Sensitivity analysis for HJB equations with an application to a coupled backward-forward system

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Abstract

In this paper, we analyse the dependence of the solution of Hamilton-Jacobi-Bellman equations on a functional parameter. This sensitivity analysis not only has the interest on its own, but also is important for the mean field games methodology, namely for solving a coupled backward-forward system. We show that the unique solution of a Hamilton-Jacobi-Bellman equation and its spacial gradient are Lipschitz continuous uniformly with respect to a functional parameter. In particular, we provide verifiable criteria for the so-called feedback regularity condition.

Key words: Hamilton-Jacobi-Bellman equation, HJB equation, sensitivity analysis, mean field control, feedback regularity

1 Introduction

Sensitivity analysis for systems governed by partial differential equations (PDEs) has been a growing interest in recent years. The results of sensitivity analysis have a wide-range of applications in science and engineering, including optimization, parameter estimation, model simplification, optimal control, experimental design. Recent progress in sensitivity analysis can be found, e.g. in [26] for Burger's equation, [25] for Navier-Stokes equation, [1, 20, 21, 22] for elliptic and parabolic equations, [2, 13, 19] for nonlinear kinetic equations, and references therein. This work contributes to the presently ongoing investigation of sensitivity analysis for optimal control problems governed by Hamilton-Jacobi-Bellman (HJB) equations.

We start with a standard stochastic optimal control problem in a finite horizon, namely one agent controls her stochastic state evolution to optimise

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certain objective function within the horizon $T > 0$. Instead of using a stochastic differential equation to describe the evolution, we associate the underlying controlled evolution to a family of linear operators, which depend on three parameters: time t, control u and a Banach space valued parameter μ. By applying techniques from operator theory, our framework covers not only diffusions, which are considered in most literature, but also a larger class of Markov evolutions.

We assume that the agent can only control her drift, but not the noise. For a given parameter curve $\{\mu_t, t \in [0, T]\}$ and certain objective function, by dynamical programming principle, the value function satisfies a HJB equation. The aim of this work is to study the Lipschitz sensitivity of the solution of the HJB equation with respect to the functional parameter $\{\mu_t, t \in [0, T]\}$. More specifically, we will show that the unique solution to a HJB equation and its spatial gradient are Lipschitz continuous uniformly with respect to $\{\mu_t, t \in [0, T]\}$ in a proper reference topology.

This sensitivity result has an important application in mean field games (MFG), which is a recently developed subject. The MFG methodology was developed independently by J.-M. Lasry and P.-L. Lions, see [17],[4] and video lectures [12], and by M. Huang, R.P. Malhamé and P. Caines, see [8], [9], [10], [11]. Mean field games methodology aims at describing control processes with a large number N of agents by studying the limit $N \to \infty$ when the contribution of each agent becomes negligible and their interaction is performed via certain mean-field characteristics, which can be expressed in terms of empirical measures. A characteristic feature of the MFG analysis is the study of a coupled system of a backward equation on functions (HJB equation) and a forward equation on probability laws (Kolmogorov equation). A feedback regularity property of the feedback control is critical for solving this system of coupled backward-forward equations. We will apply our sensitivity result to mean field games model and give verifiable conditions for the so-called feedback regularity condition.

The organisation of this paper is as follows. Section 2 recalls some basic but heavily used concepts in this paper, such as operators, propagators and Gâteaux derivatives. The main results are presented and proved in Section 3. We start by proving the well-posedness of a HJB equation. Then we show that, roughly speaking, regularity of a Hamiltonian implies similar regularity of the solutions to a HJB equation. In Section 4, as an application of this sensitivity result, we discuss a mean field games model and give verifiable conditions for the feedback regularity property (4.7), which was assumed to hold and was used as a critical condition (37) in proving Theorem 10 of [8]. Section 5 concludes the paper.

2 Preliminaries

In this section, we recall some concepts which are used throughout the paper. Let $\mathbf{B}$ and $\mathbf{D}$ denote some Banach spaces. For a function $F : \mathbf{D} \to \mathbf{B}$, its *Gâteaux derivative* $D_\chi F(\mu)$ at $\mu \in \mathbf{D}$ in the direction $\chi \in \mathbf{D}$ is defined as

$$D_\chi F(\mu) = \lim_{s \to 0} \frac{F(\mu + s\chi) - F(\mu)}{s}$$

if the limit exists. F is said to be Gâteaux differentiable at $\mu \in \mathbf{D}$ if the limit exits for all $\chi \in \mathbf{D}$. At each point $\mu \in \mathbf{D}$, the Gâteaux derivative defines a function $D_{.}F(\mu) : \mathbf{D} \to \mathbf{B}$.

Let $\mathcal{L}(\mathbf{D}, \mathbf{B})$ denote the space of linear bounded operators from $\mathbf{D}$ to $\mathbf{B}$ and it is equipped with the usual operator norm $\| \cdot \|_{\mathbf{D} \to \mathbf{B}}$. In addition, we also equip $\mathcal{L}(\mathbf{D}, \mathbf{B})$ with the strong operator topology, which is defined as the weakest topology such that the mapping $A \mapsto \|Af\|_{\mathbf{B}}$ is continuous for every $f \in \mathbf{D}$ as a mapping from $\mathcal{L}(\mathbf{D}, \mathbf{B})$ to $\mathbf{R}$. Clearly, the strong operator topology is weaker than the topology induced by the operator norm.

For the analysis of time non-homogeneous evolutions, we need the notion of a propagator. A family of mappings $\{U^{t,r}\}$ from $\mathbf{B}$ to $\mathbf{B}$, parametrized by the pairs of numbers $r \leq t$ (resp. $t \leq r$) is called a *(forward) propagator* (resp. a *backward propagator*) in $\mathbf{B}$, if $U^{t,t}$ is the identity operator in $\mathbf{B}$ for all $t \geq 0$ and the following *chain rule*, or *propagator equation*, holds for $r \leq s \leq t$ (resp. for $t \leq s \leq r$):

$$U^{t,s}U^{s,r} = U^{t,r}.$$

Sometimes, the family $\{U^{t,r}, t \leq r\}$ is also called a *two-parameter semigroup*.

A backward propagator $\{U^{t,r}, t \leq r\}$ of bounded linear operators on the Banach space $\mathbf{B}$ is called *strongly continuous* if the mappings

$$t \mapsto U^{t,r} \quad \text{for all } t \leq r, \qquad \text{and} \qquad r \mapsto U^{t,r} \quad \text{for all } t \leq r,$$

are continuous as mappings from $\mathbf{R}$ to $\mathcal{L}(\mathbf{B}, \mathbf{B})$ in the strong operator topology. By the principle of uniform boundedness if $\{U^{t,r}, t \leq r\}$ is a strongly continuous propagator of bounded linear operators, then the norms of $\{U^{t,r}, t \leq r\}$ are uniformly bounded for t, r in any compact interval.

Assume that the Banach space $\mathbf{D}$ is a dense subset of $\mathbf{B}$ and continuously embedded in $\mathbf{B}$. Suppose $\{U^{t,r}, t \leq r\}$ is a strongly continuous backward propagator of bounded linear operators on a Banach space $\mathbf{B}$ with the common invariant domain $\mathbf{D} \subset \mathbf{B}$, i.e. if $f \in \mathbf{D}$ then $U^{t,r}f \in \mathbf{D}$ for all $t \leq r$.

Let $\{L_t, t \geq 0\}$ be a family of operators $L_t \in \mathcal{L}(\mathbf{D}, \mathbf{B})$, depending continuously on t in the strong operator topology. The family $\{L_t, t \geq 0\}$ is said to *generate* $\{U^{t,r}, t \leq r\}$ on $\mathbf{D}$ if, for any $f \in \mathbf{D}$, we have

$$\frac{d}{ds} U^{t,s} f = U^{t,s} L_s f, \quad \frac{d}{ds} U^{s,r} f = -L_s U^{s,r} f, \quad \text{for all}\, t \leq s \leq r. \tag{2.1}$$

The derivatives exist in the norm topology of $\mathbf{B}$ and if $s = t$ (resp. $s = r$) they are assumed to be only a right (resp. left) derivative.

One often needs to estimate the difference of two propagators when the difference of their generators is available. To this end, we shall often use the following rather standard trick.

