A new approach to the solution of free rigid body motion for attitude maneuvers*

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***Abstract*— A Hamiltonian formulation of free rigid body motion defined on the Special Unitary Group $SU(2)$ is used to integrate the system to obtain a convenient quaternion representation for attitude engineering applications. Novel content of this paper concerns applying a modern approach, based on geometric control theory to obtain the kinematic solution in an elegant and compact form. Moreover, this integration leads to an attitude representation which is not Euler-angle-like, thus enhancing its applicability (e.g. to attitude motion design).**

I. INTRODUCTION

This paper derives an analytic solution for the natural motions of an asymmetric rigid body using a Lax pair integration on the Special Unitary Group $SU(2)$. This enables the solution to be expressed in a useful compact quaternion form. The motivation for this is to exploit the derived analytic solution, in future work, by making use of it to rapidly generate reference tracks for space attitude maneuvers or as initial guesses in numerical optimisation software. In particular, exploiting natural motions for attitude control has been undertaken for the symmetric case [1] in order to minimise torque requirement. In the literature the solution for the asymmetric case is well known and classically solved using Euler angles in terms of Jacobi elliptic functions and theta functions [2], [3], [4], [5], [6]. However, using the Hamiltonian formulation and $SU(2)$ as the configuration space it is shown that the solution can be solved independently of theta functions. In this work the solution is expressed in terms of an elliptic integral and Jacobian elliptic functions which can be evaluated using theta functions as well as many other alternative methods [7], [8]. In this respect when implementing the analytic solutions in software the user has the flexibility to choose the most appropriate method to evaluate the elliptic integral whether it be via theta functions or otherwise. An additional advantage is that the solution is in a quaternion form without being Euler-angle-like, as the quaternions are not constructed using inverse trigonometric functions of the Euler angles. Moreover, although the analytic solution to the Euler equations is expressed in terms of Jacobi elliptic functions similarly to the existing literature [9], it is the corresponding quaternion solution that is expressed in a form different to the classically stated solutions [4] [5] [6] or from the indirect application of the Euler angles solution, [2]. In this paper we derive the equations using the general Lax Pair integration for Hamiltonian systems on the Lie group $SO(3)$ [10] adapted to $SU(2)$ and specialised to the equations of the Hamiltonian of the asymmetric rigid body. Implementation issues are remarked throughout the paper to make this work accessible to practical application and which can be used as a quick reference to a compact quaternion description.

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II. MATHEMATICAL REPRESENTATIONS OF FREE RIGID BODY MOTIONS

To obtain a global description of spacecraft kinematics a rotation matrix $R(t)$ can be used which is an element of the Special Orthogonal Group $SO(3)$ [11], [12]. Often the rotation is described locally by parameterizing the rotation matrix using Euler angles [12], [13]. However, in this paper we use a rotation matrix $R(t)$ which is an element of the Special Unitary Group $SU(2)$ where $R(t) \in SU(2)$ is of the form:

$$R(t) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \quad (1)$$

with $z_1, z_2 \in \mathbb{C}$ and $\bar{z}_1, \bar{z}_2$ their complex conjugates such that $|z_1|^2 + |z_2|^2 = 1$. It seems that it was Klein [14] who discovered that for the symmetric Lagrange and toy top (a symmetric rigid body in a constant gravitaional field) simpler solutions can be obtained when $SU(2)$ is used as configuration space rather than $SO(3)$. Furthermore, a modern integration of the toy top which uses the setting of Lagrangian mechanics on the Lie group $SU(2)$ was used to find the solution in terms of hyperelliptic functions, [15]. The reason for using $SU(2)$ as the configuration space in this paper is two-fold. Firstly using a matrix representation it is possible to express the equations of motion in a convenient Lax pair form, and secondly that $SU(2)$ is isomorphic to the unit quaternions [9]. Furthermore, the mapping F from $SU(2)$ to the unit quaternions $\mathbf{Q}$ is a simple one: $F : SU(2) \leftrightarrow \mathbf{Q}$:

