such as the liveness properties, and the satisfiability checking can be performed efficiently.

Resource-constraints in RWLs are encoded by the use of resource-variables, similar to the clock-variables used in the timed logics [Alur et al, 1993; Henzinger et al, 1992; Aceto et al, 2007]. We use *resource valuations* to assign non-negative real values to resource-variables. In previous work [Larsen et al, 2014a], we restricted our attention to only one resource-variable for each type of resources. This guaranteed the decidability of the logic and the finite model property. However, this restriction bounds severely the expressiveness of the logic. Consider as an example a quantitative system performing the actions a, b, c and d in sequence, but with the additional constraint that the energy-consumption between a and c should be at most 2 joule, and the energy-consumption between b and d should be at least 3 joule. Given the overlap between the two energy-constrained phases $a - c$ and $b - d$, this property cannot be specified by the logic with only one resource-variable for each type of resources, because we need two resource-variables to measure the same resource – energy in this example.

In this paper, we allow multiple resource-variables for each type of resource, which measure the resource in different ways. For both $\mathcal{L}^w$ and $\mathcal{L}^t$, we discuss the event related resource-variables. More precisely, for each type of resource and each action, we associate one resource-variable. Whenever the system performs one action, all the resource-variables associated to this action are reset to zero after the corresponding transition, meaning that the resource valuation will map those resource-variables to zero.

RWLs are endowed with modal operators that constrain the values of resource-variables, allowing us to specify and reason about the quantitative properties related to resources, e.g., energy and time. While in an LWS we can have real-valued labels, the modalities of the logics only encode rational values. This will not limit too much the expressive power of RWLs, because a real-valued resource can be characterized by using an infinite convergent sequences of rationals approximating it.

$\mathcal{L}^w$ has operators that constrain the value of resource-variables at the current state. However, the logic does have no means of constraining the resource consumption of transitions. Whereas $\mathcal{L}^w$ can be used to encode various interesting scenarios, it is not adequate, in the sense that it is not sufficiently expressive to characterize weighted bisimulation. The logic $\mathcal{L}^t$ extends $\mathcal{L}^w$ by having quantitative constraints on the transition modalities as well. As an important result of the paper, we prove that $\mathcal{L}^t$ is adequate, i.e., the semantic equivalence induced by $\mathcal{L}^t$ coincides with the weighted bisimilarity of LWSs.

Even though $\mathcal{L}^w$ is the least expressive of the two logics discussed in this paper, we shall see that it fails to enjoy the finite model property! However, as another important result of the paper, we demonstrate how to apply a variant of the region construction technique developed for timed automata [Alur and Dill, 1990; Alur et al, 1990; Laroussinie et al, 1995], to obtain symbolic LWSs of the satisfiable formulas. These symbolic LWSs provide an abstract semantics for LWSs, allowing us to reason about satisfiability by investigating the symbolic models that are finite. We have proposed a model construction algorithm, which constructs a symbolic LWS for a given satisfiable (consistent) $\mathcal{L}^w$ formula. The symbolic model can be eventually used to determine the existence of the concrete LWSs and generate them – possibly infinite – which are models of the given formula.

The satisfiability problem of $\mathcal{L}^t$ can be solved in a similar way with $\mathcal{L}^w$. However, an attractive alternative is to first encode the problem for $\mathcal{L}^t$ into one similar to that of $\mathcal{L}^w$ by translating the given $\mathcal{L}^t$ formula into $\mathcal{L}^w$ with a special 0-cost action; then use the model construction algorithm with a minor modification to check the satisfiability of this $\mathcal{L}^w$ formula; and if the $\mathcal{L}^w$ formula is not satisfiable then the given $\mathcal{L}^t$ formula is not satisfiable either, otherwise the given $\mathcal{L}^t$ formula is satisfiable and we finally generate the model for it according to the model for the corresponding $\mathcal{L}^w$ formula.

The satisfiability problem is known to be undecidable for logics very similar to ours, such as TCTL [Alur et al, 1993], T_μ [Henzinger et al, 1992], L_ν [Laroussinie et al, 1995] and timed modal logic (TML) [Jaziri et al, 2014]. Therefore, our decidability results are quite important and, in a sense, surprising.

The paper is organized as follows: in the following section we present the notion of labelled weighted transition system together with weighted bisimulation; in Section 3 we introduce the recursive weighted logic $\mathcal{L}^w$ with its syntax and semantics; Section 4 is dedicated to the region construction technique and the symbolic models of LWSs; in Section 5 we prove the decidability of the satisfiability problem for $\mathcal{L}^w$; in Section 6 the more expressive recursive weighted logic $\mathcal{L}^t$ is introduced, together with the the adequacy problem and the satisfiability problem being solved. The paper also includes a conclusive section where we summarize the results and describe future research directions.

This paper is an extension of our previous work that has been presented at ICTAC2014 [Larsen et al, 2014b].

2 Labelled Weighted Transition Systems

A *labelled weighted transition system* (LWS) is a transition system that has several types of resources (e.g., energy, price, time, bandwidth, etc.) and has the transitions labelled both with actions and (non-negative) real numbers. In Figure 1 is represented such a system in which there are three types of resource; each number is used to represent the cost of the corresponding transitions in terms of one type of resource.

Definition 1 (Labelled Weighted Transition System) A labelled weighted transition system is a tuple

$$\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$$

where M is a non-empty set of *states*, $\mathcal{K} = \{e_1, \ldots, e_k\}$ is a finite set of (k types of) resources, Σ a non-empty set of *actions* and $\theta : M \times (\Sigma \times [\mathcal{K} \to \mathbb{R}_{\geq 0}]) \to 2^M$ is the *labelled transition function*, where $[\mathcal{K} \to \mathbb{R}_{\geq 0}]$ represents the set of functions from $\mathcal{K}$ to non-negative reals.

For simplicity, hereafter we represent the function $f : \mathcal{K} \to \mathbb{R}_{\geq 0}$ defined by $f(e_i) = r_i$ for all $i = 1, \ldots, k$ using the real values vector $\overline{u} = (r_1, \ldots, r_k) \in \mathbb{R}^k_{\geq 0}$. For $\overline{u} \in \mathbb{R}^k_{\geq 0}$, we use $\overline{u}(e_i)$ to denote the i-th component of the vector $\overline{u}$, i.e., the cost of the resource e_i during the transition.

Instead of $m' \in \theta(m, a, \overline{u})$ we write $m \xrightarrow{\overline{u}}_a m'$.

To clarify the role of the aforementioned concepts consider the following example.

Example 1 In Figure 1, we show the LWS

$$\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta),$$

where $M = \{m_0, m_1, m_2\}$, $\mathcal{K} = \{e_1, e_2, e_3\}$, $\Sigma = \{a, b\}$, and θ defined as follows: $m_0 \xrightarrow{(3,4,5)}_a m_1$, $m_0 \xrightarrow{(\pi,\pi,0)}_b m_2$ and $m_1 \xrightarrow{(\sqrt{2},1.9,7)}_a m_2$.

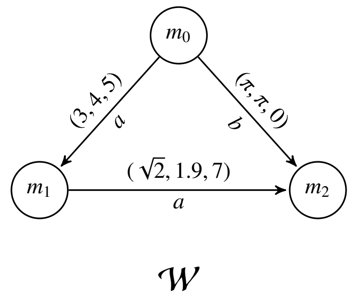

Fig. 1 Labelled Weighted Transition System

$\mathcal{W}$ has three states - m_0, m_1, m_2, three kinds of resource - e_1, e_2, e_3 and two actions - a, b. The state m_0 has two transitions: one a-transition to m_1 – which costs 3 units of e_1, 4 units of e_2 and 5 units of e_3; and one b-transition to m_2 – which costs π units of e_1 and e_2 respectively (and 0 units of e_3). If the system does an a-transition from m_0 to m_1, the amounts of the resource e_1, e_2 and e_3 increase with 3, 4 and 5 units respectively. ∎

The concept of *weighted bisimulation* is a relation between the states of a given LWS that equates states with identical (action- and weighted-) behaviors. It is defined similar to [Buchholz and Kemper, 2001; Buchholz, 2008; Boreale, 2009; Klin, 2009; Ésik, 2014] as follows.

Definition 2 (Weighted Bisimulation) Given an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$, a *weighted bisimulation* is an equivalence relation $R \subseteq M \times M$ such that whenever $(m, m') \in R$,

– if $m \xrightarrow{\bar{u}}_a m_1$, then there exists $m'_1 \in M$ s.t. $m' \xrightarrow{\bar{u}}_a m'_1$ and $(m_1, m'_1) \in R$;

– if $m' \xrightarrow{\bar{u}}_a m'_1$, then there exists $m_1 \in M$ s.t. $m \xrightarrow{\bar{u}}_a m_1$ and $(m_1, m'_1) \in R$.

If there exists a weighted bisimulation relation R such that $(m, m') \in R$, we say that m and m' are *bisimilar*, denoted by $m \sim m'$.

As for the other types of bisimulation, the previous definition can be extended to define the weighted bisimulation between distinct LWSs by considering bisimulation relations on their disjoint union. *Weighted bisimilarity* is the largest weighted bisimulation relation; if $\mathcal{W}_i = (M_i, \mathcal{K}_i, \Sigma_i, \theta_i)$, $m_i \in M_i$ for $i = 1, 2$ and m_1 and m_2 are bisimilar, we write

$$(m_1, \mathcal{W}_1) \sim (m_2, \mathcal{W}_2).$$

The following examples shows the role of the weighted bisimilarity.

Example 2 In Figure 2, $\mathcal{W}_1 = (M_1, \mathcal{K}_1, \Sigma_1, \theta_1)$ is an LWS with five states and one type of resources, where

$$M_1 = \{m_0, m_1, m_2, m_3, m_4\},$$

$\Sigma_1 = \{a, b, c, d\}$, $\mathcal{K}_1 = \{e\}$ and θ_1 is defined as: $m_0 \xrightarrow{3}_a m_1$, $m_0 \xrightarrow{2}_b m_2$, $m_1 \xrightarrow{0}_d m_2$, $m_1 \xrightarrow{3}_c m_3$, $m_2 \xrightarrow{0}_d m_1$ and $m_2 \xrightarrow{3}_c m_4$.

It is easy to see that $m_3 \sim m_4$ because neither of them can do any transition. Besides, $m_1 \sim m_2$ because both of them can do a c-transition with cost 3 to some states which are bisimilar (m_3 and m_4), and a d-action transition with cost 0 to each other. m_0 is not bisimilar to any states in $\mathcal{W}_1$.

$\mathcal{W}_2 = (M_2, \mathcal{K}_2, \Sigma_2, \theta_2)$ is an LWS with three states, where $M_2 = \{m'_0, m'_1, m'_2\}$, $\Sigma_2 = \Sigma_1$ $\mathcal{K}_2 = \mathcal{K}_1$ and θ_2 is defined as: $m'_0 \xrightarrow{3}_a m'_1$, $m'_0 \xrightarrow{2}_b m'_1$, $m'_1 \xrightarrow{0}_d m'_1$ and $m'_1 \xrightarrow{3}_c m'_2$.

We can see that:

$$(m_0, \mathcal{W}_1) \sim (m'_0, \mathcal{W}_2),$$
$$(m_1, \mathcal{W}_1) \sim (m'_1, \mathcal{W}_2),$$
$$(m_2, \mathcal{W}_1) \sim (m'_1, \mathcal{W}_2),$$
$$(m_3, \mathcal{W}_1) \sim (m'_2, \mathcal{W}_2),$$
$$(m_4, \mathcal{W}_1) \sim (m'_2, \mathcal{W}_2).$$

Notice that

$$(m''_0, \mathcal{W}_3) \nsim (m'_0, \mathcal{W}_2),$$

because

$$(m''_1, \mathcal{W}_3) \nsim (m'_1, \mathcal{W}_2).$$

Besides, $m''_1 \nsim m''_2$, because m''_1 can do a d-action with weight 2 while m''_2 cannot and m''_2 can do a d-action with weight 1 while m''_1 cannot. ∎

3 Recursive weighted logic $\mathcal{L}^w$

In this section we introduce the first recursive weighted logic (RWL) we study in this paper, denoted by $\mathcal{L}^w$, which encodes properties of LWSs.