Proposition 2.1. *For $i = 1, 2$ let $\{L_t^i, t \geq 0\}$ be a family of operators $L_t^i \in \mathcal{L}(\mathbf{D}, \mathbf{B})$, depending continuously on t in the strong operator topology, which generates a backward propagator $\{U_i^{t,r}, t \leq r\}$ in $\mathbf{B}$ satisfying*

$$a_1 := \sup_{t \leq r} \max \left\{ \|U_1^{t,r}\|_{\mathbf{B}\to\mathbf{B}}, \|U_2^{t,r}\|_{\mathbf{B}\to\mathbf{B}} \right\} < \infty.$$

If $\mathbf{D}$ is invariant under $\{U_1^{t,r}, t \leq r\}$ and

$$a_2 := \sup_{t \leq r} \|U_1^{t,r}\|_{\mathbf{D}\to\mathbf{D}} < \infty,$$

then, for each $t \leq r$,

$$U_2^{t,r} - U_1^{t,r} = \int_t^r U_2^{t,s} (L_s^2 - L_s^1) U_1^{s,r} ds \tag{2.2}$$

and

$$\|U_2^{t,r} - U_1^{t,r}\|_{\mathbf{D}\to\mathbf{B}} \leq a_1 a_2 (r - t) \sup_{t \leq s \leq r} \|L_s^2 - L_s^1\|_{\mathbf{D}\to\mathbf{B}}. \tag{2.3}$$

Proof. Define an operator-valued function $Y(s) := U_2^{t,s} U_1^{s,r}$. Since $U_i^{t,t}$, $i = 1, 2$, are identity operators, so $Y(r) = U_2^{t,r}$ and $Y(t) = U_1^{t,r}$. By (2.1), we get

$$\begin{aligned} U_2^{t,r} - U_1^{t,r} &= U_2^{t,s} U_1^{s,r} \big|_{s=t}^r = \int_t^r \frac{d}{ds} \left(U_2^{t,s} U_1^{s,r} \right) ds \\ &= \int_t^r U_2^{t,s} L_s^2 U_1^{s,r} - U_2^{t,s} L_s^1 U_1^{s,r} ds \\ &= \int_t^r U_2^{t,s} (L_s^2 - L_s^1) U_1^{s,r} ds, \end{aligned}$$

which implies both (2.2) and (2.3). □

3 Main results

Let $\mathbf{C} := C_\infty(\mathbf{R}^d)$ be the Banach space of bounded continuous functions $f : \mathbf{R}^d \to \mathbf{R}$ with $\lim_{x\to\infty} f(x) = 0$, equipped with norm $\|f\|_{\mathbf{C}} := \sup_x |f(x)|$. We shall denote by $\mathbf{C}^1 := C^1_\infty(\mathbf{R}^d)$ the Banach space of continuously differentiable and bounded functions $f : \mathbf{R}^d \to \mathbf{R}$ such that the derivative f' belongs to $\mathbf{C}$, equipped with the norm $\|f\|_{\mathbf{C}^1} := \sup_x |f(x)| + \sup_x |f'(x)|$, and by $\mathbf{C}^2 := C^2_\infty(\mathbf{R}^d)$ the Banach space of twice continuously differentiable and bounded functions $f : \mathbf{R}^d \to \mathbf{R}$ such that the first derivative f' and the second derivative f'' belong to $\mathbf{C}$, equipped with the norm $\|f\|_{\mathbf{C}^2} := \sup_x |f(x)| + \sup_x |f'(x)| + \sup_x |f''(x)|$. Let $\mathbf{C}_{Lip} := C_{Lip}(\mathbf{R}^d)$ denote the space of Lipschitz continuous functions $f : \mathbf{R}^d \to \mathbf{R}$, equipped with the norm $\|f\|_{\mathbf{C}_{Lip}} := \sup_x |f(x)| + \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|}$.

Note, that $\mathbf{C}^2 = C^2_\infty(\mathbf{R}^d)$ is a Banach space which is densely and continuously embedded in $\mathbf{C} = C_\infty(\mathbf{R}^d)$. Depending on the modelling assumption, the Banach space $\mathbf{C}^2$ can be replaced by other examples of functions spaces, such as the space of Hölder continuous functions, and our methods can be applied in a similar way.

Let $T > 0$ be fixed and $\mathcal{U}$ be a subset of a Euclidean space, interpreted as the set of admissible controls, with the Euclidean norm $|\cdot|$. Take $\mathcal{M}$ to be a bounded, convex, closed subset of another Banach space $\mathbf{S}$, equipped with the norm $\|\cdot\|_{\mathbf{S}}$. In applications, very often the Banach space $\mathbf{S}$ is taken as the dual space $(\mathbf{C}^2)^*$ of $\mathbf{C}^2$ and the set $\mathcal{M}$ is taken as the set of probability measures on $\mathbf{R}^d$, which is denoted by $\mathcal{P}(\mathbf{R}^d)$.

Let

$$\{A[t,\mu,u] : t \in [0,T], \mu \in \mathcal{M}, u \in \mathcal{U}\} \tag{3.1}$$

be a family of bounded linear operators $A[t,\mu,u] : \mathbf{C}^2 \to \mathbf{C}$, defined on the same domain $\mathbf{C}^2$ for all $(t,\mu,u) \in [0,T] \times \mathcal{M} \times \mathcal{U}$. For each $(t,\mu,u) \in [0,T]\times\mathcal{M}\times\mathcal{U}$, the linear operator $A[t,\mu,u] : \mathbf{C}^2 \mapsto \mathbf{C}$ is assumed to generate a Feller process with values in $\mathbf{R}^d$ and to be of the form

$$A[t,\mu,u]f(z) = (h(t,z,\mu,u), \nabla f(z)) + L[t,\mu]f(z), \tag{3.2}$$

where the coefficient $h : [0,T] \times \mathbf{R}^d \times \mathcal{M} \times \mathcal{U} \to \mathbf{R}^d$ is a vector-valued function. For each pair $(t,\mu) \in [0,T] \times \mathcal{M}$, the linear bounded operator $L[t,\mu] : \mathbf{C}^2 \to \mathbf{C}$ is of Lévy-Khintchine form with variable coefficients:

$$\begin{aligned} L[t,\mu]f(z) &= \frac{1}{2}(G(t,z,\mu)\nabla,\nabla)f(z) + (b(t,z,\mu),\nabla f(z)) \\ &+ \int_{\mathbf{R}^d} (f(z+y) - f(z) - (\nabla f(z), y)\mathbf{1}_{B_1}(y))\nu(t,z,\mu,dy), \end{aligned} \tag{3.3}$$

where ∇ denotes the gradient operator and $\mathbf{1}_{B_1}$ denotes the indicator function of the unit ball in $\mathbf{R}^d$; for each $(t, z, \mu) \in [0, T] \times \mathbf{R}^d \times \mathcal{M}$, $G(t, z, \mu)$ is a symmetric non-negative matrix, $b(t, z, \mu)$ is a vector, $\nu(t, z, \mu, \cdot)$ is a Lévy measure on $\mathbf{R}^d$, i.e.

$$\int_{\mathbf{R}^d} \min(1, |y|^2)\nu(t, z, \mu, dy) < \infty, \quad \nu(t, z, \mu, \{0\}) = 0. \tag{3.4}$$

We assume that the mappings $(t, z, \mu) \to G(t, z, \mu)$, $(t, z, \mu) \to b(t, z, \mu)$ and $(t, z, \mu) \to \nu(t, z, \mu, \cdot)$ are Borel measurable with respect to the Borel σ-algebra in $[0, T] \times \mathbf{R}^d \times \mathcal{M}$.

Let $(X_t^{\{\mu.\},\{u.\}} : t \in [0, T])$ be a controlled stochastic process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with values in $\mathbf{R}^d$ and generated by the family of operators $\{A[t, \mu, u] : t \in [0, T], \mu \in \mathcal{M}, u \in \mathcal{U}\}$ in (3.1) of the form (3.2). The control process is described by a stochastic process $\{u.\} = \{u_t \in \mathcal{U} : t \in [0, T]\}$. Let $C([0, T], \mathcal{M})$ be a space of continuous curves $\{\mu.\} = \{\mu_t \in \mathcal{M}, t \in [0, T]\}$. For notational brevity, in the following we write $(X_t : t \in [0, T])$ instead of $(X_t^{\{\mu.\},\{u.\}} : t \in [0, T])$.

Given a curve $\{\mu.\} \in C([0, T], \mathcal{M})$, we aim to maximize the expected total payoff

$$\mathbb{E}\left[\int_t^T J(s, X_s, \mu_s, u_s)\, ds + V^T(X_T, \mu_T)\right]$$

over a suitable class of controls $\{u_t \in \mathcal{U}, t \in [0, T]\}$ with a running cost function $J : [0, T] \times \mathbf{R}^d \times \mathcal{M} \times \mathcal{U} \to \mathbf{R}$ and a terminal cost function $V^T : \mathbf{R}^d \times \mathcal{M} \to \mathbf{R}$. Therefore, for a given curve $\{\mu_t \in \mathcal{M} : t \in [0, T]\} \in C([0, T], \mathcal{M})$, the value function $V : [0, T] \times \mathbf{R}^d \to \mathbf{R}$ starting at time t and state x is defined by

$$V(t, x; \{\mu_s : s \in [t, T]\}) := \sup_{\{u.\}} \mathbb{E}_x \left[\int_t^T J(s, X_s, \mu_s, u_s)\, ds + V^T(X_T, \mu_T)\right]. \tag{3.5}$$

By standard arguments from dynamic programming principle and assuming appropriate regularity, the value function V satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} -\frac{\partial V}{\partial t}(t, x; \{\mu.\}) &= H_t(x, \nabla V(t, x; \{\mu.\}), \mu_t) + L[t, \mu_t]V(t, x; \{\mu.\}) \\ V(T, x; \{\mu.\}) &= V^T(x; \mu_T), \end{aligned} \tag{3.6}$$

where the Hamiltonian $H : [0, T] \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{M} \to \mathbf{R}$ is defined by

$$H_t(x, p, \mu) = \max_{u \in \mathcal{U}}(h(t, x, \mu, u)p + J(t, x, \mu, u)). \tag{3.7}$$

Our main aim of this paper is to investigate the sensitivity of the solution $V(t,x;\{\mu.\})$ of the HJB equation (3.6) with respect to the functional parameter $\{\mu.\}\in C([0,T],\mathcal{M})$. In the first place, we need to show that for each fixed curve $\{\mu.\}\in C([0,T],\mathcal{M})$, the HJB equation (3.6) is well posed. Then, we discuss the sensitivity of the solution to (3.6) with respect to the parameter $\{\mu.\}\in C([0,T],\mathcal{M})$. In fact, we shall show that the unique solution V and its spatial gradient are Lipschitz continuous uniformly with respect to $\{\mu.\}$.