$$F : \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \leftrightarrow z_1 + z_2\mathbf{j} = q_0\mathbf{e} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad (2)$$

defining the coordinate change and where the complex numbers $z_1 = q_0 + i\,q_1$, $z_2 = q_2 + i\,q_3$ are regarded in their quaternion form $z_1 = q_0\mathbf{e} + q_1\mathbf{i}$, $z2 = q_2\mathbf{e} + q_3\mathbf{i}$ subject to the usual quaternionic multiplication. For more details of this isomorphism see [10]. The kinematic equations can then be expressed as [16]:

$$\frac{dR(t)}{dt} = R(t)(\Omega_1 A_1 + \Omega_2 A_2 + \Omega_3 A_3) \quad (3)$$

where $\Omega_1, \Omega_2, \Omega_3$ are the angular velocities in body fixed coordinates and A_1, A_2, A_3 describe the infinitesimal motion of the spacecraft in the roll, pitch and yaw directions respectively. Mathematically speaking they form a basis for the Lie algebra $\mathfrak{su}(2)$ of the Lie group $SU(2)$:

$$A_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ A_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4)$$

$$A_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

with its commutator defined by $[X, Y] = XY - YX$ called the Lie bracket such that $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$ and $[A_1, A_3] = -A_2$. Note the $\frac{1}{2}$ in the basis is introduced so that the commutative relation of the Lie bracket corresponds to the cross product in $\mathbb{R}^3$.

The attitude dynamics of a free rigid spacecraft can be described by the Euler equations in coordinate form:

$$\begin{aligned} \dot{\Omega}_1(t) &= \delta_1 \Omega_2(t)\Omega_3(t) \\ \dot{\Omega}_2(t) &= \delta_2 \Omega_1(t)\Omega_3(t) \\ \dot{\Omega}_3(t) &= \delta_3 \Omega_1(t)\Omega_2(t) \end{aligned} \quad (5)$$

where $\delta_1 = \frac{c_2 - c_3}{c_1}$, $\delta_2 = \frac{c_3 - c_1}{c_2}$, $\delta_3 = \frac{c_1 - c_2}{c_3}$ where c_1, c_2, c_3 are the principal moments of inertia. It is easily shown that the following quantities are conserved:

$$\begin{aligned} H &= \tfrac{1}{2}\left(c_1\Omega_1^2(t) + c_2\Omega_2^2(t) + c_3\Omega_3^2(t)\right) \\ K^2 &= c_1^2\Omega_1^2(t) + c_2^2\Omega_2^2(t) + c_3^2\Omega_3^2(t) \end{aligned} \quad (6)$$

H and K are, respectively, the kinetic energy of the system and the magnitude of the total angular momentum.

In order to integrate the equations of motion it is convenient to express Euler's equations in Lax Pair form on $SU(2)$ ([17], [18]) :

$$\frac{dM(t)}{dt} = [M(t), \Omega] = [M(t), \nabla H] \quad (7)$$

where

$$\begin{aligned} M(t) &= c_1\Omega_1(t)A_1 + c_2\Omega_2(t)A_2 + c_3\Omega_3(t)A_3 \quad (8) \\ \Omega &= \Omega_1(t)A_1 + \Omega_2(t)A_2 + \Omega_3(t)A_3 \\ \nabla H &= \frac{\partial H}{\partial M_i} = \Omega \end{aligned}$$

where the basis is defined by (4). This form is convenient as the general solution is well known to be of the form ([10]):

$$M(t) = R(t)^{-1}M(0)R(t) \quad (9)$$

III. FREE RIGID BODY ANGULAR VELOCITIES

It is well known that the Euler equations describing free rigid-body can be solved analytically in terms of Jacobi elliptic functions [9] Chapter 15, [19]. They are included here expressed in a compact form:

Lemma 1: The free rigid body angular velocities Ω_i can be expressed in the analytic form:

$$\Omega_i(t) = \frac{\sqrt{s_i}}{c_i} sn\left(\pm\sqrt{\alpha s_j}t + C_i, \frac{s_i}{s_j}\right) \quad (10)$$

when

$$\left|\ \frac{s_i}{s_j}\ \right| \leq 1$$

or

$$\Omega_i(t) = \frac{\sqrt{s_j}}{c_i} sn\left(\pm\sqrt{\alpha s_i}t + \sqrt{\frac{s_i}{s_j}}C_i, \frac{s_j}{s_i}\right) \quad (11)$$

otherwise[1].