To encode various resource-constrains in $\mathcal{L}^w$, we use resource-variables, similar to the clock-variables used in timed logics [Alur et al, 1993; Henzinger et al, 1992; Aceto et al, 2007]. In this section, we introduce event related resource-variables to measure the resources in different ways corresponding to different actions, i.e., for each action $a \in \Sigma$, we associate resource-variables $x_a^1, \ldots, x_a^k$ for each types of resource $e_1, \ldots, e_k$ respectively. In the following, we use

$$\mathcal{V}_i = \{x_a^i \mid a \in \Sigma\}$$

to denote the set of the resource-variables associated for the type of resource e_i,

$$\mathcal{V}_a = \{x_a^i \mid i = 1, \ldots, k\}$$

to denote the set of the resource-variables associated with the action a and

$$\mathcal{V} = \bigcup_{i=1,\ldots,k} \mathcal{V}_i = \bigcup_{a \in \Sigma} \mathcal{V}_a$$

to denote the set of all the resource-variables.

Note that:

1. for any i, j such that $i \neq j$, $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$, and for any a, b such that $a \neq b$, $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$;
2. $|\mathcal{V}_i| = |\Sigma|$ for any $i \in \{1, \ldots, k\}$, $|\mathcal{V}_a| = k$ for any $a \in \Sigma$, $|\mathcal{V}| = |\Sigma| \times k$.

In addition to the classic positive boolean operators, our logic $\mathcal{L}^w$ is firstly endowed with a class of recursive (formula) variables $X_1, \ldots, X_n$, which specify properties of infinite behaviours. We denote by $\mathcal{X}$ the set of recursive formula variables.

Secondly, $\mathcal{L}^w$ is endowed with a class of modalities, named *transition modalities*, of type $[a]$ or $\langle a \rangle$, for $a \in \Sigma$, which are defined as the classical transition modalities with reset operation of all the resource-variables associated with the corresponding action followed. More precisely, every time the system does an a-action, all the resource-variables $x \in \mathcal{V}_a$ will be reset after this transition, i.e., x is interpreted to zero after every a-action, for all $x \in \mathcal{V}_a$.

Besides, $\mathcal{L}^w$ is also endowed with a class of modalities of arity 0 called *state modalities* of type $x \bowtie r$, for $r \in \mathbb{Q}_{\geq 0}$, $x \in \mathcal{V}$ and $\bowtie \in \{\leq, \geq, <, >\}$, which predicates about the value of the resource-variable x at the current state.

Before proceeding with the introduction of the maximal fixed points, we firstly define the basic formulas of $\mathcal{L}^w$ and their semantics. Based on them, we will eventually introduce the recursive definitions - the maximal equation blocks - which extend the semantics of the basic formulas.

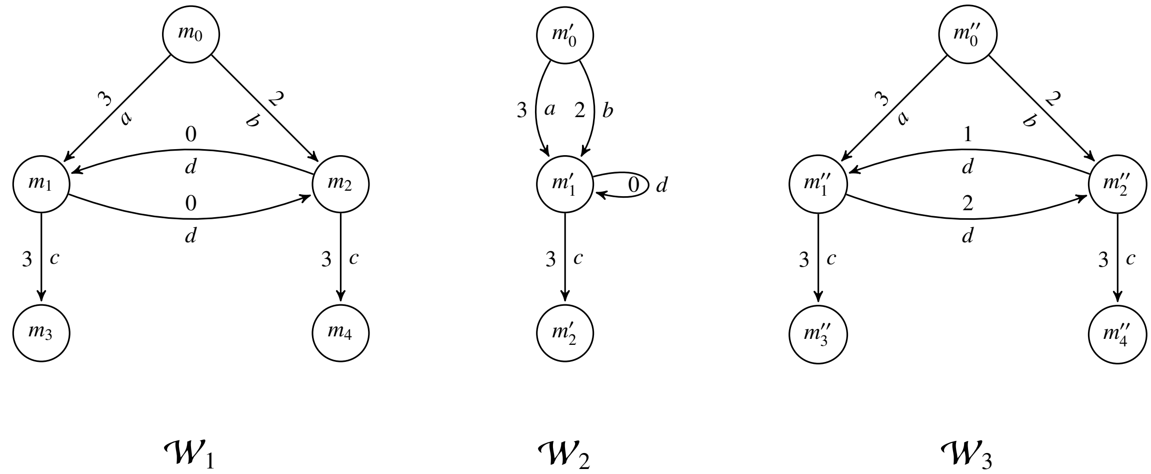

Fig. 2 Weighted Bisimulation

Definition 3 (Syntax of $\mathcal{L}^w$ Basic Formulas)
For arbitrary $r \in \mathbb{Q}_{\geq 0}$, $a \in \Sigma$, $x \in \mathcal{V}$, $\bowtie \in \{\leq, \geq, <, >\}$ and $X \in \mathcal{X}$, let

$$\mathcal{L}: \quad \phi := \top \mid \bot \mid x \bowtie r \mid \phi \wedge \phi \mid \phi \vee \phi \mid [a]\phi \mid \langle a\rangle\phi \mid X\ .$$

Before introducing the semantics for the basic formulas, we define the notion of *resource valuation* and *extended states*.

Definition 4 (Resource Valuation)
A *resource valuation* is a function $l : \mathcal{V} \to \mathbb{R}_{\geq 0}$ that assigns (non-negative) real numbers to the resource-variables in $\mathcal{V}$.

A resource valuation assigns non-negative real values to all the resource-variables and the assignment is interpreted as the amount of resources measured by the corresponding resource-variable in a given state of the system. We denote by L the class of resource valuations.

We write l_i to denote the valuation for all resource-variables $x \in \mathcal{V}_i$ under the resource valuation l, i.e.,
for any $x \in \mathcal{V}$,

$$l_i(x) = \begin{cases} l(x), & x \in \mathcal{V}_i \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Similarly, we write l_a to denote the valuation for all resource-variables $x \in \mathcal{V}_a$ under the resource valuation l, i.e.,
for any $x \in \mathcal{V}$,

$$l_a(x) = \begin{cases} l(x), & x \in \mathcal{V}_a \\ \text{undefined}, & \text{otherwise} \end{cases}$$

If l is a resource valuation and $x \in \mathcal{V}$, $s \in \mathbb{R}_{\geq 0}$ we denote by $l[x \mapsto s]$ the resource valuation that associates the same values as l to all variables except x, to which it associates the value s, i.e., for any $y \in \mathcal{V}$,

$$l[x \mapsto s](y) = \begin{cases} s, & y = x \\ l(y), & \text{otherwise} \end{cases}$$

Moreover, for $\mathcal{V}' \subseteq \mathcal{V}$ and $s \in \mathbb{R}_{\geq 0}$, we denote by $l[\mathcal{V}' \mapsto s]$ the resource valuation that associates the same values as l to all variables except those in $\mathcal{V}'$, to which it associates the value s, i.e., for any $y \in \mathcal{V}$,

$$l[\mathcal{V}' \mapsto s](y) = \begin{cases} s, & y \in \mathcal{V}' \\ l(y), & \text{otherwise} \end{cases}$$

For $\overline{u} \in \mathbb{R}^k_{\geq 0}$, $l + \overline{u}$ is defined as: for any $i \in \{1, \ldots, k\}$, for any $x \in \mathcal{V}_i$,

$$(l + \overline{u})(x) = l(x) + \overline{u}(e_i).$$

A pair (m, l) is called an *extended state* of a given LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$, where $m \in M$ and $l \in L$. Transitions between extended states are defined by:

$$(m, l) \to_a (m', l') \text{ iff } m \xrightarrow{\overline{u}}_a m' \text{ and } l' = (l + \overline{u})[\mathcal{V}_a \mapsto 0].$$

Given an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$, we interpret the $\mathcal{L}^w$ basic formulas over an extended state (m, l) and an environment ρ which maps each recursive formula variable to subsets of $M \times L$. The *LWS-semantics* of $\mathcal{L}^w$ basic formulas is defined inductively as follows.

$\mathcal{W}, (m, l), \rho \models \top$ – always;

$\mathcal{W}, (m, l), \rho \models \bot$ – never;

$\mathcal{W}, (m, l), \rho \models x \bowtie r$ iff $l(x) \bowtie r$;

$\mathcal{W}, (m, l), \rho \models \phi_1 \wedge \phi_2$ iff $\mathcal{W}, (m, l), \rho \models \phi_i$, $i = 1, 2$;

$\mathcal{W}, (m, l), \rho \models \phi_1 \vee \phi_2$ iff $\mathcal{W}, (m, l), \rho \models \phi_i$, $i = 1$ or 2;

$\mathcal{W}, (m, l), \rho \models [a]\phi$ iff for arbitrary $(m', l') \in M \times L$ such that $(m, l) \to_a (m', l')$, we have $\mathcal{W}, (m', l'), \rho \models \phi$;

$\mathcal{W}, (m, l), \rho \models \langle a\rangle\phi$ iff there exists $(m', l') \in M \times L$ such that $(m, l) \to_a (m', l')$ and $\mathcal{W}, (m', l'), \rho \models \phi$;

$\mathcal{W}, (m, l), \rho \models X$ iff $(m, l) \in \rho(X)$.

Definition 5 (Syntax of Maximal Equation Blocks) Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of recursive formula variables. A maximal equation block B is a list of (mutually recursive) equations:

$$\begin{aligned} X_1 &= \phi_1 \\ &\vdots \\ X_n &= \phi_n \end{aligned}$$

in which X_i are pairwise-distinct over $\mathcal{X}$ and ϕ_i are basic formulas over $\mathcal{X}$, for all $i = 1, \ldots, n$.

Each maximal equation block B defines an *environment* for the recursive formula variables $X_1, \ldots, X_n$, which is the weakest property that the variables satisfy.

We say that an arbitrary formula ϕ is *closed with respect to a maximal equation block B* if all the recursive formula variables appearing in ϕ are defined in B by some of its equations. If every the formula ϕ_i in the right hand side of some equation in B is closed with respect to B, we say that B is *closed*.

Given an environment ρ and $\overline{\Upsilon} = \langle \Upsilon_1, \ldots, \Upsilon_n \rangle \in (2^{M\times L})^n$, let

$$\rho_{\overline{\Upsilon}} = \rho[X_1 \mapsto \Upsilon_1, \ldots, X_n \mapsto \Upsilon_n]$$

be the environment obtained from ρ by updating the binding of X_i to Υ_i.

Given a maximal equation block B and an environment ρ, consider the function

$$f_B^\rho : (2^{M\times L})^n \longrightarrow (2^{M\times L})^n$$

defined as follows:

$$f_B^\rho(\overline{\Upsilon}) = \langle [\![\phi_1]\!]\rho_{\overline{\Upsilon}}, \ldots, [\![\phi_n]\!]\rho_{\overline{\Upsilon}} \rangle,$$

where $[\![\phi]\!]\rho = \{(m,l) \in M \times L \mid \mathcal{W}, (m,l), \rho \models \phi\}$.

Observe that $(2^{M\times L})^n$ forms a complete lattice with the ordering, join and meet operations defined as the point-wise extensions of the set-theoretic inclusion, union and intersection, respectively. Moreover, for any maximal equation block B and environment ρ, f_B^ρ is monotonic with respect to the order of the lattice and therefore, according to the Tarski fixed point theorem [Tarski, 1955], it has a greatest fixed point that we denote by $\nu\overline{X}.f_B^\rho$. This fixed point can be characterized as follows:

$$\nu\overline{X}.f_B^\rho = \bigcup\{\overline{\Upsilon} \mid \overline{\Upsilon} \subseteq f_B^\rho(\overline{\Upsilon})\}.$$

Consequently, a maximal equation block defines an environment that satisfies all its equations, i.e.,

$$[\![B]\!]\rho = \nu\overline{X}.f_B^\rho.$$

When B is closed, i.e. there is no free recursive formula variable in B, it is not difficult to see that for any ρ and ρ', $[\![B]\!]\rho = [\![B]\!]\rho'$. So, we just take $\rho = 0$ and write $[\![B]\!]$ instead of $[\![B]\!]0$. In the rest of the paper we will only discuss this kind of equation blocks. (For those that are not closed, we only need to have the initial environment which maps the free recursive variables to subsets of $M \times L$.)

Now we are ready to define the general semantics of $\mathcal{L}^w$: for an arbitrary LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$ with $m \in M$, an arbitrary resource valuation $l \in L$ and arbitrary $\mathcal{L}^w$-formula ϕ closed w.r.t. a maximal equation block B,

$$\mathcal{W}, (m,l) \models_B \phi \text{ iff } \mathcal{W}, (m,l), [\![B]\!] \models \phi.$$

The symbol $\models_B$ is interpreted as satisfiability for the block B. Whenever it is not the case that $\mathcal{W}, (m,l) \models_B \phi$, we write $\mathcal{W}, (m,l) \not\models_B \phi$. We say that a formula ϕ is *B-satisfiable* if there exists at least one LWS that satisfies it for the block B in one of its states under at least one resource valuation; ϕ is a *B-validity* if it is satisfied in all states of any LWS under any resource valuation - in this case we write $\models_B \phi$.