3.1 Main assumptions

For any $\mu\in\mathcal{M}$, define the set $\mathcal{M}-\mu := \{\eta-\mu : \eta\in\mathcal{M}\}$, which, as a subset of $\mathbf{S}$, is equipped with the norm $\|\cdot\|_{\mathbf{S}}$. In the analysis below,we need the following assumptions:

(A1): the Hamiltonian $H : [0,T]\times\mathbf{R}^d\times\mathbf{R}^d\times\mathcal{M}\to\mathbf{R}$ is continuous in t and Lipschitz continuous uniformly in x on bounded subsets of p. Furthermore, it is Lipschitz continuous uniformly in p, that is there exists a constant c_1 such that for all $x\in\mathbf{R}^d$, $\mu\in\mathcal{M}$ and $t\in[0,T]$ we have

$$|H_t(x,p,\mu)-H_t(x,p',\mu)|\le c_1|p-p'| \qquad \text{for } p,p'\in\mathbf{R}^d. \tag{3.8}$$

It is bounded in $p=0$, that is there exists a constant $c_2>0$ such that

$$|H_t(x,0,\mu)|\le c_2 \qquad \text{for all } x\in\mathbf{R}^d,\ \mu\in\mathcal{M},\ t\in[0,T]. \tag{3.9}$$

For each $t\in[0,T]$ and $x,p\in\mathbf{R}^d$, the function $\mu\mapsto H_t(x,p,\mu)$ is Gâteaux differentiable in any direction $\chi\in\mathcal{M}-\mu$, such that $(t,x,p,\mu)\mapsto D_\chi H_t(x,p,\mu)$ is continuous and satisfies that for each bounded set $B\subset\mathbf{R}^d$ there exists a constant $c_3>0$ such that

$$\sup_{p\in B}|D_\chi H_t(x,p,\mu)|\le c_3\|\chi\|_{\mathbf{S}} \qquad \text{for all } t\in[0,T], x\in\mathbf{R}^d, \mu\in\mathcal{M}. \tag{3.10}$$

(A2): the mapping

$$[0,T]\times\mathcal{M}\to\mathcal{L}(\mathbf{C}^2,\mathbf{C}), \qquad (t,\mu)\mapsto L[t,\mu]$$

is continuous in the strong operator topology. For any $\{\mu.\}\in C([0,T],\mathcal{M})$, the operator curve $\{L[t,\mu_t] : t\in[0,T]\}$ generates a strongly continuous backward propagator $\{U^{t,s}_{\{\mu.\}}, t\le s\}$ of operators $U^{t,s}_{\{\mu.\}}\in\mathcal{L}(\mathbf{C},\mathbf{C})$ with the common invariant domains $\mathbf{C}^2$ and $\mathbf{C}^1$. There exists a constant $c_4>0$ such that for all $0\le t\le s\le T$ we have

$$\max\left\{\|U^{t,s}_{\{\mu.\}}\|_{\mathbf{C}\to\mathbf{C}}, \|U^{t,s}_{\{\mu.\}}\|_{\mathbf{C}^1\to\mathbf{C}^1}, \|U^{t,s}_{\{\mu.\}}\|_{\mathbf{C}^2\to\mathbf{C}^2}\right\}\le c_4. \tag{3.11}$$

The propagator has a *smoothing property*, that is for each $0 \le t < s \le T$ we have

$$U^{t,s}_{\{\mu.\}} : \mathbf{C} \to \mathbf{C}^1, \quad U^{t,s}_{\{\mu.\}} : \mathbf{C}_{Lip} \to \mathbf{C}^2, \tag{3.12}$$

and there exists a $\beta \in (0,1)$ and constants $c_5, c_6 > 0$ such that

$$\|U^{t,s}_{\{\mu.\}}\phi\|_{\mathbf{C}^1} \le c_5(s-t)^{-\beta}\|\phi\|_{\mathbf{C}}, \qquad \|U^{t,s}_{\{\mu.\}}\psi\|_{\mathbf{C}^2} \le c_6(s-t)^{-\beta}\|\psi\|_{\mathbf{C}_{Lip}} \tag{3.13}$$

for all $\phi \in \mathbf{C}$ and $\psi \in \mathbf{C}_{Lip}$.

(ii) for any $t \in [0,T]$, the mapping $\mu \mapsto L[t,\mu]$ is Gâteaux differentiable in any direction $\chi \in \mathcal{M} - \mu$, such that the mapping $\mu \mapsto D_\chi L[t,\mu]$ is continuous in the strong operator topology of $\mathcal{L}(\mathbf{C}^2, \mathbf{C})$. There exists a constant $c_7 > 0$ such that for each $\mu \in \mathcal{M}$ and $\chi \in \mathcal{M} - \mu$ we have

$$\|D_\chi L[t,\mu]\|_{\mathbf{C}^2 \to \mathbf{C}} \le c_7 \|\chi\|_{\mathbf{S}} \quad \text{for all}\, t \in [0,T]; \tag{3.14}$$

(A3): for any $\mu \in \mathcal{M}$, the mapping $x \mapsto V^T(x;\mu)$ is twice continuously differentiable, and for each $x \in \mathbf{R}^d$ the mapping $\mu \mapsto V^T(x;\mu)$ is Gâteaux differentiable in any direction $\chi \in \mathcal{M} - \mu$ such that the mapping $(x,\mu) \mapsto D_\chi V^T(x;\mu)$ is continuous. There exists a constant $c_8 > 0$ such that

$$\|D_\chi V^T(\cdot;\mu)\|_{\mathbf{C}^1} \le c_8 \|\chi\|_{\mathbf{S}}. \tag{3.15}$$

Remark 3.1. *If the Banach space* $\mathbf{S}$ *is given as the Euclidean space* $\mathbf{R}$*, then* D_χ *corresponds to the standard partial derivatives and are denoted by* $\partial/\partial\alpha$ *for* $\alpha \in \mathbf{R}$.

The smoothing conditions (3.12) and (3.13) in assumption **(A2)** are essential and critical in the following analysis. Let us show two basic examples which satisfy assumption **(A2)**: the diffusion operator

$$L[t,\mu]f(x) = \frac{1}{2}(\sigma^2(t,x,\mu)\nabla,\nabla)f(x) + (b(t,x,\mu),\nabla f)(x) \tag{3.16}$$

with smooth enough functions b, σ, see e.g. in [23] and references therein. The operators $\{L[t,\mu], t \in [0,T]\}$ generate the stochastic process $(X(t), t \in [0,T])$ which obeys the stochastic differential equation

$$dX_t = b(t, X_t, \mu_t)dt + \sigma(t, X_t, \mu_t)dW_t,$$

where W is a standard Brownian motion.

Another example is given by stable-like processes with the generating family

$$L[t,\mu]f(x) = a(t,x)|\Delta|^{\alpha(x)/2} + (b(t,x,\mu),\nabla f)(x) \tag{3.17}$$

with smooth enough functions a, α such that the range of a is a compact interval of positive numbers and the range of α is a compact subinterval of $(1,2)$. In both cases, each operator $U_\mu^{t,s}$, $t \le s$, has a kernel, e.g. it is given by

$$U_\mu^{t,s} f(x) = \int G_\mu(t,s,x,y) f(y) dy \tag{3.18}$$

with a certain Green's function G_μ, such that for every $x \in \mathbf{R}^d$ and $t \le s$,

$$\sup_{\mu \in \mathcal{M}} \int_{\mathbf{R}^d} |\nabla_x G_\mu(t,s,x,y)| dy \le c(s-t)^{-\beta} \tag{3.19}$$

for a constant $c > 0$. Here, in the first case (3.16), we have $\beta = \frac{1}{2}$ and in the second case (3.17), we have $\beta = (\inf_x \alpha(x))^{-1}$. In both cases the smoothing conditions (3.12) and (3.13) are satisfied, see [14] and references therein.

3.2 Well-posedness of HJB equation

In this subsection, we prove the well-posedness of the HJB equation (3.6). For this purpose, we can fix $\{\mu_t \in \mathcal{M} : t \in [0,T]\} \in C([0,T], \mathcal{M})$ and thus, we omit the dependence of the functions H, L, V^T on the parameter $\mu \in \mathcal{M}$, and we consider the Cauchy problem

$$\begin{aligned} -\frac{\partial V}{\partial t}(t,x) &= H_t(x, \nabla V(t,x)) + L_t V(t,x) \\ V(T,x) &= V^T(x) \end{aligned} \tag{3.20}$$

with the Hamiltonian $H : [0,T] \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ defined by

$$H_t(x,p) = \max_{u \in \mathcal{U}} (h(t,x,u)p + J(t,x,u)). \tag{3.21}$$

and the operator $L_t : \mathbf{C}^2 \to \mathbf{C}$ for each $t \in [0,T]$. By Duhamel's principle, if V is a classical solution of (3.20), then V is also a *mild solution* of (3.20), i.e. it satisfies

$$V(t,x) = (U^{t,T} V^T(\cdot))(x) + \int_t^T U^{t,s} H_s(\,\cdot\,, \nabla V(s,\cdot))(x) ds \tag{3.22}$$

for all $t \in [0,T]$ and $x \in \mathbf{R}^d$.

For the sensitivity analysis of this work, it is sufficient to consider only a mild solution, which exists under weaker conditions than a classical solution. For this reason, we will establish the existence of a unique mild solution. In this subsection, we mostly follow Chapter 7 in [14], where one also can find details for existence of a classical solution. We present this result for completeness on the level of generality which is required by what follows.