$sn(\cdot,\cdot)$ is a Jacobi elliptic function and the constants C_i are defined[2] by

$$C_i = sn^{-1}\left(\frac{c_i\Omega_i(0)}{\sqrt{s_i}}, \frac{s_i}{s_j}\right) \quad (12)$$

with

$$s_i = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\chi}}{2\alpha} \qquad s_j = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\chi}}{2\alpha} \quad (13)$$

and

$$\begin{aligned} \alpha &= -\tfrac{(c_i - c_j)(c_i - c_k)}{c_j c_k} \\ \beta &= \tfrac{4c_j c_k H - 2c_i(c_j + c_k)H + 2c_i K^2 - (c_j + c_k)K^2}{c_i c_j c_k} \\ \chi &= -\tfrac{\left(2c_j H - K^2\right)\left(2c_k H - K^2\right)}{c_i^2 c_j c_k} \end{aligned} \quad (14)$$

where the indexes do not represent a sum; conversely i, j and k follow a "circular notation", which means they appear in a consecutive recursion (e.g. i=1,j=2,k=3 or i=2,j=3,k=1 etc. ...) with the conserved quantities H and K defined in terms of the initial angular velocities:

$$\begin{aligned} H &= \tfrac{1}{2}\left(c_1\Omega_1^2(0) + c_2\Omega_2^2(0) + c_3\Omega_3^2(0)\right) \\ K^2 &= c_1^2\Omega_1^2(0) + c_2^2\Omega_2^2(0) + c_3^2\Omega_3^2(0) \end{aligned} \quad (15)$$

Proof. [11]

Remark 1: The sign $\pm$ in (11) is dependent on the initial conditions. For implementation the sign has to be chosen so that the sign of the first derivative of the analytically defined angular velocity at the initial time $t = 0$:

$$\dot{\Omega}_i(0) = \pm\sqrt{\alpha s_j}\frac{\sqrt{s_i}}{c_i} cn\left(C_i, \frac{s_i}{s_j}\right) dn\left(C_i, \frac{s_i}{s_j}\right) \quad (16)$$

when $s_j \geq s_i$ or otherwise

$$\dot{\Omega}_i(0) = \pm\sqrt{\alpha s_i}\frac{\sqrt{s_j}}{c_i} cn\left(\sqrt{\frac{s_i}{s_j}}\,C_i, \frac{s_j}{s_i}\right) dn\left(\sqrt{\frac{s_i}{s_j}}\,C_i, \frac{s_j}{s_i}\right)$$

is matched to the sign of the original equations at $t = 0$:

$$\dot{\Omega}_i(0) = \delta_i\Omega_l(0)\Omega_k(0) \text{ ,with } l \neq k \neq i \quad (17)$$

[1]Using the relation:

$$\sqrt{\mu}\, sn(\,\zeta, \mu\,) = sn(\,\sqrt{\mu}\,\zeta, \mu^{-1}\,)$$

[2]Note that the domain of an inverse elliptic function is $\mathbb{C}^2$ as well.