To exemplify the expressiveness of $\mathcal{L}^w$, we propose the following example of a system with recursively-defined properties.

Example 3 Consider a system which only has one type of resource, e.g., energy. It involves three actions: a, b and c, to which three resource-variables x_a, x_b and x_c are associated respectively. Those resource-variables are used to measure the amount of energy in the system. The specifications of the system are as follows:

1. The system cannot cost one or more than one unit of energy;
2. The system has the following (action) trace: $abcbcbc\ldots$, i.e., it does an a-action followed by infinitely repeating the sequence bc of actions, during which both b and c will have some non-zero cost.

In our logic the above mentioned requirements can be encoded as follows:

$$\phi = \langle a \rangle X,$$
$$B = \left\{ \begin{array}{l} X = x_a < 1 \wedge \langle b \rangle (Y \wedge x_c > 0), \\ Y = x_a < 1 \wedge \langle c \rangle (X \wedge x_b > 0) \end{array} \right\}$$

■

4 Regions and Symbolic Models

Before proceeding with the definitions of regions and symbolic models, we take a further look at Example 3 in the above section. It is not difficult to see that there exists a model satisfying the formula ϕ under the maximal equation block B, but it must be infinite. This is because x_a is synchronised with x_b and x_c, which are constantly growing, while x_a is bounded by 1. This example proves that $\mathcal{L}^w$ does not enjoy the finite model property.

In this section we introduce the region technique for LWS, which is inspired by that for timed automata of Alur and Dill [Alur and Dill, 1990; Alur et al, 1990]. It provides an abstract semantics of LWSs in the form of finite labelled transition systems with the truth value of the $\mathcal{L}^w$ formulas being maintained.

Here we introduce the regions defined for a given maximal constant $N \in \mathbb{N}$. For the case where the maximal constant is a rational number $\frac{p}{q}$, where $p, q \in \mathbb{N}$, we only need to get the regions for the maximal constant p_i first and divide all the regions by q. In fact for this case we could, alternatively, assume that all the constraints involve natural numbers, since the constraints that occur in one formula are finitely many (for instance, we can multiply all the rationals with the same well-chosen integer; by this operation the truth values of the correspondingly modified formulas are preserved).

For $r \in \mathbb{R}_{\geq 0}$, let $\lfloor r \rfloor \overset{\text{def}}{=} \max\{z \in \mathbb{Z} \mid z \leq r\}$ denote the integral part of r, and let $\{r\} = r - \lfloor r \rfloor$ denote its fractional part. Moreover, we have $\lceil r \rceil \overset{\text{def}}{=} \min\{z \in \mathbb{Z} \mid z \geq r\}$.

Definition 6 Let $N \in \mathbb{N}$ be a given maximal constant and let $\mathcal{V}_i$ be a set of resource-variables for resource e_i. Then $l_i, l'_i : \mathcal{V}_i \to \mathbb{R}_{\geq 0}$ are equivalent with respect to N, denoted by $l_i \doteq l'_i$ iff:

1. $\forall x \in \mathcal{V}_i$, $l_i(x) > N$ iff $l'_i(x) > N$;
2. $\forall x \in \mathcal{V}_i$ s.t. $0 \leq l_i(x) \leq N$,
 $\lfloor l_i(x) \rfloor = \lfloor l'_i(x) \rfloor$ and $\{l_i(x)\} = 0 \Leftrightarrow \{l'_i(x)\} = 0$;
3. $\forall x, y \in \mathcal{V}_i$ s.t. $0 \leq l_i(x), l_i(y) \leq N$,
 $\{(l_i(x)\} \leq \{l_i(y)\} \Leftrightarrow \{(l'_i(x)\} \leq \{l'_i(y)\}$.

The equivalence classes under $\doteq$ are called *regions*. $[l_i]$ denotes the region which contains the labelling l_i for resource-variables $x \in \mathcal{V}_i$ and $R^{\mathcal{V}_i}_{N_i}$ denotes the set of all regions for the set $\mathcal{V}_i$ of resource-variables for resource e_i and the constant N_i. Notice that for a given $N_i \in \mathbb{N}$, $R^{\mathcal{V}_i}_{N_i}$ is finite.

For a region $\delta \in R^{\mathcal{V}_i}_{N_i}$, we define the *successor* region as the region δ' – denoted by $\delta \rightsquigarrow \delta'$ – iff:

$$\text{for any } l_i \in \delta, \text{ there exists } d \in \mathbb{R}_{\geq 0} \text{ s.t. } l_i + d \in \delta'.$$

For $\mathcal{V}' \subseteq \mathcal{V}$ and $r \in \mathbb{R}_{\geq 0}$, $\overline{\delta}[\mathcal{V}' \mapsto r]$ is defined as:

$$\overline{\delta}[\mathcal{V} \mapsto r](x) = \begin{cases} r, & x \in \mathcal{V}' \\ \overline{\delta}(x), & \text{otherwise} \end{cases}$$

As we mentioned before, for the case where the maximal constant is a rational number $\frac{p_i}{q_i}$ where $p_i, q_i \in \mathbb{N}$, we only need to get the regions for the maximal constant p first and divide all the regions by q_i to achieve the set of all regions for the set $\mathcal{V}_i$ of resource-variables for resource e_i and the constant $\frac{p_i}{q_i}$ – denoted by $R^{\mathcal{V}_i}_{p_i/q_i}$. Let

$$\mathcal{R}^{\mathcal{V}} = \{\overline{[l]} = ([l_1], \ldots, [l_k]) \mid [l_i] \in R^{\mathcal{V}_i}_{p_i/q_i}, \frac{p_i}{q_i} \in \mathbb{Q}_{\geq 0} \text{ for any } i \in \{1, \ldots, k\}\}$$

For $r \in \mathbb{R}_{\geq 0}$, we use $r \in \overline{[l]}(x)$ to denote $r \in [l_i](x)$, for any $i \in \{1, \ldots, k\}$ and $x \in \mathcal{V}_i$.

We will now define the fundamental concept of a *symbolic model* of LWS. Every extended state (m, l) is replaced by a so-called *extended symbolic state* $(m, \overline{[l]})$. Whenever we have transition between two extended states, there should also be a transition between the corresponding symbolic states, i.e.:

$$(m, \overline{[l]}) \to_a (m', \overline{[l']}) \text{ iff } (m, l) \to_a (m', l').$$

Definition 7 Given $\mathcal{R}^{\mathcal{V}}$ and a non-empty set of states M^s, a *symbolic LWS* is a tuple

$$\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$$

where $\Pi^s \subseteq M^s \times \mathcal{R}^{\mathcal{V}}$ is a non-empty set of *symbolic states* $\pi^s = (m, \overline{\delta})$, Σ^s a non-empty set of actions and

$$\theta^s : \Pi^s \times \Sigma^s \to 2^{\Pi^s}$$

is the *symbolic labelled transition function*, which satisfies the following:

$$\text{if } (m', \overline{\delta'}) \in \theta^s((m, \overline{\delta}), a), \text{ then } \overline{\delta} \rightsquigarrow \overline{\delta'}.$$

Given a symbolic LWS, we can define the symbolic satisfiability relation $\models^s$ with $\pi = (m, \overline{\delta}) \in \Pi^s$ as follows:

$\mathcal{W}^s, \pi, \rho^s \models^s \top$ – always;

$\mathcal{W}^s, \pi, \rho^s \models^s \bot$ – never;

$\mathcal{W}^s, \pi, \rho^s \models^s x \bowtie r$ iff for any $w \in \overline{\delta}(x)$, $w \bowtie r$;

$\mathcal{W}^s, \pi, \rho^s \models^s \phi_1 \wedge \phi_2$ iff $\mathcal{W}^s, \pi, \rho^s \models^s \phi_i$, $i = 1, 2$

$\mathcal{W}^s, \pi, \rho \models^s \phi_1 \vee \phi_2$ iff $\mathcal{W}^s, \pi, \rho^s \models^s \phi_i$, $i = 1$ or 2;

$\mathcal{W}^s, \pi, \rho^s \models^s [a]\phi$ iff for any $\pi' \in \Pi^s$ s.t. $\pi \to_a \pi'$, we have $\mathcal{W}^s, \pi', \rho^s \models^s \phi$;

$\mathcal{W}^s, \pi, \rho^s \models^s \langle a \rangle \phi$ iff there exists $\pi' \in \Pi^s$ s.t. $\pi \to_a \pi'$ and $\mathcal{W}^s, \pi', \rho^s \models^s \phi$;

$\mathcal{W}^s, \pi, \rho^s \models^s X$ iff $m \in \rho^s(X)$.

Similarly we can define the symbolic B-satisfiability relation $\models^s_B$ as in Section 3:

$$\mathcal{W}^s, \pi \models^s_B \phi \;\text{ iff }\; \mathcal{W}^s, \pi, [\![B]\!] \models^s \phi.$$

Lemma 1 *Let ϕ be a $\mathcal{L}^w$ formula closed w.r.t a maximal equation block B. If it is satisfied by a symbolic LWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ i.e. $\mathcal{W}^s, \pi \models^s_B \phi$ with $\pi = (m, \overline{\delta}) \in \Pi^s$, then there exists an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$ and a resource valuation $l \in L$ such that $\mathcal{W}, (m, l) \models_B \phi$ with $m \in M$.*

Proof. Let $\Sigma = \Sigma^s$, $\mathcal{K}$ be the set of the resources appearing in $\mathcal{R}^{\mathcal{V}}$ and $l \in \overline{\delta}$. The transition function is defined as: $(m_1, \overline{\delta_1}, l_1) \xrightarrow{\overline{u}}_a (m_2, \overline{\delta_2}, l_2)$ iff,

1. $(m_1, \overline{\delta_1}) \to_a (m_2, \overline{\delta_2})$,
2. for $i = 1, 2, l_i \in \overline{\delta_i}$,
3. and $l_2 = (l_1 + \overline{u})[\mathcal{V}_a \mapsto 0]$.

We define the transition relation starting from $(m, \overline{\delta}, l)$. Let M be the set of all the states in the form of $(m', \overline{\delta'}, l')$ defined for the transitions as the above. Note that it might be infinite. It is easy to verify that $\mathcal{W}, ((m, \overline{\delta}, l), l) \models_B \phi$. ■

5 Satisfiability of $\mathcal{L}^w$

In this section, we prove that it is decidable whether a given $\mathcal{L}^w$ formula ϕ which is closed w.r.t. a maximal equation block B is satisfiable. We also present a decision procedure for the satisfiability problem of $\mathcal{L}^w$. The results rely on a syntactic characterization of satisfiability that involves a notion of *mutually-consistent sets* that we define later.

Consider an arbitrary formula $\phi \in \mathcal{L}^w$ which is closed w.r.t. a maximal equation block B. In this context we define the following notions:

– Let $\Sigma[\phi, B]$ be the set of all actions $a \in \Sigma$ such that a appears in some transition modality of type $\langle a \rangle$ or $[a]$ in ϕ or B.

– For any $e_i \in \mathcal{K}$ and $x \in \mathcal{V}_i$, let $Q_i[\phi, B] \subseteq \mathbb{Q}_{\geq 0}$ be the set of all $r \in \mathbb{Q}_{\geq 0}$ such that r is in the label of some state or transition modality of type $x \bowtie r$ that appears in the syntax of ϕ or B.

– We denote by g_i the *granularity of e_i in ϕ*, defined as the least common denominator of the elements of $Q_i[\phi, B]$. Let $R^{\mathcal{V}_i}_{p_i/g_i}[\phi, B]$ be the set of all regions for resource e_i, where $\frac{p_i}{g_i} = \max Q_i[\phi, B]$. Let
$$\mathcal{R}^{\mathcal{V}}[\phi, B] = \{\overline{\delta} = (\delta_1, \ldots, \delta_k) \mid \delta_i \in R^{\mathcal{V}_i}_{p_i/g_i}[\phi, B] \text{ for any } i \in \{1, \ldots, k\}\}.$$

Observe that $\Sigma[\phi, B]$, $Q_i[\phi, B]$, $R^{\mathcal{V}_i}_{p_i/g_i}[\phi, B]$ and $\mathcal{R}^{\mathcal{V}}[\phi, B]$ are all finite (or empty).