Theorem 3.1. *Assume conditions* **(A1)** *and* **(A2)**. *If the terminal data* $V^T(\cdot)$ *is in* $\mathbf{C}^1$, *then there exists a unique mild solution* V *of* (3.20), *satisfying* $V(t,\cdot) \in \mathbf{C}^1$ *for all* $t \in [0,T]$.

Proof. Let $C^T_{V^T}([0,T],\mathbf{C}^1)$ be the set of functions $\phi : [0,T] \times \mathbf{R}^d \to \mathbf{R}$, which satisfy $\phi(T,x) = V^T(x)$ for all $x \in \mathbf{R}^d$, $\phi(t,\cdot) \in \mathbf{C}^1$ for each $t \in [0,T]$ and the mapping $t \mapsto \phi(t,\cdot)$ is continuous as a mapping from $[0,T]$ to $\mathbf{C}$. We equip this space with the norm

$$\|\phi\|_{C^T_{V^T}([0,T],\mathbf{C}^1)} := \sup_{t\in[0,T]} \|\phi(t,\cdot)\|_{\mathbf{C}^1}.$$

Note this definition of the set $C^T_{V^T}([0,T],\mathbf{C}^1)$ is not standard in the sense that it the continuity is considered from $[0,T]$ to $\mathbf{C}$, but not from $[0,T]$ to $\mathbf{C}^1$.

Define an operator Ψ acting on $C^T_{V^T}([0,T],\mathbf{C}^1)$ by

$$\Psi(\phi)(t,x) := (U^{t,T}V^T(\cdot))(x) + \int_t^T U^{t,s}H_s(\cdot,\nabla\phi(s,\cdot))(x)ds. \tag{3.23}$$

Clearly, the mapping $t \to \Psi(\phi)(t,\cdot)$ is continuous since the propagator $U^{t,T}$ is strongly continuous in t and the integral term is continuous in t.

Since $V^T(\cdot) \in \mathbf{C}^1$ and the family $\{U^{t,T}, 0 \le t \le T\}$ is bounded as a family of mappings from $\mathbf{C}^1$ to $\mathbf{C}^1$, we have $U^{t,T}V^T(\cdot) \in \mathbf{C}^1$ and it is uniformly bounded on $0 \le t \le T$. By the triangle inequality and (3.8),(3.9), for each $t \in [0,T]$

$$\begin{aligned}\|H_t(\cdot,\nabla\phi(t,\cdot))\|_{\mathbf{C}} &\le \|H_t(\cdot,0)\|_{\mathbf{C}} + \|H_t(\cdot,\nabla\phi(t,\cdot)) - H_t(\cdot,0)\|_{\mathbf{C}} \\ &\le c_2 + c_1\|\nabla\phi(t,\cdot)\|_{\mathbf{C}} \\ &\le c_2 + c_1\|\phi(t,\cdot)\|_{\mathbf{C}^1}. \end{aligned} \tag{3.24}$$

The smoothing condition (3.12) guarantees that $U^{t,s}H_s(\cdot,\nabla\phi(s,\cdot)) \in \mathbf{C}^1$ for each $0 \le t < s \le T$. The conditions (3.11), (3.13) and the inequality (3.24) imply for each $t \in [0,T]$ that

$$\begin{aligned}\|\Psi(\phi)(t,\cdot)\|_{\mathbf{C}^1} \le& \|U^{t,T}V^T(\cdot)\|_{\mathbf{C}^1} + \int_t^T \|U^{t,s}H_s(\cdot,\nabla\phi(s,\cdot))\|_{\mathbf{C}^1}\ ds \\ \le& c_4\|V^T(\cdot)\|_{\mathbf{C}^1} + c_5\int_t^T (s-t)^{-\beta}\|H_s(\cdot,\nabla\phi(s,\cdot))\|_{\mathbf{C}}\ ds \\ \le& c_4\|V^T(\cdot)\|_{\mathbf{C}^1} + c_5\left(c_2 + c_1\sup_{t\le s\le T}\|\phi(s,\cdot)\|_{\mathbf{C}^1}\right)\frac{(T-t)^{1-\beta}}{1-\beta}.\end{aligned}$$

It follows that the operator Ψ maps $C^T_{V^T}([0,T],\mathbf{C}^1)$ to itself, i.e.

$$\Psi : C^T_{V^T}([0,T],\mathbf{C}^1) \mapsto C^T_{V^T}([0,T],\mathbf{C}^1).$$

Conditions (3.8) and (3.13) imply for every $\phi^1,\phi^2 \in C^T_{V^T}([0,T],\mathbf{C}^1)$ and $t \in [0,T]$ that

$$\begin{aligned}
&\|\Psi(\phi^1)(t,\cdot) - \Psi(\phi^2)(t,\cdot)\|_{\mathbf{C}^1} \\
\le& \int_t^T \|U^{t,s}[H_s(\cdot,\nabla\phi^1(s,\cdot)) - H_s(\cdot,\nabla\phi^2(s,\cdot))]\|_{\mathbf{C}^1}\ ds \\
\le& \int_t^T c_5(s-t)^{-\beta} c_1 \|\nabla\phi^1(s,\cdot) - \nabla\phi^2(s,\cdot)\|_{\mathbf{C}}\ ds \\
\le& c_1 c_5 \frac{(T-t)^{1-\beta}}{1-\beta} \sup_{t\le s\le T} \|\phi^1(s,\cdot) - \phi^2(s,\cdot)\|_{\mathbf{C}^1}.
\end{aligned} \tag{3.25}$$

By choosing t_0 small enough, it follows that the mapping Ψ is a contraction in $C^T_{V^T}([0,T],\mathbf{C}^1)$ for time $t \in [T-t_0,T]$. Consequently, by the contraction mapping principle, Ψ has a unique fixed point for $t \in [T-t_0,T]$. The well-posedness on the whole interval $[0,T]$ is proved, as usual, by iterations. □

By the wellposedness of equation (3.22), its solution defines a propagator in $\mathbf{C}^1$. Standard arguments, see e.g. [3], show that this solution is a viscosity solution to the original equation (3.20) and it solves the corresponding optimization problem.

3.3 Sensitivity analysis of HJB

In this subsection, we analyse the dependency of the solution of the HJB equation (3.6) on the functional parameter $\{\mu_\cdot\} \in C([0,T],\mathcal{M})$. Under the conditions **(A1)** and **(A2)**, Theorem 3.1 guarantees the existence of a unique mild solution $V(\cdot,\cdot;\{\mu_\cdot\})$ of (3.6) for each fixed curve $\{\mu_\cdot\} \in C([0,T],\mathcal{M})$.

The following observation plays an important role in this work. Let $\{\mu^1_\cdot\},\{\mu^2_\cdot\}$ be in $C([0,T],\mathcal{M})$ and let $\alpha \in [0,1]$. Since $\mathcal{M}$ is convex, the curve

$$\{\mu^1_\cdot\} + \alpha\{(\mu^2-\mu^1)_\cdot\} := \{\mu^1_t + \alpha(\mu^2_t - \mu^1_t), t \in [0,T]\}$$

belongs to $C([0,T],\mathcal{M})$. Thus, we can define the function

$$V_\alpha : [0,T]\times\mathbf{R}^d \to \mathbf{R}, \qquad V_\alpha(t,x) := V\big(t,x;\{\mu^1_\cdot\} + \alpha\{(\mu^2-\mu^1)_\cdot\}\big). \tag{3.26}$$

and have the relation

$$V(t,x;\{\mu^2_\cdot\}) - V(t,x;\{\mu^1_\cdot\}) = V_1(t,x) - V_0(t,x). \tag{3.27}$$

Furthermore, for $\{\mu_\cdot^1\}, \{\mu_\cdot^2\} \in C([0,T], \mathcal{M})$ define

$$\begin{aligned} H_{\alpha,t}(x,p) :&= H_t(x,p,\mu_t^1 + \alpha(\mu_t^2 - \mu_t^1)) \\ &= \max_{u \in \mathcal{U}} (h(t,x,\mu_t^1 + \alpha(\mu_t^2 - \mu_t^1),u)p + J(t,x,\mu_t^1 + \alpha(\mu_t^2 - \mu_t^1),u)) \end{aligned} \tag{3.28}$$

$$L_\alpha[t] := L(t, \mu_t^1 + \alpha(\mu_t^2 - \mu_t^1)) \tag{3.29}$$

$$V_\alpha^T(x) := V^T(x; \mu_T^1 + \alpha(\mu_T^2 - \mu_T^1)) \tag{3.30}$$

with $\alpha \in [0,1]$, $t \in [0,T]$ and $(x,p) \in \mathbf{R}^d \times \mathbf{R}^d$. Then the sensitivity analysis of the solution of (3.6) with respect to a function parameter $\{\mu.\} \in C([0,T], \mathcal{M})$ can be reduced to the one of the solution to the following Cauchy problem with respect to a real parameter $\alpha \in [0,1]$:

$$\begin{aligned} &\frac{\partial V_\alpha}{\partial t}(t,x) = -H_{\alpha,t}(x, \nabla V_\alpha(t,x)) - L_\alpha[t] V_\alpha(t,x) \\ &V_\alpha(T,x) = V_\alpha^T(x). \end{aligned} \tag{3.31}$$

The sensitivity analysis with respect to $\alpha \in [0,1]$ consists of two steps. First, we omit the Hamiltonian term in (3.31) and only consider the sensitivity of the evolution $V_\alpha(t,\cdot) = U_\alpha^{t,T} V_\alpha^T(\cdot)$, where for each $\alpha \in [0,1]$, the propagator $\{U_\alpha^{t,s} : t \leq s\}$ is generated by the family of operators $\{L_\alpha[t] : t \in [0,T]\}$. This intermediate step gives us some interesting results. Then we take into account the Hamiltonian term and complete the analysis.