IV. QUATERNION SOLUTION

The main result of this paper, that is, to express the quaternion solution of natural rigid body motion independently of theta functions, is stated as a Theorem:

Theorem 1: The natural motions of a free rigid body in quaternion form with $\bar{q} = [q_0, q_1, q_2, q_3]^T$ is:

$$
\begin{aligned}
q_0 &= \mathcal{F}_1 \left(\cos(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_3 - \sin(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_4\right) \\
q_1 &= \mathcal{F}_1 \left(\sin(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_3 + \cos(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_4\right) \\
q_2 &= \mathcal{F}_2 \left(\cos(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_3 + \sin(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_4\right) \\
q_3 &= \mathcal{F}_2 \left(\sin(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_3 - \cos(\tfrac{\varphi_1(t)}{2})\, \mathcal{F}_4\right)
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
&\varphi_1(t) = \tfrac{K}{c_1} t + \kappa \Pi(n; \vartheta |\, m) + D, \\
&\mathcal{F}_1 = \mathcal{S}_1 \sqrt{\tfrac{1+x(t)}{2}}, \\
&\mathcal{F}_2 = \mathcal{S}_2 \sqrt{\tfrac{1-x(t)}{2}}, \\
&\mathcal{F}_3 = \mathcal{S}_3 \tfrac{1}{\sqrt{1+y^\star(t)^2}}, \\
&\mathcal{F}_4 = \mathcal{S}_4 \tfrac{y^\star(t)}{\sqrt{1+y^\star(t)^2}}, \\
&\mathcal{S}_i = \pm 1, \text{ for } i = 1,2,3,4, \\
&y^\star(t) = \tfrac{y(t)}{1+\sqrt{1+y(t)^2}}, \\
&x(t) = \tfrac{c_1 \Omega_1(t)}{K}, \\
&y(t) = \tfrac{c_2 \Omega_2(t)}{c_3 \Omega_3(t)}.
\end{aligned} \tag{19}
$$

where $\Pi(n; \vartheta |\, m)$ is the incomplete elliptic integral of the third kind with

$$
\begin{aligned}
&n = s_1/K^2 \\
&f(t) = \pm\sqrt{s_2 \alpha}\ t + C_{11} \\
&m = s_1/s_2 \\
&\vartheta = am(f(t), m) \\
&\kappa = \pm \left(2Hc_1 - K^2\right) / \left(Kc_1 \sqrt{s_2 \alpha}\right)
\end{aligned} \tag{20}
$$

when $\mid s_1/s_2 \mid \leq 1$. When $\mid s_1/s_2 \mid > 1$ it is given:

$$
\begin{aligned}
&n = s_2/K^2 \\
&f(t) = \pm\sqrt{s_1 \alpha}\ t + \sqrt{\tfrac{s_1}{s_2}} C_{11} \\
&m = s_2/s_1 \\
&\vartheta = am(f(t), m) \\
&\kappa = \pm \left(2Hc_1 - K^2\right) / \left(Kc_1 \sqrt{s_1 \alpha}\right)
\end{aligned} \tag{21}
$$

$am(\cdot, m)$ is the Jacobi amplitude, D is a constant of integration and $\Omega_1(t), \Omega_2(t), \Omega_3(t)$ are defined in Lemma 1. Note that in $f(t)$ and κ the sign must be chosen in accordance with the sign of $\sqrt{s_2 \alpha}$ in the Ω_1 expression.

Remark 2: The $\mathcal{S}_i$ functions are sign functions, so they can be either +1 or -1. To implement the equations it is not necessary to take into account all their combinations. Indeed, excluding all the equivalent rotations[3], it is enough to consider only two of them: $\mathcal{S}_1$ or $\mathcal{S}_2$ and $\mathcal{S}_3$ or $\mathcal{S}_4$ respectively. The sign functions can be set by comparison with the known initial first derivative of the quaternions:

$$\dot{q}_0 = \bar{\Omega}_0(\, \Omega_1(0), \Omega_2(0), \Omega_3(0)\,)\, q_0$$

Note that the theorem is given for an arbitrary initial condition without loss of generality as the quaternions can be rotated to any initial condition.

Proof.