At this point we can start our model construction. We fix a formula $\phi_0 \in \mathcal{L}^w$ that is closed w.r.t. a given maximal equation block B and, supposing that the formula admits a model, we construct a model for it. Let

$$\mathcal{L}[\phi_0, B] = \{\phi \in \mathcal{L}^w \mid \Sigma[\phi, B] \subseteq \Sigma[\phi_0, B], Q_i[\phi, B] \subseteq Q_i[\phi_0, B]\}.$$

Here we are going to construct a *symbolic model* first. To construct the symbolic model we will use as symbolic states tuples of the type $(\Gamma, \overline{\delta}) \in 2^{\mathcal{L}[\phi_0, B]} \times \mathcal{R}^{\mathcal{V}}[\phi_0, B]$, which are required to be maximal in a precise way. The intuition is that a state $(\Gamma, \overline{\delta}) \subseteq 2^{\mathcal{L}[\phi_0, B]} \times \mathcal{R}^{\mathcal{V}}[\phi_0, B]$ shall symbolically satisfy the formula ϕ in our model, whenever $\phi \in \Gamma$. From this symbolic model we can generalize an LWS - might be infinite - which is a model of the given $\mathcal{L}^w$ formula. Our construction is inspired from the region construction proposed in [Laroussinie et al, 1995] for timed automata, which adapts the classical filtration-based model construction used in modal logics [Hughes and Cresswell, 1996; Harel et al, 2001; Walukiewicz, 2000].

Let $\Omega[\phi_0, B] \subseteq 2^{\mathcal{L}[\phi_0, B]} \times \mathcal{R}^{\mathcal{V}}[\phi_0, B]$. Since $\mathcal{L}[\phi_0, B]$ and $\mathcal{R}^{\mathcal{V}}[\phi_0, B]$ are both finite, $\Omega[\phi_0, B]$ is finite.

Definition 8 (Maximal Pair of $\mathcal{L}^w$)
For any $(\Gamma, \overline{\delta}) \in \Omega[\phi_0, B]$, $(\Gamma, \overline{\delta})$ is said to be *maximal* iff:

1. $\top \in \Gamma$, $\bot \notin \Gamma$;
2. $x \bowtie r \in \Gamma$ iff for any $w \in \mathbb{R}_{\geq 0}$ s.t. $w \in \overline{\delta}(x)$, $w \bowtie r$;
3. $\phi \wedge \psi \in \Gamma$ implies $\phi \in \Gamma$ and $\psi \in \Gamma$;
   $\phi \vee \psi \in \Gamma$ implies $\phi \in \Gamma$ or $\psi \in \Gamma$;
4. $X \in \Gamma$ implies $\phi \in \Gamma$, for $X = \phi \in B$.

The following definition establishes the framework on which we will define our model.

Definition 9 (Mutually-Consistent Set of $\mathcal{L}^w$)
Let $C \subseteq 2^{\Omega[\phi_0, B]}$. C is said to be *mutually-consistent* if for any $(\Gamma, \overline{\delta}) \in C$, whenever $\langle a \rangle \psi \in \Gamma$, then there exists $(\Gamma', \overline{\delta'}) \in C$ s.t.:

1. there exists $\overline{\delta''}$ s.t. $\overline{\delta} \rightsquigarrow \overline{\delta''}$ and $\overline{\delta'} = \overline{\delta''}[\mathcal{V}_a \mapsto 0]$;
2. $\psi \in \Gamma'$;
3. for any $[a]\psi' \in \Gamma$, $\psi' \in \Gamma'$.

We say that $(\Gamma, \overline{\delta})$ is *consistent* if it belongs to some mutually-consistent set.

The following lemma proves a necessary precondition for the model construction.

Lemma 2 *Let $\phi \in \mathcal{L}[\phi_0, B]$ be a formula closed w.r.t. a maximal equation block B. Then ϕ is satisfiable iff there exist $\Gamma \subseteq \mathcal{L}[\phi_0, B]$ and $\overline{\delta} \in \mathcal{R}^{\mathcal{V}}[\phi_0, B]$ s.t.*

$$(\Gamma, \overline{\delta}) \text{ is consistent and } \phi \in \Gamma.$$

Proof. ($\Longrightarrow$): Suppose ϕ is satisfied in the LWS

$$\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$$

under the resource valuation $l \in L$, i.e., there exists $m \in M$ s.t. $\mathcal{W}, (m, l) \models_B \phi$. We construct

$$C = \{(\Gamma, \overline{\delta}) \in \Omega[\phi_0, B] \mid \exists m \in M \text{ s.t. for any } \psi \in \Gamma, \exists l \in \overline{\delta} \text{ s.t. } \mathcal{W}, (m, l) \models_B \psi\}.$$

It is not difficult to verify that C is a mutually-consistent set.

($\Longleftarrow$): Let C be a mutually-consistent set.

We construct a symbolic LWS

$$\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s),$$

where $\Pi^s = C$, $\Sigma^s = \Sigma[\phi_0, B]$ and for $(\Gamma, \overline{\delta}), (\Gamma', \overline{\delta'}) \in C$, the transition relation $(\Gamma, \overline{\delta}) \rightarrow_a (\Gamma', \overline{\delta'})$ is defined iff

1. there exists $\overline{\delta''}$ s.t. $\overline{\delta} \rightsquigarrow \overline{\delta''}$ and $\overline{\delta'} = \overline{\delta''}[\mathcal{V}_a \mapsto 0]$;

2. whenever $[a]\psi \in \Gamma$ then $\psi' \in \Gamma'$.

Let $\rho^s(X) = \{(\Gamma, \overline{\delta}) \mid X \in \Gamma\}$ for $X \in \mathcal{X}$.

With this construction we can prove the following implication by a simple induction on the structure of ϕ, where $(\Gamma, \overline{\delta}) \in \Pi^s$:

$$\phi \in \Gamma \text{ implies } \mathcal{W}^s, (\Gamma, \overline{\delta}), \rho^s \models^s \phi.$$

We prove that ρ^s is a fixed point of B under the assumption that $X = \phi_X \in B$ as follows:

> $\mathcal{W}^s, (\Gamma, \delta), \rho^s \models X$ implies $\Gamma \in \rho^s(X)$ implies $X \in \Gamma$ by the construction of ρ^s, which implies $\phi_X \in \Gamma$ by the definition of $\Omega[\phi_0, B]$.
>
> Then $\mathcal{W}^s, (\Gamma, \overline{\delta}), \rho^s \models^s \phi_X$ by the implication we just proved.

Thus ρ^s is a fixed point of B. Since $[\![B]\!]$ is the maximal fixed point, $\rho^s \subseteq [\![B]\!]$.

Therefore, for any $\phi \in \Gamma$, $(\Gamma, \overline{\delta}) \in C$, we have $\mathcal{W}^s, (\Gamma, \overline{\delta}), \rho^s \models^s \phi$, which implies $\mathcal{W}^s, (\Gamma, \overline{\delta}), [\![B]\!] \models^s \phi$ because $\rho^s \subseteq [\![B]\!]$.

Hence, $\phi \in \Gamma$ and $(\Gamma, \overline{\delta}) \in C$ implies $\mathcal{W}^s, \Gamma \models^s_B \phi$.

By Lemma 1, there exists an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$ and a resource valuation $l \in L$ such that $\mathcal{W}, (m, l) \models_B \phi$ with $m \in M$. ∎

To summarize, the above lemmas allow us to conclude the model constructions.

Theorem 1 *For any satisfiable $\mathcal{L}^w$ formula ϕ closed w.r.t. a maximal equation block B, there exists a finite symbolic LWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ such that $\mathcal{W}^s, \pi \models^s_B \phi$ for some $\pi \in \Pi^s$. Reversely, if a $\mathcal{L}^w$ formula ϕ is satisfied by a symbolic model, then it is satisfiable, i.e., there exists an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$ and a resource valuation $l \in L$ such that $\mathcal{W}, (m, l) \models_B \phi$ for some $m \in M$.*

Lemma 2 and Theorem 1 provide a decision procedure for the satisfiability problem of $\mathcal{L}^w$. Given a $\mathcal{L}^w$ formula ϕ_0 closed w.r.t. a maximal equation block B, the algorithm constructs the model

$$\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta).$$

To do this, we first construct the symbolic LWS

$$\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$$

such that ϕ_0 is satisfied in some state $\pi \in \Pi^s$, i.e. $\mathcal{W}^s, \pi \models^s_B \phi_0$, where $\Sigma^s = \Sigma[\phi_0, B]$.

If ϕ_0 is satisfiable, then it is contained in some maximal set Γ, where $(\Gamma, \overline{\delta})$ is consistent with $\overline{\delta} \in \mathcal{R}^{\mathcal{V}}[\phi_0, B]$. Hence, ϕ_0 will be satisfied at some state π of $\mathcal{W}^s$. If ϕ_0 is not satisfiable, then the attempt to construct a model will fail; in this case the algorithm will halt and report the failure.

We start with a superset of the set of states of $\mathcal{W}$, then repeatedly delete states when we discover some inconsistency. This will give a sequence of approximations

$$\mathcal{W}^s_0 \supseteq \mathcal{W}^s_1 \supseteq \mathcal{W}^s_2 \supseteq \ldots$$

converging to $\mathcal{W}^s$.

The domains Π^s_i, $i = 0, 1, 2, \ldots$, of these structures are defined below and they are s.t.

$$\Pi^s_0 \supseteq \Pi^s_1 \supseteq \Pi^s_2 \supseteq \ldots.$$

The transition relation for $\mathcal{W}^s_i$ are defined as follows: for any $(\Gamma, \overline{\delta}), (\Gamma', \overline{\delta'}) \in \Pi^s_i$, $(\Gamma, \overline{\delta}) \rightarrow_a (\Gamma', \overline{\delta'})$ iff

1. there exists $\overline{\delta''}$ s.t. $\overline{\delta} \rightsquigarrow \overline{\delta''}$ and $\overline{\delta'} = \overline{\delta''}[\mathcal{V}_a \mapsto 0]$;
2. whenever $[a]\psi \in \Gamma$ then $\psi \in \Gamma'$.

The following is the algorithm for constructing the domains Π^s_i of $\mathcal{W}^s_i$.

Algorithm

– **Step 1:** Construct $\Pi^s_0 = \Omega[\phi_0, B]$.

– **Step 2:** Repeat the following for $i = 0, 1, 2, \ldots$ until no more states are deleted.

  Find a formula $[a]\phi \in \mathcal{L}[\phi_0, B]$ and a state $(\Gamma, \overline{\delta}) \in \Pi^s_i$ violating the following property

  $$[\forall (\Gamma', \overline{\delta'}) \in \Pi^s_i, (\Gamma, \overline{\delta}) \rightarrow_a (\Gamma', \overline{\delta'}) \Rightarrow \phi \in \Gamma'] \text{ implies } [a]\phi \in \Gamma.$$

  that is: there exists $\langle a \rangle \neg\phi \in \Gamma$, but for no Γ' and $\overline{\delta'}$ such that $(\Gamma, \overline{\delta}) \rightarrow_a (\Gamma', \overline{\delta'})$, it is the case that $\neg\phi \in \Gamma'$.

  Pick such an $[a]\phi$ and $(\Gamma, \overline{\delta})$.
  Delete $(\Gamma, \overline{\delta})$ from Π^s_i to get Π^s_{i+1}.

Step 2 can be justified intuitively as follows. To say that $(\Gamma, \overline{\delta})$ violates the above mentioned condition, it means that $(\Gamma, \overline{\delta})$ requires an a-transition to some state that does not satisfy ϕ; however, the left-hand side of the condition above guarantees that all the outcomes of an a-transition satisfy ϕ. This demonstrates that $(\Gamma, \overline{\delta})$ can not be in Π^s, since every state $(\Gamma, \overline{\delta})$ in Π^s satisfies ψ, whenever $\psi \in \Gamma$.

The algorithm must terminate, since there are only finitely many states initially, and at least one state must be deleted during each iteration of step 2 in order to continue. Then, ϕ is satisfiable if and only if, upon termination there exists $(\Gamma, \overline{\delta}) \in \Pi^s$ such that $\phi \in \Gamma$. Obviously, Π^s is a mutually-consistent set upon termination.

The correctness of this algorithm follows from the proof of Lemma 2. The direction ($\Leftarrow$) of the proof guarantees that all formulas in any Γ with $(\Gamma, \overline{\delta}) \in \Pi^s$ are satisfiable. The direction ($\Rightarrow$) of the proof guarantees that all satisfiable Γ will not be deleted from Π^s.

After we get the symbolic LWS $\mathcal{W}^s$, we can use the technique in Lemma 1 to generalize an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$, which might be infinite.

Suppose ϕ_0 is satisfied by $(\Gamma, \overline{\delta}) \in \Pi^s$, i.e., $\mathcal{W}^s, (\Gamma, \overline{\delta}) \models^s_B \phi_0$. Let $\Sigma = \Sigma^s$, $\mathcal{K}$ be the set of the resources appearing in $\mathcal{R}^{\mathcal{V}}[\phi_0, B]$ and $l \in \overline{\delta}$. The transition function is defined as: $(\Gamma_1, \overline{\delta_1}, l_1) \xrightarrow{\overline{u}}_a (\Gamma_2, \overline{\delta_2}, l_2)$ iff,

1. $\Gamma_1, \overline{\delta_1}) \rightarrow_a (\Gamma_2, \overline{\delta_2})$,
2. for $i = 1, 2, l_i \in \overline{\delta_i}$,
3. and $l_2 = (l_1 + \overline{u})[\mathcal{V}_a \mapsto 0]$.