Theorem 3.2. *Assume conditions* **(A2),(A3)** *and define*

$$W : [0,1] \times [0,T] \times \mathbf{R}^d \to \mathbf{R}^d, \qquad W_\alpha(t,x) = U_\alpha^{t,T} V_\alpha^T(x). \tag{3.32}$$

Then for each $t \in [0,T]$ and $x \in \mathbf{R}^d$, the mapping $\alpha \to W_\alpha(t,x)$ is Lipschitz continuous with uniformly bounded Lipschitz constants, more precisely for every $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$, there exists a constant $c > 0$ such that

$$\begin{aligned} &\frac{\|W_{\alpha_1}(t,\cdot) - W_{\alpha_2}(t,\cdot)\|_{\mathbf{C}^1}}{|\alpha_1 - \alpha_2|} \\ &\leq c \left(\sup_{\gamma \in [\alpha_1,\alpha_2]} \left\| \frac{\partial V_\gamma^T}{\partial \alpha} \right\|_{\mathbf{C}^1} + (T-t)^{1-\beta} \sup_{\substack{s \in [t,T] \\ \gamma \in [\alpha_1,\alpha_2]}} \left\| \frac{\partial L_\gamma}{\partial \alpha}[s] \right\|_{\mathbf{C}^2 \to \mathbf{C}} \|V_{\alpha_2}^T(\cdot)\|_{\mathbf{C}^1} \right) \end{aligned}$$

for every $t \in [0,T]$.

Proof. From (3.32), for each $\alpha_1, \alpha_2 \in [0,1]$, $t \in [0,T]$ and $x \in \mathbf{R}^d$, we have

$$\begin{aligned} &W_{\alpha_1}(t,x) - W_{\alpha_2}(t,x) \\ &= U_{\alpha_1}^{t,T}\left(V_{\alpha_1}^T - V_{\alpha_2}^T\right)(x) + \left(U_{\alpha_1}^{t,T} - U_{\alpha_2}^{t,T}\right)V_{\alpha_2}^T(x). \end{aligned} \tag{3.33}$$

By the condition **(A3)**, for each $x \in \mathbf{R}^d$ the mapping $\alpha \to V_\alpha^T(x)$ is differentiable and the derivative $\frac{\partial V_\alpha^T}{\partial \alpha}(\cdot)$ belongs to $\mathbf{C}^1$. Since for any $0 \le t \le T$ and $\alpha_1 \in [0,1]$, $U_{\alpha_1}^{t,T} : \mathbf{C}^1 \to \mathbf{C}^1$, together with (3.11) we have

$$\begin{aligned} \|U_{\alpha_1}^{t,T}\left(V_{\alpha_1}^T - V_{\alpha_2}^T\right)(\cdot)\|_{\mathbf{C}^1} &= \left\| U_{\alpha_1}^{t,T} \int_{\alpha_2}^{\alpha_1} \frac{\partial V_\gamma^T}{\partial \alpha}(\cdot) d\gamma \right\|_{\mathbf{C}^1} \\ &\le c_4 |\alpha_1 - \alpha_2| \sup_{\gamma \in [\alpha_1, \alpha_2]} \left\| \frac{\partial V_\gamma^T}{\partial \alpha}(\cdot) \right\|_{\mathbf{C}^1}. \end{aligned} \tag{3.34}$$

By (2.2) in Proposition 2.1 and the smoothing property (3.13), we have

$$U_{\alpha_1}^{t,T} - U_{\alpha_2}^{t,T} = \int_t^T U_{\alpha_1}^{t,s}(L_{\alpha_1}[s] - L_{\alpha_2}[s]) U_{\alpha_2}^{s,T} ds : \mathbf{C}^2 \to \mathbf{C}^1.$$

Together with the condition **A2** (ii) that for each $t \in [0,T]$ the mapping $\alpha \mapsto L_\alpha[t]$ is differentiable and $\frac{\partial L_\alpha}{\partial \alpha}[t] : \mathbf{C}^2 \to \mathbf{C}$, we have

$$\begin{aligned} &\left\| \left(U_{\alpha_1}^{t,T} - U_{\alpha_2}^{t,T}\right) V_{\alpha_2}^T(\cdot) \right\|_{\mathbf{C}^1} \\ &\le c_4 c_5 \int_t^T (s-t)^{-\beta} ds \sup_{s \in [t,T]} \|L_{\alpha_1}[s] - L_{\alpha_2}[s]\|_{\mathbf{C}^2 \to \mathbf{C}} \|V_{\alpha_2}^T(\cdot)\|_{\mathbf{C}^2} \\ &\le c_4 c_5 \frac{(T-t)^{1-\beta}}{1-\beta} |\alpha_1 - \alpha_2| \sup_{\substack{s \in [t,T] \\ \gamma \in [\alpha_1, \alpha_2]}} \left\| \frac{\partial L_\gamma}{\partial \alpha}[s] \right\|_{\mathbf{C}^2 \to \mathbf{C}} \|V_{\alpha_2}^T(\cdot)\|_{\mathbf{C}^2}. \end{aligned} \tag{3.35}$$

Therefore, from (3.33) together with (3.34) and (3.35), we complete the proof. □

In this work, we are only concerned with the Lipschitz continuity of the solution of the HJB with respect to the parameter. It is interesting to know whether the mapping $\alpha \mapsto W_\alpha(t,\cdot)$ is differentiable for each $t \in [0,T]$. For the completeness, the next proposition will show the existence of the derivative $\frac{\partial W_\alpha}{\partial \alpha}(t,\cdot)$ in $\mathbf{C}$ and present its' explicit expression.

Proposition 3.1. *Assume conditions* **(A2)** *and* **(A3)**. *Then*

(i) *for each* $0 \le t < s \le T$*, the mapping* $\alpha \mapsto U_\alpha^{t,s}$ *is differentiable in the sense of the strong operator topology in* $\mathcal{L}(\mathbf{C}^2, \mathbf{C})$ *and the derivative* $\frac{\partial U_\alpha^{t,s}}{\partial \alpha}$ *has the representation*

$$\frac{\partial U_\alpha^{t,s}}{\partial \alpha} = \int_t^s U_\alpha^{t,r} \frac{\partial L_\alpha}{\partial \alpha}[r] U_\alpha^{r,s}\, dr, \quad \text{for } 0 \le t < s \le T. \tag{3.36}$$

(ii) *for each* $t \in [0,T]$*, the mapping* $\alpha \mapsto W_\alpha(t,\cdot)$ *defined in* (3.32) *is differentiable as a function from* $[0,1]$ *to* $\mathbf{C}$ *and the partial derivative* $\frac{\partial W_\alpha}{\partial \alpha}(t,\cdot)$ *can be represented by*

$$\frac{\partial W_\alpha}{\partial \alpha}(t,\cdot) = U_\alpha^{t,T} \frac{\partial V_\alpha^T}{\partial \alpha}(\cdot) + \frac{\partial U_\alpha^{t,T}}{\partial \alpha} V_\alpha^T(\cdot), \quad \text{for } t \in [0,T]. \tag{3.37}$$

Proof. (i) Since the operator $L_\alpha[t]$ is differentiable in α for each $t \in [0,T]$, together with (2.2) in Proposition 2.1, for $\alpha_1, \alpha \in [0,1]$ with $\alpha_1 \neq \alpha$, we have

$$\begin{aligned} \frac{U_{\alpha_1}^{t,s} - U_\alpha^{t,s}}{\alpha_1 - \alpha} &= \frac{1}{\alpha_1 - \alpha} \int_t^s U_{\alpha_1}^{t,r} (L_{\alpha_1}[r] - L_\alpha[r]) U_\alpha^{r,s} dr \\ &= \int_t^s U_{\alpha_1}^{t,r} \frac{\partial L_\gamma}{\partial \alpha}[r] U_\alpha^{r,s} dr \end{aligned}$$

with some $\gamma \in [\alpha_1, \alpha]$. By (2.3) in Proposition 2.1, for each $0 \le t < r \le T$ the mapping $\alpha \mapsto U_\alpha^{t,r}$ is continuous as a function from $[0,1]$ to $\mathcal{L}(\mathbf{C}^2, \mathbf{C})$ in the strong operator topology. Then by standard density arguments, for each $0 \le t < r \le T$, the mapping $\alpha \mapsto U_\alpha^{t,r}$ is also continuous as a function from $[0,1]$ to $\mathcal{L}(\mathbf{C}, \mathbf{C})$ in the strong operator topology. Together with the smoothing property (3.12) and the condition that for each $r \in [0,T]$ and $\gamma \in [0,1]$, $\frac{\partial L_\gamma}{\partial \alpha}[r] : \mathbf{C}^2 \to \mathbf{C}$, we have that the derivative

$$\frac{\partial U_\alpha^{t,r}}{\partial \alpha} := \lim_{\alpha_1 \to \alpha} \frac{U_{\alpha_1}^{t,s} - U_\alpha^{t,s}}{\alpha_1 - \alpha}$$

exists in the strong topology in $\mathcal{L}(\mathbf{C}^2, \mathbf{C})$ and equals to

$$\frac{\partial U_\alpha^{t,r}}{\partial \alpha} = \int_t^s U_\alpha^{t,r} \frac{\partial L_\alpha}{\partial \alpha}[r] U_\alpha^{r,s} dr.$$