From equation (9):

$$R(t) M(t) R(t)^{-1} = M(0) \tag{22}$$

where $M(0)$ is a matrix of constant entries and $R(t)M(t)R(t)^{-1}$ describes the conjugacy class of $M(t)$ and therefore an initial $R(0)$ can be chosen such that $M(0) = KA_1$. Therefore, it suffices to integrate the particular solution:

$$\bar{R}(t) M(t) \bar{R}(t)^{-1} = KA_1 \tag{23}$$

where K is the constant defined in (6) therefore

$$M(t) = K\bar{R}(t)^{-1} A_1 \bar{R}(t) \tag{24}$$

as $\exp(\varphi_1(t) A_1)$ is a stabiliser of A_1 it is convenient to introduce the coordinate form [10]:

$$\bar{R}(t) = \exp(\varphi_1(t)A_1)\exp(\varphi_2(t)A_2)\exp(\varphi_3(t)A_1) \tag{25}$$

and substituting into (24) yields:

$$M(t) = \frac{iK}{2} \begin{pmatrix} \cos\varphi_2(t) & e^{-i\varphi_3(t)} \sin\varphi_2(t) \\ e^{i\varphi_3(t)} \sin\varphi_2(t) & -\cos\varphi_2(t) \end{pmatrix} \tag{26}$$

then equating (26) with $M(t)$ in (8) yields:

$$c_1 \Omega_1(t) = K \cos\varphi_2(t) \tag{27}$$

which gives $\varphi_2(t)$ and

$$
\begin{aligned}
c_2\Omega_2(t) + ic_3\Omega_3(t) &= iK\, e^{-i\varphi_3(t)} \sin\varphi_2(t) \\
-c_2\Omega_2(t) + ic_3\Omega_3(t) &= iK\, e^{i\varphi_3(t)} \sin\varphi_2(t)
\end{aligned} \tag{28}
$$

which gives $\varphi_3(t)$:

$$\tan\varphi_3(t) = \frac{c_2\Omega_2(t)}{c_3\Omega_3(t)} \tag{29}$$

then let $x(t) = \cos\varphi_2(t)$ and $y(t) = \tan\varphi_3(t)$ to yield (19).

It remains to solve for $\varphi_1(t)$. Using the coordinate representation of $\bar{R}(t)$, (25) yields:

$$
\begin{aligned}
&\bar{R}(t)^{-1} \frac{d\bar{R}(t)}{dt} = \\
&\frac{\dot{\varphi}_1(t)}{2} \begin{pmatrix} i\cos\varphi_2(t) & ie^{-i\varphi_3(t)} \sin\varphi_2(t) \\ ie^{i\varphi_3(t)} \sin\varphi_2(t) & -i\cos\varphi_2(t) \end{pmatrix} \\
&+ \frac{\dot{\varphi}_2(t)}{2} \begin{pmatrix} 0 & e^{-i\varphi_3(t)} \\ -e^{i\varphi_3(t)} & 0 \end{pmatrix} + \frac{\dot{\varphi}_3(t)}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\end{aligned}
$$

then equating (30) to (3) yields:

[3]Depending of whether the orthonormal frame is positively or negatively oriented.

$$\Omega_2(t) + i\Omega_3(t) = \dot{\varphi}_1(t) i e^{-i\varphi_3(t)} \sin\varphi_2(t) + \dot{\varphi}_2(t) e^{-i\varphi_3(t)}$$
$$-\Omega_2(t) + i\Omega_3(t) = \dot{\varphi}_1(t) i e^{i\varphi_3(t)} \sin\varphi_2(t) - \dot{\varphi}_2(t) e^{i\varphi_3(t)} \quad (30)$$

the two equations in (30) can be rearranged to give:

$$\frac{\Omega_2(t)}{e^{-i\varphi_3(t)}} + \frac{i\Omega_3(t)}{e^{-i\varphi_3(t)}} = \dot{\varphi}_1(t) i \sin\varphi_2(t) + \dot{\varphi}_2(t)$$
$$-\frac{\Omega_2(t)}{e^{i\varphi_3}} + \frac{i\Omega_3(t)}{e^{i\varphi_3}} = \dot{\varphi}_1(t) i \sin\varphi_2(t) - \dot{\varphi}_2(t) \quad (31)$$