We define the transition relation starting from $(\Gamma, \overline{\delta}, l)$. Let M be the set of all the states in the form of $(\Gamma', \overline{\delta'}, l')$ defined for the transitions as above. Note that the model defined this way might be infinite. It is easy to verify that $\mathcal{W}, ((\Gamma, \overline{\delta}, l), l) \models_B \phi_0$.

Theorem 1, also supported by the above algorithm, demonstrates the decidability of the B-satisfiability problem for $\mathcal{L}^w$.

Theorem 2 (Decidability of B-satisfiability)
For an arbitrary maximal equation block B, the B-satisfiability problem for $\mathcal{L}^w$ is decidable.

Example 4 Now we can discuss the satisfiability of the formula ϕ in Example 3.

$$\phi = \langle a \rangle X,$$
$$B = \left\{ \begin{array}{l} X = x_a < 1 \wedge \langle b \rangle (Y \wedge x_c > 0), \\ Y = x_a < 1 \wedge \langle c \rangle (X \wedge x_b > 0) \end{array} \right\}$$

In Figure3 is the symbolic LWS for the above formula ϕ w.r.t B obtained by applying our algorithm. Here the details of using the algorithm to get the model are not presented, limited by the length of the paper, which is very technical.

$\Gamma_0 = \{\phi, \langle a \rangle X\}$
$\Gamma_1 = \{X, x_a < 1, \langle b \rangle (Y \wedge x_c > 0)\}$
$\Gamma_2 = \{Y, x_c > 0, x_a < 1, \langle c \rangle (X \wedge x_b > 0)\}$
$\Gamma_3 = \{X, x_b > 0, x_a < 1, \langle b \rangle (Y \wedge x_c > 0)\}$

$\delta_0 = [x_a = x_b = x_c = 0]$
$\delta_1 = [x_a = 0, 0 < x_b = x_c < 1]$
$\delta_2 = [x_b = 0, 0 < x_a = x_c < 1]$
$\delta_3 = [x_b = 0, 0 < x_a < x_c < 1]$
$\delta_4 = [x_c = 0, 0 < x_b < x_a < 1]$
$\delta_5 = [x_b = 0, 0 < x_c < x_a < 1]$

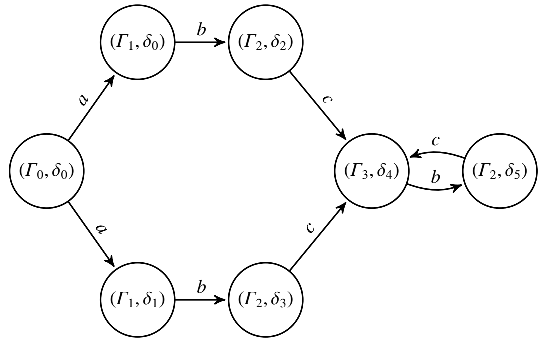

Fig. 3 Symbolic LWS for ϕ

From the symbolic model in Figure 3, one can generate an LWS, which in this case is infinite, where ϕ is satisfied in some state of it. In Figure 4 (on the next page), we show part of this infinite model.

$l_0 = (0, 0, 0)$	$l_5 = (0.2, 0, 0.3)$
$l_1 = (0.1, 0.1, 0)$	$l_6 = (0.5, 0.2, 0)$
$l_2 = (\frac{\pi}{4}, \frac{\pi}{4}, 0)$	$l_7 = (0.4, 0.2, 0)$
$l_3 = (0.3, 0, 0.3)$	$l_8 = (0.6, 0, 0.1)$
$l_4 = (\frac{\pi}{4}, 0, \frac{\pi}{4})$	$l_9 = (0.5, 0, 0.1)$
......	

■

6 Extension of $\mathcal{L}^w$

The recursive weighted modal logic $\mathcal{L}^w$ introduced in Section 3 can be used to encode various interesting scenarios. However, it is not adequate in the sense that bisimilarity of the models does not coincide with the semantic equivalence

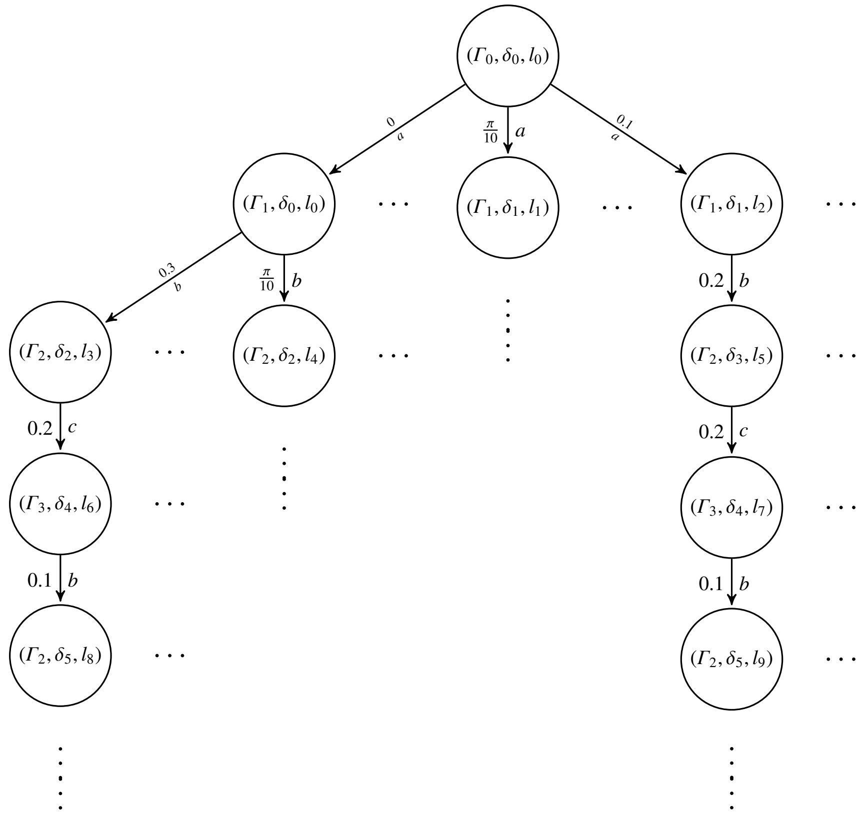

Fig. 4 Generalizing LWS from the symbolic model

induced by the logic over the class of models (Hennessy-Milner property). For example, consider the following two simple systems in Figure 5 (on the next page). Both of them have one type of resource e and one action a - therefore, only one resource-variable x_a^e. It is not difficult to see that they cannot be distinguished by any formula or theory (set of formulas) of $\mathcal{L}^w$.

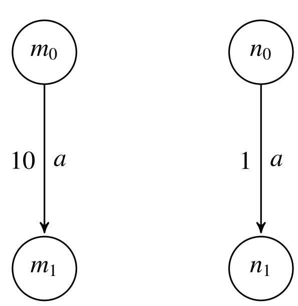

Fig. 5 Counter example of $\mathcal{L}^w$ Adequacy

In the following we introduce the recursive weighted logic $\mathcal{L}^t$, which extends $\mathcal{L}^w$ by allowing resource constraints in the transition modalities. For $\mathcal{L}^t$ the maximal equation blocks are defined similarly to for $\mathcal{L}^w$, on top of a class of basic formulas. We show that $\mathcal{L}^t$ is adequate and that its satisfiability problem is still decidable.

In $\mathcal{L}^t$, the resource-variables are still event related, as in $\mathcal{L}^w$, i.e., after each action, the corresponding resource-variables will be reset to zero. The same as $\mathcal{L}^w$, we use $\mathcal{V}_i = \{x_a^i \mid a \in \Sigma\}$ to denote the set of the resource-variables associated for the type of resource e_i, $\mathcal{V}_a = \{x_a^i \mid i = 1, \ldots, k\}$ to denote the set of the resource-variables associated with the action a and $\mathcal{V} = \bigcup_{i=1,\ldots,k} \mathcal{V}_i = \bigcup_{a \in \Sigma} \mathcal{V}_a$ to denote the set of all the resource-variables.

Definition 10 (Syntax of $\mathcal{L}^t$ Basic Formulas)
For arbitrary $r \in \mathbb{Q}_{\geq 0}$, $a \in \Sigma$, $x \in \mathcal{V}$, $\bowtie \in \{\leq, \geq, <, >\}$, $I \subseteq \{1, \ldots, k\}$ and $X \in \mathcal{X}$, let

$$\begin{aligned}\mathcal{L}^t: \quad \phi^t &:= \top \mid \bot \mid x \bowtie r \mid \phi^t \wedge \phi^t \mid \phi^t \vee \phi^t \\ &\mid [\textstyle\bigwedge_{i \in I}(e_i \bowtie r_i)]_a \phi^t \mid \langle \textstyle\bigwedge_{i \in I}(e_i \bowtie r_i)\rangle_a \phi^t \mid X\,.\end{aligned}$$

The semantics of $\mathcal{L}^t$ basic formulas is defined similar to $\mathcal{L}^w$ except for that of the transition modalities, which is defined as follows:

$\mathcal{W}, (m, l), \rho \models^t [\bigwedge_{i \in I}(e_i \bowtie r_i)]_a \phi^t$ iff for arbitrary $(m', l') \in M \times L$ such that $(m, l) \rightarrow_a (m', l')$ and $(l' - l)(e_i) \bowtie r_i$ for any $i \in I$, we have $\mathcal{W}, (m', l'[\mathcal{V}_a \mapsto 0]), \rho \models^t \phi^t$,

$\mathcal{W},(m,l),\rho \models^t \langle \bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a \phi^t$ iff there exists $(m',l') \in M\times L$ such that $(m,l)\to_a (m',l')$,$(l'-l)(e_i)\bowtie r_i$ for any $i\in I$ and $\mathcal{W},(m',l'[\mathcal{V}_a\mapsto 0]),\rho \models^t \phi^t$.

As for $\mathcal{L}^w$, we can define the semantics of $\mathcal{L}^t$ with maximal equation blocks B^t:

$$\mathcal{W},(m,l)\models_{B^t} \phi^t \text{ iff } \mathcal{W},(m,l),[\![B^t]\!]\models^t \phi^t.$$

6.1 Adequacy of $\mathcal{L}^t$

It is clear that $\mathcal{L}^t$ is more expressive than $\mathcal{L}^w$. We will show in the following that $\mathcal{L}^t$ is sufficiently expressive to characterize weighted bisimilarity, i.e., $\mathcal{L}^t$ enjoys the Hennessy-Milner property. We do this by proving that the logic without fixed points is already adequate. Since we do not consider the fixed points, the environment is not necessary for the semantics. So, in the following lemma, we only write $\models$ instead of $\models_{B^t}$ and we use $\mathcal{L}^t/X$ to denote $\mathcal{L}^t$ without fixed points.

Lemma 3 *Let $\mathcal{W}=(M,\mathcal{K},\Sigma,\theta)$ be an image-finite labelled weighted transition system. Then, for any $m,m'\in M$:*

$$m\sim m' \;\; iff \;\; \forall\phi^t\in\mathcal{L}^t/X \text{ and } l\in L \quad \mathcal{W},(m,l)\models\phi^t \Leftrightarrow \mathcal{W},(m',l)\models\phi^t.$$

Proof. ($\Longrightarrow$) Induction on ϕ^t. The cases $\top$, $\bot$ and $\phi^t\wedge\psi^t$, $\phi^t\vee\psi^t$ are easy.

- *Case $x\bowtie r$*:
  $\mathcal{W},(m,l)\models x\bowtie r$ implies $l(x)\bowtie r$, which implies $\mathcal{W},(m',l)\models x\bowtie r$.
  Hence, $\mathcal{W},(m,l)\models x\bowtie r$ implies $\mathcal{W},(m',l)\models x\bowtie r$.
  Similarly $\mathcal{W},(m',l)\models x\bowtie r$ implies $\mathcal{W},(m,l)\models x\bowtie r$.
- *Case $[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$*:
  $\mathcal{W},(m,l)\models[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$ implies that for any $(m_1,l_1)\in M\times L$ such that: 1. $(m,l)\to_a(m_1,l_1)$, 2. $(l_1-l)(e_i)\bowtie r_i$ for any $i\in I$, we have $\mathcal{W},(m_1,l_1)\models\phi^t$.
  $(m,l)\to_a(m_1,l_1)$ implies $m\xrightarrow{\overline{u}}_a m_1$ and $l_1=l+\overline{u}$.
  Since $m\sim m'$, for any $m_1'\in M$ s.t. $m'\xrightarrow{\overline{u}}_a m_1'$, there exists $m_1\in M$ s.t. $m\xrightarrow{\overline{u}}_a m_1$ and $m_1\sim m_1'$.
  $\mathcal{W},(m_1,l_1)\models\phi^t$ implies $\mathcal{W},(m_1',l_1)\models\phi^t$ by inductive hypothesis.
  So for any $(m_1',l_1)\in M$ such that: 1. $(m',l)\to_a(m_1',l_1)$, 2. $(l_1-l)(e_i)\bowtie r_i$ for any $i\in I$, we have $\mathcal{W},(m_1',l_1)\models\phi^t$.
  Then $\mathcal{W},(m',l)\models[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$.
  Hence, $\mathcal{W},(m,l)\models[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$ implies
  $\mathcal{W},(m',l)\models[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$.