(ii) By condition **(A3)**, the mapping $\alpha \mapsto V_\alpha^T(\cdot)$ is differentiable and the derivative exists in $\mathbf{C}^1$, namely, $\lim_{\alpha_1 \to \alpha} \frac{V_{\alpha_1}^T - V_\alpha^T}{\alpha_1 - \alpha}$ exists and belongs to

$\mathbf{C}^1$. Since the mapping $\alpha \mapsto U_\alpha^{t,r}$ is continuous as a function from $[0,1]$ to $\mathcal{L}(\mathbf{C},\mathbf{C})$ in the strong operator topology, hence for any $t \in [0,T]$,

$$\lim_{\alpha_1 \to \alpha} \left(U_{\alpha_1}^{t,T} \frac{V_{\alpha_1}^T - V_\alpha^T}{\alpha_1 - \alpha} \right) = U_\alpha^{t,T} \frac{V_\alpha^T}{\partial \alpha} \in \mathbf{C}.$$

Finally, from (3.32) , the derivative

$$\begin{aligned}
\frac{\partial W_\alpha}{\partial \alpha}(t,\cdot) &:= \lim_{\alpha_1 \to \alpha} \frac{W_{\alpha_1}(t,\cdot) - W_\alpha(t,\cdot)}{\alpha_1 - \alpha} \\
&= \lim_{\alpha_1 \to \alpha} \left(U_{\alpha_1}^{t,T} \frac{V_{\alpha_1}^T(\cdot) - V_\alpha^T(\cdot)}{\alpha_1 - \alpha} \right) + \lim_{\alpha_1 \to \alpha} \frac{U_{\alpha_1}^{t,T} - U_\alpha^{t,T}}{\alpha_1 - \alpha} V_\alpha^T(\cdot) \\
&= U_\alpha^{t,T} \frac{\partial V_\alpha^T}{\partial \alpha}(\cdot) + \frac{\partial U_\alpha^{t,T}}{\partial \alpha} V_\alpha^T(\cdot)
\end{aligned}$$

exists in $\mathbf{C}$ for each $t \in [0,T]$ and any $\alpha \in [0,1]$. □

Remark 3.2. *If one would have that, for each $0 \le t < s \le T$, the mapping $\alpha \mapsto U_\alpha^{t,s}$ is continuous from $[0,1]$ to $\mathcal{L}(\mathbf{C},\mathbf{C}^1)$, then $\lim_{\alpha_1 \to \alpha} U_\alpha^{t,s}$ exists as an operator from $\mathbf{C}$ to $\mathbf{C}^1$. Then one would have that the mapping $\alpha \mapsto W_\alpha(t,\cdot)$ is differentiable as a function from $[0,1]$ to $\mathbf{C}^1$.*

Theorem 3.3. *Assume the conditions* **(A1), (A2)** *and* **(A3)**. *Then we have the following:*

(a) *For any $T > 0$, the mild solution V_α of (3.31) is Lipschitz continuous with respect to α i.e. there exists a constant $c = c(T) > 0$ such that for each $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$,*

$$\begin{aligned}
&\sup_{t \in [0,T]} \frac{\|V_{\alpha_1}(t,\cdot) - V_{\alpha_2}(t,\cdot)\|_{\mathbf{C}^1}}{|\alpha_1 - \alpha_2|} \\
&\le c \Bigg(\sup_{\gamma \in [\alpha_1,\alpha_2]} \left\| \frac{\partial V_\gamma^T}{\partial \alpha}(\cdot) \right\|_{\mathbf{C}^1} + \sup_{\substack{(t,p) \in \emptyset \\ \gamma \in [\alpha_1,\alpha_2]}} \left\| \frac{\partial H_{\gamma,t}(\cdot,p)}{\partial \alpha} \right\|_{\mathbf{C}} \qquad (3.38) \\
&\qquad + \sup_{\substack{t \in [0,T] \\ \gamma \in [\alpha_1,\alpha_2]}} \left\| \frac{\partial L_\gamma}{\partial \alpha}[t] \right\|_{\mathbf{C}^2 \to \mathbf{C}} \left(\|V_{\alpha_2}^T(\cdot)\|_{\mathbf{C}^2} + 1 \right) \Bigg),
\end{aligned}$$

where $\emptyset = \{(t,p) : t \in [0,T], |p| \le \sup_{t \in [0,T]} \|V_\alpha(t,\cdot)\|_{\mathbf{C}^1}\}$.

(b) *The mild solution V of (3.20) and its spacial derivative ∇V are Lipschitz continuous uniformly with respect to $\{\mu_\cdot\}$, that is, for each $\{\mu^1_\cdot\}, \{\mu^2_\cdot\} \in C([0,T], \mathcal{M})$, there exists a constant $k > 0$ such that*

$$\sup_{t\in[0,T]} \|V(t,\cdot;\{\mu^1_\cdot\}) - V(t,\cdot;\{\mu^2_\cdot\})\|_{\mathbf{C}^1} \le k \sup_{t\in[0,T]} \|\mu^1_t - \mu^2_t\|_{\mathbf{S}} \quad (3.39)$$

and

$$\sup_{t\in[0,T]} \|\nabla V(t,\cdot;\{\mu^1_\cdot\}) - \nabla V(t,\cdot;\{\mu^2_\cdot\})\|_{\mathbf{C}} \le k \sup_{t\in[0,T]} \|\mu^1_t - \mu^2_t\|_{\mathbf{S}}. \quad (3.40)$$

Proof. (i) Recall in the proof of Theorem 3.1, for any $\alpha \in [0,1]$, the unique solution V_α is the unique fixed point of the mapping

$$\phi \mapsto \Psi_\alpha(\phi), \quad C^T_{V^T_\alpha}([0,T], \mathbf{C}^1) \to C^T_{V^T_\alpha}([0,T], \mathbf{C}^1)$$

defined by, for each $t \in [0,T]$

$$\Psi_\alpha(\phi)(t,\cdot) = U^{t,T}_\alpha V^T_\alpha(\cdot) + \int_t^T U^{t,s}_\alpha H_{\alpha,s}(\cdot, \nabla\phi(s,\cdot))ds. \quad (3.41)$$

For any $\alpha_i \in [0,1]$, $i = 1,2$, let V_{α_i} be the unique fixed point of the mapping Ψ_{α_i}, i.e.

$$V_{\alpha_i} = \Psi_{\alpha_i}(V_{\alpha_i}), \quad \text{for } i = 1,2.$$

Then from (3.41) we have

$$\begin{aligned}
&V_{\alpha_1}(t,\cdot) - V_{\alpha_2}(t,\cdot) = \Phi_{\alpha_1}(V_{\alpha_1}(t,\cdot)) - \Phi_{\alpha_2}(V_{\alpha_2}(t,\cdot)) \\
&= U^{t,T}_{\alpha_1} V^T_{\alpha_1}(\cdot) - U^{t,T}_{\alpha_2} V^T_{\alpha_2}(\cdot) \\
&\quad + \int_t^T U^{t,s}_{\alpha_1} H_{\alpha_1,s}(\cdot, \nabla V_{\alpha_1}(s,\cdot))ds - \int_t^T U^{t,s}_{\alpha_2} H_{\alpha_2,s}(\cdot, \nabla V_{\alpha_2}(s,\cdot))ds \\
&= U^{t,T}_{\alpha_1} V^T_{\alpha_1}(\cdot) - U^{t,T}_{\alpha_2} V^T_{\alpha_2}(\cdot) \\
&\quad + \int_t^T \left(U^{t,s}_{\alpha_1} - U^{t,s}_{\alpha_2}\right) H_{\alpha_1,s}(\cdot, \nabla V_{\alpha_1}(s,\cdot))ds \\
&\quad + \int_t^T U^{t,s}_{\alpha_2} \left(H_{\alpha_1,s}(\cdot, \nabla V_{\alpha_1}(s,\cdot)) - H_{\alpha_2,s}(\cdot, \nabla V_{\alpha_1}(s,\cdot))\right) ds \\
&\quad + \int_t^T U^{t,s}_{\alpha_2} \left(H_{\alpha_2,s}(\cdot, \nabla V_{\alpha_1}(s,\cdot)) - H_{\alpha_2,s}(\cdot, \nabla V_{\alpha_2}(s,\cdot))\right) ds \\
&=: \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4. \qquad (3.42)
\end{aligned}$$

By theorem 3.2, there exists a constant $c > 0$ such that

$$\begin{aligned}\|\Lambda_1\|_{\mathbf{C}^1} \le c|\alpha_1-\alpha_2| &\sup_{\gamma\in[\alpha_1,\alpha_2]} \left\|\frac{\partial V^T_\gamma}{\partial\alpha}\right\|_{\mathbf{C}^1} \\ &+ c|\alpha_1-\alpha_2|(T-t)^{1-\beta} \sup_{\substack{s\in[t,T]\\ \gamma\in[\alpha_1,\alpha_2]}} \left\|\frac{\partial L_\gamma}{\partial\alpha}[s]\right\|_{\mathbf{C}^2\to\mathbf{C}} \|V^T_{\alpha_2}(\cdot)\|_{\mathbf{C}^2}. \quad (3.43)\end{aligned}$$