  Similarly, $\mathcal{W},(m',l)\models[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$ implies
  $\mathcal{W},(m,l)\models[\bigwedge_{i\in I}(e_i\bowtie r_i)]_a\phi^t$.
- *Case $\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$*:
  $\mathcal{W},(m,l)\models\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$ implies that there exists $(m_1,l_1)\in M\times L$ such that: 1. $(m,l)\to_a(m_1,l_1)$, 2. $(l_1-l)(e_i)\bowtie r_i$ for any $i\in I$ and 3. $\mathcal{W},(m_1,l_1)\models\phi^t$.
  $(m,l)\to_a(m_1,l_1)$ implies $m\xrightarrow{\overline{u}}_a m_1$ and $l_1=l+\overline{u}$.
  Since $m\sim m'$, there exists m_1' such that $m'\xrightarrow{\overline{u}}_a m_1'$ and $m_1\sim m_1'$.
  $\mathcal{W},(m_1,l_1)\models\phi^t$ implies $\mathcal{W},(m_1',l_1)\models\phi^t$ by inductive hypothesis.
  So we have that there exists $(m_1',l_1)\in M\times L$ such that: 1. $(m',l)\to_a(m_1',l_1)$, 2. $(l_1-l)(e_i)\bowtie r_i$ for any $i\in I$ and 3. $\mathcal{W},(m_1',l_1)\models\phi^t$.
  Therefore, $\mathcal{W},(m',l)\models\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$.
  Hence, $\mathcal{W},(m,l)\models\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$ implies
  $\mathcal{W},(m',l)\models\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$.

  Similarly, $\mathcal{W},(m',l)\models\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$ implies
  $\mathcal{W},(m,l)\models\langle\bigwedge_{i\in I}(e_i\bowtie r_i)\rangle_a\phi^t$.

($\Longleftarrow$) Let $R=\{(m,m')\mid$
$\forall\phi^t\in\mathcal{L}^t/X,\mathcal{W},(m,l)\models\phi^t\Leftrightarrow\mathcal{W},(m',l)\models\phi^t\}$.
We prove that R is a weighted bisimulation relation.

- If $m\xrightarrow{\overline{u}}_a m_1$:
  If for any $I\subseteq\{1,\ldots,k\}$ and for any $i\in I$, $r_i\in\mathbb{Q}$ s.t. $\overline{u}(e_i)\bowtie r$, there exists no $m_1'\in M$ s.t. $m'\xrightarrow{\overline{u}}_a m_1'$ and $\mathcal{W},(m',l)\models[\bigwedge_{i\in I}(e_i\bowtie r)]_a\bot$.
  Then, $\mathcal{W},(m,l)\models[\bigwedge_{i\in I}(e_i\bowtie r)]_a\bot$ since $(m,m')\in R$.
  This contradicts the premise! Hence, there exists at least one $m_1'\in M$ s.t $m'\xrightarrow{\overline{u}}_a m_1'$.

  Suppose $S=\{m_h'\mid m'\xrightarrow{\overline{u}}_a m_h'\}$ and $(m_1,m_h')\notin R$ for any h, i.e. for any h, there exists l_h and ϕ_h^t s.t.
  $\mathcal{W},(m_1,l_h)\models\phi_h^t$ and $\mathcal{W},(m_h',l_h)\not\models\phi_h^t$.

  For any h and for any $x_b^i\in\mathcal{V}(\phi_h^t)$ with $b\in\Sigma$ and $i\in\{1,\ldots,k\}$, we introduce one new action b_h and one new resource-variable $x_{b_h}^i$.
  Let $\phi_h^{t\,\prime}=\phi_h^t\{x_{b_h}^i/x_b^i\}$ for every ϕ_h^t; let $l'(x_{b_h}^i)=l_h(x_b^i)$ for any $x_{b_h}^i$ and $l'(x_a^i)=0$ for any x_a^i with $a\in\Sigma$.
  Then for all h, we have:
  $\mathcal{W},(m_1,l')\models\bigwedge_h\phi_h^{t\,\prime}$ and $\mathcal{W},(m_h',l')\not\models\phi_h^{t\,\prime}$.

  Then there exists l such that: 1. $l'=(l+\overline{u})[\mathcal{V}\mapsto 0]$, 2. $\mathcal{W},(m,l)\models[]_a\bigwedge_h\phi_h^{t\,\prime}$ and 3. $\mathcal{W},(m',l)\not\models[]_a\bigwedge_h\phi_h^{t\,\prime}$.
  Contradiction!
  Hence, there exists $m_1'\in M$ s.t. $m'\xrightarrow{\overline{u}}_a m_1'$ and $m_1\sim m_1'$.
- If $m'\xrightarrow{\overline{u}}_a m_1'$: similar as above. ∎

Theorem 3 (Adequancy of $\mathcal{L}^t$)
Let
$$\mathcal{W}=(M,\mathcal{K},\Sigma,\theta)$$
be an image-finite labelled weighted transition system. Then,

for any $m, m' \in M$:

$$m \sim m' \quad \textit{iff for any } \phi^t \in \mathcal{L}^t \textit{ under } B^t \textit{ and } l \in L, \\ \mathcal{W}, (m, l) \models_{B^t} \phi^t \Leftrightarrow \mathcal{W}, (m', l) \models_{B^t} \phi^t.$$

6.2 Satisfiability of $\mathcal{L}^t$

We can use a similar technique to the one used in Section 5 for checking satisfiability of $\mathcal{L}^w$, to decide the satisfiability of $\mathcal{L}^t$ by adding the requirements of the weights on the transitions. More precisely, we define the symbolic semantics for $\mathcal{L}^t$ and add the requirements of the weights on the transitions and the definitions of maximal pair $(\Gamma, \overline{\delta})$ and mutually-consistent set C as follows:

$\mathcal{W}^s, \pi, \rho^s \models^s [\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi$ iff:
for any $\pi' = (m', \overline{\delta'}) \in \Pi^s$ s.t.
(1) $\pi \to_a \pi'$ and
(2) $w' - w \bowtie r_i$ for any $i \in I$ and $w \in \overline{\delta}(e_i), w' \in \overline{\delta'}(e_i)$,
we have $\mathcal{W}^s, \pi', \rho^s \models^s \phi$,

$\mathcal{W}^s, \pi, \rho^s \models^s \langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi$ iff:
there exists $\pi' = (m', \overline{\delta'}) \in \Pi^s$ s.t.
(1) $\pi \to_a \pi'$,
(2) $w' - w \bowtie r_i$ for any $i \in I$ and $w \in \overline{\delta}(e_i), w' \in \overline{\delta'}(e_i)$
and
(3) $\mathcal{W}^s, \pi', \rho^s \models^s \phi$.

Definition 11 (Maximal Pair of $\mathcal{L}^t$)
For any $(\Gamma, \overline{\delta}) \in \Omega[\phi_0^t, B^t]$, $(\Gamma, \overline{\delta})$ is said to be *maximal* iff:

1. $\top \in \Gamma$, $\bot \notin \Gamma$;
2. $x \bowtie r \in \Gamma$ iff for any $w \in \mathbb{R}_{\geq 0}$ s.t. $w \in \overline{\delta}(x)$, $w \bowtie r$;
3. $\phi \wedge \psi \in \Gamma$ implies $\phi \in \Gamma$ and $\psi \in \Gamma$;
$\phi \vee \psi \in \Gamma$ implies $\phi \in \Gamma$ or $\psi \in \Gamma$;
4. $\langle\bigwedge_{i\in I} e_i < r_i\rangle_a\phi \in \Gamma$ implies $\langle\bigwedge_{i\in I} e_i \leq r_i\rangle_a\phi \in \Gamma$,
$\langle\bigwedge_{i\in I} e_i > r_i\rangle_a\phi \in \Gamma$ implies $\langle\bigwedge_{i\in I} e_i \geq r_i\rangle_a\phi \in \Gamma$;
5. $\langle\bigwedge_{i\in I} e_i \leq r_i\rangle_a\phi \in \Gamma$ implies $\langle\bigwedge_{i\in I} e_i < r_i + s_i\rangle_a\phi \in \Gamma$,
$\langle\bigwedge_{i\in I} e_i \geq r_i\rangle_a\phi \in \Gamma$ implies $\langle\bigwedge_{i\in I} e_i > r_i - s_i\rangle_a\phi \in \Gamma$,
for any $s_i \in \mathbb{Q}$ and $s_i > 0$;
6. $X \in \Gamma$ implies $\phi \in \Gamma$, for $X = \phi \in B$.

Definition 12 (Mutually-Consistent Set of $\mathcal{L}^t$)
Let $C \subseteq 2^{\Omega[\phi_0^t, B^t]}$. C is said to be *mutually-consistent* if for any $(\Gamma, \overline{\delta}) \in C$, whenever $\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\psi \in \Gamma$, then there exists $(\Gamma', \overline{\delta'}) \in C$ s.t.:

1. there exists $\overline{\delta''}$ s.t.
(1) $\overline{\delta} \rightsquigarrow \overline{\delta''}$,
(2) $\overline{\delta'} = \overline{\delta''}[\mathcal{V}_a \mapsto 0]$ and
(3) $w' - w \bowtie r_i$ for any $i \in I$ and $w \in \overline{\delta}(e_i), w' \in \overline{\delta'}(e_i)$;
2. $\psi \in \Gamma'$;
3. for any $[\bigwedge_{j\in J}(e_j \bowtie r_j)]_a\psi' \in \Gamma$,
(1) $w' - w \bowtie r_j$ for any $j \in J$ and $w \in \overline{\delta}(e_j), w' \in \overline{\delta'}(e_j)$
and
(2) $\psi' \in \Gamma'$.

Based on the above definitions, we can prove similar lemma and theorem as Lemma 2 and Theorem 1. Therefore, we prove that the satisfiability problem of $\mathcal{L}^t$ is decidable.

Alternatively, one can solve this problem in the following way:

- firstly encode this problem into a satisfiability problem similar to that of $\mathcal{L}^w$, by translating the given $\mathcal{L}^t$ formula into $\mathcal{L}^w$ with a special 0-cost action;
- secondly, check the satisfiability of the new $\mathcal{L}^w$ formula by using the model construction algorithm of Section 5 with a minor modification;
- and thirdly, if the $\mathcal{L}^w$ formula is not satisfiable then the given $\mathcal{L}^t$ formula is not satisfiable either, otherwise the given $\mathcal{L}^t$ formula is satisfiable and we can finally generate the model for it according to the model for the corresponding $\mathcal{L}^w$ formula.

In the following we show how to do this in details.

Firstly, we define the function that translate a $\mathcal{L}^t$ formula into $\mathcal{L}^w$ with a special 0-cost action (ε).

Let $F : \mathcal{L}^t \to \mathcal{L}^w$ be a function encoding $\mathcal{L}^t$ formulas into $\mathcal{L}^w$ defined inductively. The basic cases are trivial. We only encode the transition modalities as follows, where ε is a new introduced action which always cost 0 unit of all types of resources and x_ε^i for all $i \in \{1, \ldots, k\}$ are the new introduced resource-variables:

$$F([\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t) = [\varepsilon][a](\bigwedge_{i\in I}(x_\varepsilon^i \bowtie r_i) \to F(\phi^t)),$$

$$F(\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t) = \langle\varepsilon\rangle\langle a\rangle(\bigwedge_{i\in I}(x_\varepsilon^i \bowtie r_i) \wedge F(\phi^t)).$$

Based on these, we can also extend F to the maximal equation blocks, defined as:

$$F(B) = \{X = F(\phi_X) \mid X = \phi_X \in B^t\}.$$

Lemma 4 *Let* $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$ *be an LWS and* $\mathcal{W}' = (M, \mathcal{K}, \Sigma', \theta')$ *be an LWS extended from* $\mathcal{W}$*, which is defined by taking*

$$\Sigma' = \Sigma \cup \varepsilon, \theta' = \theta \cup \{m \xrightarrow{\overline{0}}_\varepsilon m) \mid m \in M\}.$$

Then, for any $\phi^t \in \mathcal{L}^t$ *closed w.r.t* B^t*, any* $m \in M$ *and* $l \in L$*,*

$$\mathcal{W}, (m, l) \models_{B^t} \phi^t \textit{ iff } \mathcal{W}', (m, l') \models_B F(\phi^t),$$

where $F(\phi^t)$ is the corresponding $\mathcal{L}^w$ formula closed w.r.t $B = F(B^t)$ and $l' : \mathcal{V}' = \mathcal{V} \cup \{x^i_\varepsilon \mid i \in \{1,\dots,k\}\} \to \mathbb{R}_{\geq 0}$ is a variable valuation satisfying $l'(x) = l(x)$ for any $x \in \mathcal{V}$.

Proof. ($\Rightarrow$) Induction on the structure of ϕ^t. The basic cases are trivial.

- *Case* $[\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t$:
  $\mathcal{W},(m,l) \models_{B'} [\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t$ implies that for any $(m_1,l_1) \in M \times L$ such that: 1. $(m,l) \to_a (m_1,l_1)$, 2. $(l_1 - l)(e_i) \bowtie r_i$ for any $i \in I$, we have $\mathcal{W},(m_1,l_1[\mathcal{V}_a \mapsto 0]) \models_{B'} \phi^t$.

  By the construction of $\mathcal{W}'$, $m \xrightarrow{\overline{0}}_\varepsilon m$, which implies $(m,l') \to_\varepsilon (m,l')$.
  Suppose 1. $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$, 2. $l'_1(x^i_\varepsilon) \bowtie r_i$ for any $i \in I$.

  $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$ implies $m \xrightarrow{\overline{u}}_a m_1$ and $l'_1 - l'[\mathcal{V}_\varepsilon \mapsto 0] = \overline{u}$. For any $i \in I$, $l'[\mathcal{V}_\varepsilon \mapsto 0](x^i_\varepsilon) = 0$, so $(l'_1 - l'[\mathcal{V}_\varepsilon \mapsto 0])(e_i) \bowtie r_i$.
  Let $l_1 : \mathcal{V} \to \mathbb{R}_{\geq 0}$ be a variable valuation defined as: for any $x \in \mathcal{V}$, $l_1(x) = l'_1(x)$. Then $l_1 - l = \overline{u}$ and $(l_1 - l)(e_i) \bowtie r_i$ for any $i \in I$.
  Hence $(m,l) \to_a (m_1,l_1)$ and $(l_1 - l)(e_i) \bowtie r_i$ for any $i \in I$, which implies $\mathcal{W},(m_1,l_1[\mathcal{V}_a \mapsto 0]) \models_{B'} \phi^t$. By inductive hypothesis, $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\phi^t)$.
  Hence $\mathcal{W}',(m,l') \models_B F([\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t)$.

- *Case* $\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t$:
  $\mathcal{W},(m,l) \models_{B'} \langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t$ implies that there exists $(m_1,l_1) \in M\times L$ s.t.: 1. $(m,l) \to_a (m_1,l_1)$, 2. $(l_1 - l)(e_i) \bowtie r_i$ for any $i \in I$, 3. $\mathcal{W},(m_1,l_1[\mathcal{V}_a \mapsto 0]) \models_{B'} \phi^t$.

  By the construction of $\mathcal{W}'$, $m \xrightarrow{\overline{0}}_\varepsilon m$, which implies $(m,l') \to_\varepsilon (m,l')$.

  $(m,l) \to_a (m_1,l_1)$ implies $m \xrightarrow{\overline{u}}_a m_1$ and $l_1 - l = \overline{u}$.
  Let $l'_1 : \mathcal{V}' \to \mathbb{R}_{\geq 0}$ be a variable valuation defined as:
  (1) $l'_1(x) = l_1(x)$ for any $x \in \mathcal{V}$, (2) $l'_1(x^i_\varepsilon) = \overline{u}(e_i)$ for any $x^i_\varepsilon \in \mathcal{V}' - \mathcal{V}$. Then $l'_1 - l'[\mathcal{V}_\varepsilon \mapsto 0] = \overline{u}$.
  So $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$.
  For any $i \in I$, $l'_1[\mathcal{V}_a \mapsto 0](x^i_\varepsilon) = l'_1(x^i_\varepsilon) = \overline{u}(e_i)$, which implies $l'_1[\mathcal{V}_a \mapsto 0](x^i_\varepsilon) \bowtie r_i$.
  By inductive hypothesis $\mathcal{W},(m_1,l_1[\mathcal{V}_a \mapsto 0]) \models_{B'} \phi^t$ implies $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\phi^t)$.
  Hence $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t)$.

($\Leftarrow$) Induction on the structure of ϕ^t. The basic cases are trivial.

- *Case* $[\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t$:

  By the construction of $\mathcal{W}'$, $m \xrightarrow{\overline{0}}_\varepsilon m$, which implies $(m,l') \to_\varepsilon (m,l')$. Hence $\mathcal{W}',(m,l') \models_B F([\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t)$ implies for any $(m_1,l'_1) \in M \times L$ such that: 1. $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$, 2. $l'_1(x^i_\varepsilon) \bowtie r_i$ for any $i \in I$, we have $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\phi^t)$.

  Suppose 1. $(m,l) \to_a (m_1,l_1)$ with $l_1 - l = \overline{u}$, 2. $\overline{u}(e_i) \bowtie r_i$ for any $i \in I$.

  $(m,l) \to_a (m_1,l_1)$ with $l_1 - l = \overline{u}$ implies $m \xrightarrow{\overline{u}}_a m_1$.
  Let $l'_1 : \mathcal{V}' \to \mathbb{R}_{\geq 0}$ be a variable valuation defined as:
  (1) $l'_1(x) = l_1(x)$ for any $x \in \mathcal{V}$, (2) $l'_1(x^i_\varepsilon) = \overline{u}(e_i)$ for any $x^i_\varepsilon \in \mathcal{V}' - \mathcal{V}$. Then $l'_1 - l'[\mathcal{V}_\varepsilon \mapsto 0] = \overline{u}$.
  So $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$.
  And for any $i \in I$, $l'_1(x^i_\varepsilon) = \overline{u}(e_i) \bowtie r_i$.
  Hence $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\phi^t)$. By inductive hypothesis, $\mathcal{W},(m_1,l_1[\mathcal{V}_a \mapsto 0]) \models_{B'} \phi^t$.
  Therefore, $\mathcal{W},(m,l') \models_{B'} [\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t$.

- *Case* $\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t$:
  By construction of $\mathcal{W}'$, $m \xrightarrow{\overline{0}}_\varepsilon m$, which implies $(m,l') \to_\varepsilon (m,l')$. So $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t)$ implies there exists $(m_1,m'_1) \in M \times L$ such that:
  1. $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$, 2. $l'_1[\mathcal{V}_a \mapsto 0](x^i_\varepsilon) \bowtie r_i$ for any $i \in I$, 3. $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\phi^t)$.

  $(m,l'[\mathcal{V}_\varepsilon \mapsto 0]) \to_a (m_1,l'_1)$ implies that 1. $m \xrightarrow{\overline{u}}_a m_1$, 2. $l'_1 - l'[\mathcal{V}_\varepsilon \mapsto 0] = \overline{u}$.
  For any $i \in I$, $l'[\mathcal{V}_\varepsilon \mapsto 0](x^i_\varepsilon) = 0$. So $l'_1[\mathcal{V}_a \mapsto 0](x^i_\varepsilon) = l'_1(x^i_\varepsilon) = (l'_1 - l'[\mathcal{V}_\varepsilon \mapsto 0])(x^i_\varepsilon) \bowtie r_i$, which implies $\overline{u}(e_i) \bowtie r_i$.
  Let $l_1 : \mathcal{V} \to \mathbb{R}_{\geq 0}$ be a variable valuation defined as for any $x \in \mathcal{V}$, $l_1(x) = l'_1(x)$. Then $l_1 - l = \overline{u}$ and $(l_1 - l)(e_i) \bowtie r_i$ for any $i \in I$.
  By inductive hypothesis, $\mathcal{W}',(m_1,l'_1[\mathcal{V}_a \mapsto 0]) \models_B F(\phi^t)$ implies $\mathcal{W},(m_1,l_1[\mathcal{V}_a \mapsto 0]) \models_{B'} \phi^t$.
  Hence $\mathcal{W},(m,l) \models_{B'} \langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t$. ∎

In the following we show how to construct the symbolic model of ϕ^t. Firstly we construct a symbolic model for $F(\phi^t)$ by using the algorithm of $\mathcal{L}^w$ with a minor modification in the definition of mutually-consistent set as follows.

Definition 13 (ε-Mutually-Consistent of $\mathcal{L}^w$)
Let $C \subseteq 2^{\Omega[F(\phi^t),H(B^t)]}$. C is said to be *ε-mutually-consistent* if C is mutually consistent (Definition 9) and for any $(\Gamma,\overline{\delta}) \in C$, whenever $\langle\varepsilon\rangle\psi \in \Gamma$, then there exists $(\Gamma',\overline{\delta'}) \in C$ s.t.:

1. $\overline{\delta'} = \overline{\delta}[\mathcal{V}_\varepsilon \mapsto 0]$;
2. $\psi \in \Gamma'$;
3. for any $[\varepsilon]\psi' \in \Gamma$, $\psi' \in \Gamma'$.

It is not difficult to prove similar results as the ones stated in Lemma 2 and Theorem 1. If $F(\phi^t)$ is not satisfiable, then by Lemma 4, ϕ^t is not satisfiable either.

Otherwise, we get a symbolic model $\mathcal{W}^s$ for $F(\phi^t)$. In the following we construct a symbolic model for ϕ^t without ε-transitions according to $\mathcal{W}^s$.

Given a symbolic LWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ which is constructed according to the definition of ε-mutually-consistent set, we generate

$$\mathcal{W}^s_t = (\Pi^s_t, \Sigma^s_t, \theta^s_t)$$

with $\Sigma^s_t = \Sigma^s/\varepsilon$. Π^s_t and θ^s_t are defined according to the following three steps that must be applied, consecutively, in the order they are described:

1. as shown in Figure 6, for any $\pi_1 \overset{0}{\rightarrow}_\varepsilon \pi_2 \overset{0}{\rightarrow}_\varepsilon \pi_3$
let $\pi' = \{\pi_2, \pi_3\}$ and
(1) $\pi'_2 \rightarrow_a \pi_2$ implies $\pi'_2 \rightarrow_a \pi'$, $\pi'_3 \rightarrow_b \pi_3$ implies $\pi'_3 \rightarrow_b \pi'$, $a, b \in \Sigma^s_t$,
(2) $\pi_2 \rightarrow_a \pi''_2$ implies $\pi' \rightarrow_a \pi''_2$, $\pi_3 \rightarrow_b \pi''_3$ implies $\pi' \rightarrow_b \pi''_3$, $a, b \in \Sigma^s_t$;

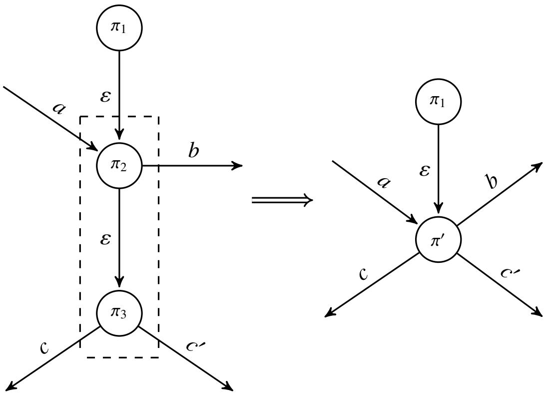

Fig. 6 Step 1

2. as shown in Figure 7, for any $\pi_1 \overset{0}{\rightarrow}_\varepsilon \pi_2$
for any $\pi_1 \rightarrow_a \pi'_1$, delete this transition, $a, b \in \Sigma^s_t$;

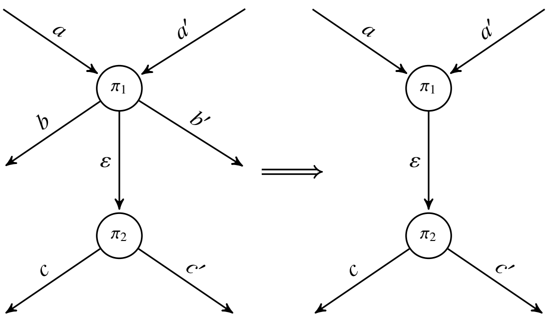

Fig. 7 Step 2

3. as shown in Figure 8 (on the next page), for any $\pi_1 \overset{0}{\rightarrow}_\varepsilon \pi_2, \ldots, \pi_1 \overset{0}{\rightarrow}_\varepsilon \pi_n$
let $\pi' = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and
(1) $\pi_0 \rightarrow_a \pi_1$ implies $\pi_0 \rightarrow_a \pi'$, $a \in \Sigma^s_t$,
(2) for any π_j with $j = \{2, \ldots, n\}$, $\pi_j \rightarrow_a \pi'_j$ implies $\pi' \rightarrow_a \pi'_j$, $a \in \Sigma^s_t$.