By proposition 2.1 and the differentiability of $L_\alpha[t]$ in α for each $t \in [0,T]$, we have

$$\begin{aligned}\|\Lambda_2\|_{\mathbf{C}^1} &= \left\|\int_t^T \left(\int_t^s U^{t,r}_{\alpha_1}(L_{\alpha_1}[r]-L_{\alpha_2}[r])U^{r,s}_{\alpha_2}dr\right) H_{\alpha_1,s}(\cdot,\nabla V_{\alpha_1}(s,\cdot))ds\right\|_{\mathbf{C}^1} \\ &= \left\|\int_t^T \left(\int_t^s U^{t,r}_{\alpha_1}(\alpha_1-\alpha_2)\frac{\partial L_\theta}{\partial\alpha}[r]U^{r,s}_{\alpha_2}dr\right) H_{\alpha_1,s}(\cdot,\nabla V_{\alpha_1}(s,\cdot))ds\right\|_{\mathbf{C}^1} \\ &\le |\alpha_1-\alpha_2|c_5c_6\int_t^T\int_t^s (r-t)^{-\beta}(s-r)^{-\beta}\|\frac{\partial L_\theta}{\partial\alpha}[r]\|_{\mathbf{C}^2\to\mathbf{C}}dr \\ &\qquad\qquad \|H_{\alpha_1,s}(\cdot,\nabla V_{\alpha_1}(s,\cdot))\|_{\mathbf{C}_{Lip}}ds \\ &\le |\alpha_1-\alpha_2|c_5c_6m\int_t^T (s-t)^{1-2\beta}\sup_{r\in[t,s]}\left\|\frac{\partial L_\theta}{\partial\alpha}[r]\right\|_{\mathbf{C}^2\to\mathbf{C}}ds \\ &\le |\alpha_1-\alpha_2|c_5c_6m\frac{1}{2-2\beta}(T-t)^{2-2\beta}\sup_{r\in[t,T]}\left\|\frac{\partial L_\theta}{\partial\alpha}[r]\right\|_{\mathbf{C}^2\to\mathbf{C}} \quad (3.44)\end{aligned}$$

with a $\theta \in [0,1]$ and $m := \sup_{(s,p)\in\Omega}\|H_{\alpha,s}(\cdot,p)\|_{\mathbf{C}_{Lip}} < \infty$ where $\Omega = \{(s,p) : s\in[0,T], |p| \le \sup_{s\in[0,T]}\|V_\alpha(s,\cdot)\|_{\mathbf{C}^1}\}$. By the condition **(A1)**, for each $t \in [0,T]$ and $x, p \in \mathbf{R}^d$ the mapping $\alpha \mapsto H_{\alpha,s}(x,p)$ is differentiable and the derivative is continuous, we have

$$\begin{aligned}\|\Lambda_3\|_{\mathbf{C}^1} &= \left\|\int_t^T U^{t,s}_{\alpha_2}(\alpha_1-\alpha_2)\frac{\partial H_{\alpha,s}}{\partial\alpha}(\cdot,\nabla V_{\alpha_1}(s,\cdot))ds\right\|_{\mathbf{C}^1} \\ &\le |\alpha_1-\alpha_2|\int_t^T c_5(s-t)^{-\beta}\left\|\frac{\partial H_{\theta,s}}{\partial\alpha}(\cdot,\nabla V_{\alpha_1}(s,\cdot))\right\|_{\mathbf{C}}ds \\ &\le |\alpha_1-\alpha_2|c_5\frac{(T-t)^{1-\beta}}{1-\beta}\sup_{(s,p)\in\emptyset}\left\|\frac{\partial H_{\theta,s}}{\partial\alpha}(\cdot,p)\right\|_{\mathbf{C}}, \quad (3.45)\end{aligned}$$

where $\emptyset = \{(s,p) : s\in[0,T], |p| \le \sup_{s\in[0,T]}\|V_{\alpha_1}(s,\cdot)\|_{\mathbf{C}^1}\}$. By (3.13) and

(3.8), we get

$$\begin{aligned}
\|\Lambda_4\|_{\mathbf{C}^1} &= \left\| \int_t^T U_{\alpha_2}^{t,s} \left(H_{\alpha_2,s}(\cdot, \nabla V_{\alpha_1}(s,\cdot)) - H_{\alpha_2,s}(\cdot, \nabla V_{\alpha_2}(s,\cdot)) \right) ds \right\|_{\mathbf{C}^1} \\
&\le c_1 c_5 \int_t^T (s-t)^{-\beta} \|\nabla V_{\alpha_1}(s,\cdot) - \nabla V_{\alpha_2}(s,\cdot)\|_{\mathbf{C}}\, ds \\
&\le c_1 c_5 \int_t^T (s-t)^{-\beta} \|V_{\alpha_1}(s,\cdot) - V_{\alpha_2}(s,\cdot)\|_{\mathbf{C}^1}\, ds \\
&\le c_1 c_5 \frac{1}{1-\beta} (T-t)^{1-\beta} \sup_{s\in[t,T]} \|V_{\alpha_1}(s,\cdot) - V_{\alpha_2}(s,\cdot)\|_{\mathbf{C}^1}. \qquad (3.46)
\end{aligned}$$

It follows, from (3.42) together with the estimates (3.43), (3.44), (3.45) and (3.46), that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|V_{\alpha_1}(t,\cdot) - V_{\alpha_2}(t,\cdot)\|_{\mathbf{C}^1} \qquad (3.47) \\
&\le c|\alpha_1 - \alpha_2| \sup_{\gamma\in[\alpha_1,\alpha_2]} \left\| \frac{\partial V_\gamma^T}{\partial \alpha} \right\|_{\mathbf{C}^1} \\
&\quad + c|\alpha_1 - \alpha_2| (T-t)^{1-\beta} \sup_{\substack{t\in[0,T] \\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial L_\gamma}{\partial \alpha}[t] \right\|_{\mathbf{C}^2\to\mathbf{C}} \|V_{\alpha_2}^T(\cdot)\|_{\mathbf{C}^2} \\
&\quad + |\alpha_1 - \alpha_2| c_5 c_6 m \frac{1}{2-2\beta} T^{2-2\beta} \sup_{\substack{t\in[0,T] \\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial L_\gamma}{\partial \alpha}[t] \right\|_{\mathbf{C}^2\to\mathbf{C}} \\
&\quad + |\alpha_1 - \alpha_2| c_5 \frac{T^{1-\beta}}{1-\beta} \sup_{\substack{(t,p)\in\emptyset \\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial H_{\gamma,t}}{\partial \alpha}(\cdot, p) \right\|_{\mathbf{C}} \\
&\quad + c_1 c_5 \frac{1}{1-\beta} T^{1-\beta} \sup_{t\in[0,T]} \|V_{\alpha_1}(t,\cdot) - V_{\alpha_2}(t,\cdot)\|_{\mathbf{C}^1}. \qquad (3.48)
\end{aligned}$$

For $c_1 c_5 \frac{1}{1-\beta} T^{1-\beta} < 1$, i.e. $T < \left(\frac{1-\beta}{c_1 c_5}\right)^{\beta-1}$, we have inequality (3.38). For any finite $T > 0$, the proof follows by iterations.

(ii) By the definitions of $L_\alpha[t]$, $H_{\alpha,t}(x,p)$, $V_\alpha^T(x)$ in (3.28), (3.29), (3.30) respectively and the assumptions (3.10), (3.14), (3.15), for any $\{\mu_\cdot^1\}, \{\mu_\cdot^2\} \in C([0,T], \mathcal{M})$, the statement follows from the equation (3.27) and the inequality (3.38) by setting $\alpha_1 = 1$ and $\alpha_2 = 0$. □

4 Application to mean field games

In this section, we apply the sensitivity results in Theorem 3.3 to a mean field games model and give verifiable conditions for the so-called feedback regularity condition.

Let us consider a continuous time dynamic game with a continuum of players and a terminal time $T > 0$. Take $\mathbf{S} = (\mathbf{C}^2)^*$, as the dual Banach space of $\mathbf{C}^2$, and $\mathcal{M} = \mathcal{P}(\mathbf{R}^d)$, as the set of probability measures on $\mathbf{R}^d$. In this game, all players are identical so it is symmetric with respect to permutation of the players. Choose one of the players and call it the *reference player*. We use the controlled stochastic process $(X_t : t \in [0, T])$ to model the controlled state dynamics of the reference player. At each time $t \in [0, T]$, the reference player knows only his own position X_t and the empirical distribution of all players $\mu_t \in \mathcal{P}(\mathbf{R}^d)$.

Remark 4.1. *For a better understanding of the stochastic process $(X_t : t \in [0, T])$, one may think that the controlled dynamics is described by a stochastic differential equation. For a very particular case of our model, set $L[t, \mu] = \frac{1}{2}(\sigma\nabla, \nabla)$ with a constant σ, i.e. the operator $L[t, \mu]$ generates a Brownian motion $\{\sigma W_t : t \geq 0\}$. Then one can write a stochastic differential equation corresponding to the generator* (3.2) *as*

$$dX_t = h(t, X_t, \mu_t, u_t)\, dt + \sigma dW_t \quad \textit{for all } t \geq 0.$$

In fact, this is exactly the case which was considered in the initial work on the mean field games [7, 8, 9, 18]. In our framework, this controlled dynamics of each player is extended to an arbitrary Markov process with a generator (3.2) *depending on a probability measures μ.*

The empirical distribution evolution of all players in the state space $\mathbf{R}^d$, denoted by $\{\mu_t \in \mathcal{P}(\mathbf{R}^d) : t \in [0, T]\}$, is described by the evolution equation

$$\frac{d}{dt}\int_{\mathbf{R}^d} g(y)\ \mu_t(dy) = \int_{\mathbf{R}^d} (A[t, \mu_t, u_t]g(y))\ \mu_t(dy) \quad \text{for all}\, g \in \mathbf{C}^2 \tag{4.1}$$

with a given initial value $\mu_0 \in \mathcal{P}(\mathbf{R}^d)$. The equation (4.1) is a controlled version of a *general kinetic equation* in weak form and very often written in the compact form

$$\frac{d}{dt}(g, \mu_t) = (A[t, \mu_t, u_t]g, \mu_t). \tag{4.2}$$

See [15, 16] for more discussion on equation (4.2) and its well posedness with open-loop controls under rather general technical assumptions. A solution