Notice that: for any $\pi' = \{\pi_1, \ldots, \pi_n\} \in \Pi^s_t/\Pi$ and for any $x^i_a \in \mathcal{V} = \mathcal{V}' - \{x^i_\varepsilon \mid i = 1, \ldots, k\}$,

$$\overline{\delta_1}(x^i_a) = \ldots = \overline{\delta_n}(x^i_a).$$

Hence, we denote the region related to π' as $\overline{\delta'}$ which associates the same region to each resource-variable in $\mathcal{V}$.

Lemma 5 *Given any $\mathcal{L}^t$ formula ϕ^t closed w.r.t B^t, if $F(\phi^t)$ is satisfiable in symbolic LWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$, i.e., there exists $\pi = (\Gamma, \overline{\delta}) \in \Pi^s$ s.t. $\mathcal{W}^s, \pi \models_{F(B^t)} F(\phi^t)$, then there exists symbolic LWS $\mathcal{W}^s_t = (\Pi^s_t, \Sigma^s_t, \theta^s_t)$ constructed as above, with $\pi' \in \Pi^s_t$ s.t. either $\pi' = \pi$ or $\pi \in \pi'$ and $\mathcal{W}^s_t, \pi' \models_{B^t} \phi^t$.*

Proof. Induction on the structure of ϕ^t. The cases $\bot, \top, \phi^t \wedge \psi^t, \phi^t \vee \psi^t$ and X are trivial.

- *Case $x^i_a \bowtie r$:*
$\mathcal{W}^s, \pi \models_{F(B^t)} F(x^i_a \bowtie r)$ implies for all $w \in \overline{\delta}(x^i_a)$, $w \bowtie r$.
If $\pi \in \Pi^s_t$, then it is obvious that $\mathcal{W}^s_t, \pi \models_{B^t} x^i_a \bowtie r$.
Otherwise, there exists $\pi' \in \Pi^s_t$ s.t. $\pi \in \pi'$.
As mentioned above, $\overline{\delta'}(x^i_a) = \overline{\delta}(x)$ for any $x^i_a \in \mathcal{V}$.
Hence, $\mathcal{W}^s_t, \pi' \models_{B^t} x^i_a \bowtie r$ also holds for this case.

- *Case $[\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t$:*
$F([\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t) = [\varepsilon][a](\bigwedge_{i\in I}(x^i_\varepsilon \bowtie r_i) \rightarrow F(\phi^t))$.
Suppose there exist $\pi', \pi'_1 \in \Pi^s_t$ s.t.: (1) $\pi' \rightarrow_a \pi'_1$, (2) $w'_1 - w' \bowtie r_i$ for any $i \in I$, $w'_1 \in \overline{\delta'_1}(e^i)$ and $w' \in \overline{\delta'}(e^i)$.
Then there exists $\pi = (\Gamma, \overline{\delta}), \pi_1 = (\Gamma_1, \overline{\delta_1}), \pi_2 = (\Gamma_2, \overline{\delta_2}) \in \Pi^s$ s.t.: 1. $\pi \overset{0}{\rightarrow}_\varepsilon \pi_1 \rightarrow_a \pi_2$, 2. $w_2 - w_1 \bowtie r_i$ for any $i \in I$, $w_2 \in \overline{\delta_2}(e_i)$ and $w_1 \in \overline{\delta_1}(e_i)$.
$\pi \overset{0}{\rightarrow}_\varepsilon \pi_1$ implies that $\overline{\delta_1}(x^i_\varepsilon) = 0$ for any $x^i_\varepsilon \in \mathcal{V}_\varepsilon$. Hence for any $x^i_\varepsilon \in \mathcal{V}_\varepsilon$ and any $w_2 \in \overline{\delta_2}(x^i_\varepsilon)$, we have $w_2 \bowtie r_i$.
That is equivalent to: for any $w_2 \in \overline{\delta_2}(e_i)$ and $w_1 \in \overline{\delta_1}(e_i)$, we have $w_2 - w_1 \bowtie r$.
Since $\mathcal{W}^s, \pi \models_{F(B^t)} F([\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t)$, we have that $\mathcal{W}^s, \pi_2 \models_{F(B^t)} F(\phi^t)$.
By inductive hypothesis, there exists $\pi'_2 \in \Pi^s_t$ s.t.: 1. either $\pi'_2 = \pi_2$ or $\pi_2 \in \pi'_2$, 2. $\mathcal{W}^s_t, \pi'_2 \models_{B^t} \phi^t$.
Hence $\mathcal{W}^s_t, \pi' \models_{B^t} [\bigwedge_{i\in I}(e_i \bowtie r_i)]_a\phi^t$.

- *Case $\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t$:*
$F(\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t) = \langle\varepsilon\rangle\langle a\rangle(\bigwedge_{i\in I}(x^i_\varepsilon \bowtie r_i) \wedge F(\phi^t))$.
$\mathcal{W}^s, \pi \models_{F(B^t)} F(\langle\bigwedge_{i\in I}(e_i \bowtie r_i)\rangle_a\phi^t)$ implies that there exists $\pi_1 = (\Gamma_1, \overline{\delta_1}), \pi_2 = (\Gamma_2, \overline{\delta_2}) \in \Pi^s$ s.t.: 1. $\pi \overset{0}{\rightarrow}_\varepsilon \pi_1 \rightarrow_a \pi_2$, 2. for any $i \in I$, $\overline{\delta_2}(x^i_\varepsilon) \bowtie r_i$, 3. $\mathcal{W}^s, \pi_2 \models_{F(B^t)} F(\phi^t)$.
$\pi \overset{0}{\rightarrow}_\varepsilon \pi_1$ implies that $\overline{\delta_1}(x^i_\varepsilon) = 0$ for any $x^i_\varepsilon \in \mathcal{V}_\varepsilon$.
Hence $w_2 - w_1 \bowtie r_i$ for any $x^i_\varepsilon \in \mathcal{V}_\varepsilon$, $w_2 \in \overline{\delta_2}(x^i_\varepsilon)$ and $w_1 \in \overline{\delta_1}(x^i_\varepsilon)$. That is equivalent to: $w_2 - w_1 \bowtie r_i$ for any $w_2 \in \overline{\delta_2}(e_i)$ and $w_1 \in \overline{\delta_1}(e_i)$.
There exists $\pi' \in \Pi^s_t$ s.t. $\pi, \pi_1 \in \pi'$, according to the construction of $\mathcal{W}^s_t$.

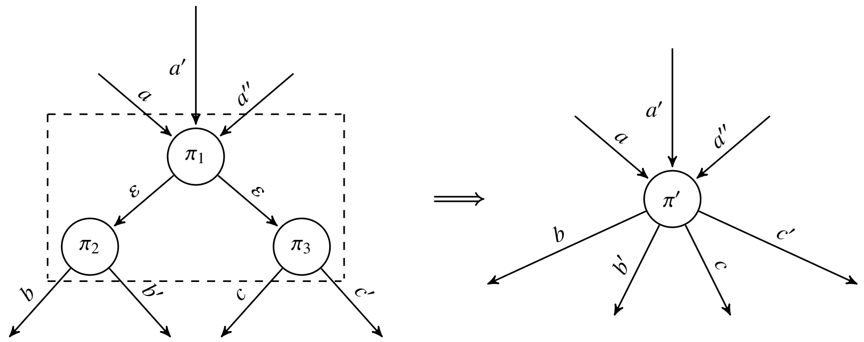

Fig. 8 Step 3

By inductive hypothesis, $\mathcal{W}^s, \pi_2 \models_{F(B^t)} F(\phi^t)$ implies there exists $\pi_2' \in \Pi_t^s$ s.t.: 1. either $\pi_2' = \pi_2$ or $\pi_2 \in \pi_2'$, 2. $\mathcal{W}_t^s, \pi_2' \models_{B^t} \phi^t$.
Hence $\mathcal{W}_t^s, \pi' \models_{B^t} \langle \bigwedge_{i \in I}(e_i \bowtie r_i) \rangle_a \phi^t$. ∎

By the above-introduced method, one can construct the symbolic model for a given $\mathcal{L}^t$ formula – if satisfiable. Then, we can use the technique in Lemma 1 to generalize an LWS $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$, which is a model for the given $\mathcal{L}^t$ formula.

Theorem 4 (Decidability of B^t-satisfiability for $\mathcal{L}^t$) *For an arbitrary maximal equation block B^t, the B^t-satisfiability problem for $\mathcal{L}^t$ is decidable.*

7 Conclusion

In this paper we developed two recursive versions of the weighted modal logic [Larsen and Mardare, 2014] $\mathcal{L}^w$ and $\mathcal{L}^t$. They use a semantics based on labelled weighted transition systems (LWSs). This type of transition systems is used to describe systems involving various types of quantitative information. These models involve a number of resources that label both transitions and states. In articular, the transitions are labelled simultaneously with both actions and real values representing the costs of the corresponding transitions in term of resources.

The two RWLs, $\mathcal{L}^w$ and $\mathcal{L}^t$, encode qualitative and quantitative properties of LWSs. With respect to the weighted logics studied before, $\mathcal{L}^w$ and $\mathcal{L}^t$ make use of recursive variables that allow us to encode circular properties and infinite behaviours, including safety and cost-bounded liveness properties. The RWLs use first-order resource-variables to measure local costs. The main syntactic operators are similar to the ones of timed logics for real-time systems. $\mathcal{L}^w$ has operators that constrain the value of resource-variables at the current state. $\mathcal{L}^t$ extends $\mathcal{L}^w$ by having quantitative constraints on the transition modalities as well. This extension makes sure that $\mathcal{L}^t$ is adequate (while $\mathcal{L}^w$ is not), i.e., the semantic equivalence induced by $\mathcal{L}^t$ coincides with the weighted bisimilarity of LWSs.

Even though $\mathcal{L}^w$ is the least expressive of the two logics discussed in this paper, it fails to enjoy the finite model property. However, we proved that the satisfiability problem for $\mathcal{L}^w$ is still decidable. By applying a variant of the region construction technique developed for timed automata, we obtained symbolic LWSs of the satisfiable $\mathcal{L}^w$ formulas. These symbolic LWSs provide an abstract semantics for LWSs, allowing us to reason about satisfiability by investigating the symbolic models that are finite. We have proposed a model construction algorithm, which constructs a symbolic LWS for a given satisfiable (consistent) $\mathcal{L}^w$ formula. The symbolic model can be eventually used to determine the existence of the concrete LWSs and generate them. The systems are possibly infinitely.

The satisfiability problem of $\mathcal{L}^t$ can be solved in a similar way with $\mathcal{L}^w$. However, we provided an attractive alternative: firstly, encode the problem for $\mathcal{L}^t$ into one similar to that of $\mathcal{L}^w$ by translating the given $\mathcal{L}^t$ formula into $\mathcal{L}^w$ with a special 0-cost action; secondly, use the model construction algorithm with a minor modification to check the satisfiability of this $\mathcal{L}^w$ formula; and finally, if the $\mathcal{L}^w$ formula is not satisfiable then the given $\mathcal{L}^t$ formula is not satisfiable either, otherwise the given $\mathcal{L}^t$ formula is satisfiable and the model for it can be eventually generated according to the model for the corresponding $\mathcal{L}^w$ formula.

This work opens a few future research directions. It worth investigating further the possibility of implementing the algorithms and developing tools for satisfiability checking. A theoretical extension of this work can be oriented towards the models also involving transitions with negative labels; in this case a major challenge is to understand how reset operator behaves and how one could adapt the region construction to fit these features. Another theoretical research direction extending this paper we proposed in [Larsen et al, 2015], where we study the alternation-free quantitative Mu-calculus that extends the logics studied here by also considering minimal fixed points and general reset operators instead of event related reset operations. However, in [Larsen et al, 2015] we had to use a different proof strategy to demonstrate that satisfiability is decidable, in particular we had to adapt a tableaux method. This proof strategy does not allow us to reuse the ideas presented here and consequently we did not obtain similar algorithms. We intend for future to investigate the possibility of extending our algorithms to more expressive logics.

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