$\{\mu_t : t \in [0,T]\}$ of equation (4.2) is called the (probability) *measure flow*. Let $C_{\mu_0}([0,T], \mathcal{P}(\mathbf{R}^d))$ be the set of continuous functions $t \to \mu_t$ with $\mu_t \in \mathcal{P}(\mathbf{R}^d)$ for each $t \in [0,T]$ and with the norm

$$\|\mu\|_{(\mathbf{C}^2)^*} := \sup_{\|g\|_{\mathbf{C}^2} \leq 1} |(g, \mu)| = \sup_{\|g\|_{\mathbf{C}^2} \leq 1} \left| \int_{\mathbf{R}^d} g(x)\mu(dx) \right|. \tag{4.3}$$

In this game, the reference player faces an optimisation problem described by the HJB equation (3.6). If the max is achieved only at one point, i.e. for any $(t, x, \mu, p) \in [0,T] \times \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d) \times \mathbf{R}^d$,

$$\arg\max_{u \in \mathcal{U}} (h(t, x, \mu, u)p + J(t, x, \mu, u))$$

is a singleton, then one can derive the unique optimal control strategy from the solution of (3.6). For any given curve $\{\mu_t : t \in [0,T]\} \in C_{\mu_0}([0,T], \mathcal{P}(\mathbf{R}^d))$, let the resulting unique optimal control strategy be denoted by

$$\hat{u}(t, x; \{\mu_s : s \in [t,T]\}) \quad \text{for all } t \in [0,T], x \in \mathbf{R}^d. \tag{4.4}$$

Substituting the feedback control strategy (4.4) into (4.2) yields the closed-loop evolution equation for the distributions μ_t

$$\frac{d}{dt}(g, \mu_t) = (A[t, \mu_t, \hat{u}(t, x; \{\mu_s : s \in [t,T]\})]g, \mu_t).$$

The mean field game methodology amounts to find an optimal control strategy $\{\hat{u}_\cdot\}$ for each agent and a measure flow $\{\mu_\cdot\}$ such that the following two coupled equations

$$\begin{aligned} &\frac{d}{dt}(g, \mu_t) = (A[t, \mu_t, \hat{u}(t, x; \{\mu_s : s \in [t,T]\})]g, \mu_t) \\ &\mu|_{t=0} = \mu_0 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &-\frac{\partial V}{\partial t}(t, x) = \max_{u_\cdot \in \mathcal{U}} (h(t, x, \mu_t, u_t)\nabla V + J(t, x, \mu_t, u_t)) + L[t, \mu_t]V(t, x) \\ &V(T, \cdot; \mu_T) = V^T(\cdot; \mu_T) \end{aligned} \tag{4.6}$$

hold. The control strategy $\hat{u}$ applied in (4.5) is derived from the HJB equation (4.6). Since the controlled kinetic equation (4.5) is forward and the

HJB equation (4.6) is backward, this system of coupled equations is referred to as a *coupled backward-forward system.*

To solve this coupled backward-forward system (4.5)-(4.6), it is critical that the resulting control mapping $\hat{u}$ (4.4) satisfies the so-called *feedback regularity* condition (see e.g. [8]), i.e. for any $\{\eta_t : t \in [0,T]\}$, $\{\xi_t : t \in [0,T]\} \in C_{\mu_0}([0,T], \mathcal{P}(\mathbf{R}^d))$, there exists a constant $k_1 > 0$ such that

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times\mathbf{R}^d} & |\hat{u}(t,x;\{\eta_s : s\in[t,T]\}) - \hat{u}(t,x;\{\xi_s : s\in[t,T]\})| \\ & \le k_1 \sup_{s\in[0,T]} ||\eta_s - \xi_s||_{(\mathbf{C}^2)^*}. \end{aligned} \tag{4.7}$$

Theorem 4.1. *Suppose L, H and V^T in* (3.6)-(3.7) *satisfy the conditions* **(A1)**, **(A2)**, **(A3)** *respectively. Assume additionally the max in* (3.7) *is achieved only at one point, i.e. for any $(t,x,\mu,p) \in [0,T]\times\mathbf{R}^d\times\mathcal{P}(\mathbf{R}^d)\times\mathbf{R}^d$*

$$\arg\max_{u\in\mathcal{U}}(h(t,x,\mu,u)p + J(t,x,\mu,u)) \tag{4.8}$$

is a singleton and the resulting control as a function of (t,x,μ,p) is continuous in $t\in[0,T]$ and Lipschitz continuous in $(x,\mu,p)\in\mathbf{R}^d\times\mathcal{P}(\mathbf{R}^d)\times\mathbf{R}^d$ uniformly with respect to t, x, μ and bounded p. Then, given a trajectory $\{\mu_\cdot\} \in C_{\mu_0}([0,T],\mathcal{P}(\mathbf{R}^d))$, the feedback control mapping

$$\hat{u}(t,x;\{\mu_s : s\in[t,T]\})$$

defined via equations (3.6)-(3.7)*, is Lipschitz continuous uniformly in $\{\mu_\cdot\}$, i.e. for every $\{\eta_\cdot\}$, $\{\xi_\cdot\} \in C_{\mu_0}([0,T],\mathcal{P}(\mathbf{R}^d))$,*

$$\begin{aligned} \sup_{(t,x)\in[0,T]\times\mathbf{R}^d} & |\hat{u}(t,x;\{\eta_s : s\in[t,T]\}) - \hat{u}(t,x;\{\xi_s : s\in[t,T]\})| \\ & \le k_1 \sup_{s\in[0,T]} ||\eta_s - \xi_s||_{(\mathbf{C}^2)^*} \end{aligned} \tag{4.9}$$

with some constant $k_1 > 0$.

Proof. By Theorem 3.3, together with the assumption that the resulting unique control mapping is Lipschitz continuous in (x,μ,p), we conclude that the unique point of maximum in the expression

$$\max_{u\in\mathcal{U}}\{h(t,x,\mu_t,u)\nabla V(t,x;\{\mu_\cdot\}) + J(t,x,\mu_t,u)\}$$

has the claimed properties. □

We give two examples where the uniqueness condition (4.8) holds.

Example 4.1 (H_∞-optimal control, see [24] for its systematic presentation)**.** *Assume that the running cost function J is quadratic in u, i.e.*

$$J(t, x, \mu, u) = \alpha(t, x, \mu) - \theta(t, x, \mu)u^2$$

and the drift coefficient h is linear in u, i.e.

$$h(t, x, \mu, u) = \beta(t, x, \mu)u,$$

where the functions $\alpha, \beta, \theta : [0, T] \times \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d) \to \mathbf{R}$ and $\theta(t, x, \mu) > 0$ for any (t, x, μ). Thus, we are maximising a quadratic function over control u. It is easy to get an explicit formula of the unique point of maximum, i.e.

$$\hat{u} = \frac{\beta}{2\theta}(t, x, \mu)p.$$

Thus, the HJB equation (3.6) *rewrites as*

$$\frac{\partial V}{\partial t}(t, x; \{\mu.\}) + \frac{\beta^2}{4\theta}(t, x, \mu_t)(\nabla V)^2(t, x; \{\mu.\}) + \alpha(t, x, \mu_t) + L[t, \mu_t]V(t, x; \{\mu.\}) = 0,$$

which is a generalized backward Burger's equation.

Example 4.2. *Assume $h(t, x, \mu, u) = u$ and $J(t, x, \mu, u)$ is a strictly concave smooth function of u. Then H_t is the Legendre transform of $-J$ as a function of u, and the unique point of maximum in* (3.7) *is*

$$\hat{u} = \frac{\partial H_t}{\partial p}.$$

If $J(t, x, \mu, u)$ has the decomposition

$$J(t, x, \mu, u) = \tilde{V}(x, \mu) + \tilde{J}(x, u)$$

for $\tilde{V} : \mathbf{R}^d \times \mathcal{P}(\mathbf{R}^d) \to \mathbf{R}$, $\tilde{J} : \mathbf{R}^d \times \mathcal{U} \to \mathbf{R}$ and $L(t, \mu) = \Delta$, the corresponding coupled backward-forward system (4.5)-(4.6) *turns to system (2) of [18] (only there the kinetic equation is written in the strong form and in reverse time).*

Remark 4.2. *Let us stress again that in* (3.39), (3.40) *the space* $\mathbf{S}$ *is an abstract Banach space, but in application to control depending on empirical measures, we have in mind the norm of the dual space* $(\mathbf{C}^2)^*$, *where* $\mathbf{C}^2$ *is the domain of the generating family* $A[t, \mu, u]$.

5 Conclusion

In this paper, our main aim is to analysis the sensitivity of the solution to a Hamilton-Jacobi-Bellman equation (3.6) with respect to a functional parameter $\{\mu.\} \in C([0,T], \mathcal{M})$. This problem was first reduced to the sensitivity analysis with respect to a real-valued parameter $\alpha \in [0,1]$. Then we proved in Theorem 3.2 and Theorem 3.3 that the unique mild solution is Lipschitz continuous with respect to α. Finally, as an application of our sensitivity results, we gave verifiable conditions for the feedback regularity condition which is needed in mean field games model for solving the coupled backward-forward system (4.5)-(4.6).

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