then adding the two equations in (31) and using the expressions in (28) and simplifying gives:

$$\dot{\varphi}_1(t) = K\left(\frac{\Omega_2^2(t)c_2 + c_3\Omega_3^2(t)}{(c_2\Omega_2(t))^2 + (c_3\Omega_3(t))^2}\right) \quad (32)$$

and using the conserved quantities (6) can be expressed as:

$$\varphi_1(t) = \int K\left(\frac{2H - c_1\Omega_1^2(t)}{K^2 - (c_1\Omega_1(t))^2}\right) dt \quad (33)$$

Similarly as shown in [2], such expression can be taken to a known form after algebraic manipulation and a proper change of variable. Indeed, taking into account the general expression of the angular velocity:

$$\Omega_1(t) = \zeta sn(\xi t + \gamma, \mu)$$

it yields:

$$K\left(\frac{2H - c_1\Omega_1^2(t)}{K^2 - (c_1\Omega_1(t))^2}\right) = \quad (34)$$
$$\frac{K}{c_1} + \frac{2Hc_1 - K^2}{Kc_1}\frac{1}{1 - (\frac{c_1\zeta}{K})^2 sn^2(\xi t + \gamma, \mu)}$$

The integration of the first term is trivial, while to the integration of the second term the following change of variable is considered:

$$\vartheta = am(\xi t + \gamma, \mu) \quad (35)$$
$$d\vartheta = \xi\sqrt{1 - \mu^2 sn^2(\xi t + \gamma, \mu)}\, dt$$

Such step, takes the integral to the form

$$\dot{\varphi}_1(t) = \int \frac{K}{c_1} dt + \quad (36)$$

$$\sigma \int \frac{1}{1 - (\frac{c_1\zeta}{K})^2 sn^2(\xi t + \gamma, \mu)} \frac{1}{\sqrt{1 - \mu^2 sn^2(\xi t + \gamma, \mu)}} d\vartheta$$

with

$$\sigma = \frac{2Hc_1 - K^2}{Kc_1\xi}$$

whose solution is a linear term plus an indefinite elliptic integral of the third kind ([2], [7]) and an integration constant, as stated in *Theorem* 1.

The isomorphism (2) is used to map $\bar{R}(t)$ onto the quaternions $\bar{q}$ in (18)

Remark 3: To obtain the initial state through the origin R_{int}, the quaternions (18) have to be calculated by pulling $\bar{R}(t)$ in (25) back to the identity via:

$$R(t) = R_{int}\bar{R}(0)^{-1}\bar{R}(t) \quad (37)$$

and again, to obtain the final expression, the isomorphism (2) has to be used to map onto the quaternions.

V. CONCLUSIONS

A Hamiltonian formulation of free rigid body motion defined on the Special Unitary Group $SU(2)$ is used to integrate the system to obtain a compact quaternion representation. In contrast to the classical solution of the evolution of the configuration space, this modern geometric approach leads to a solution that involves neither theta functions nor Euler angles defined by inverse trigonometric functions. Moreover the solution is presented in a form of analytically defined quaternions depending only on the angular velocities, therefore, they are convenient for a variety of engineering applications (e.g. space attitude motion planning and control) as they constitute a non-singular set of variables describing the attitude of a free rigid body. As an example, some of the practical advantages are shown in Figures 1 and 2. Here the same natural motion is described first using the solution in Euler angle form, found in literature, [2] Eq. (B5) and (B10); then using the quaternions of Theorem 1 of Section IV in this paper. While the curves in Figure 2 are smooth, Figure 1 shows discontinuities which make them unfeasible for direct application, e.g. two different Euler angle sets should be used at the same time in order to switch from one to another to avoid singularities. Future work will investigate the potential use of these analytic solutions in spacecraft attitude dynamics and control problems to rapidly generate reference tracks for attitude control and as quick initial guesses for numerical optimisation processes on-board. Figure 2 and 3 illustrate one of the potential applications introduced above.

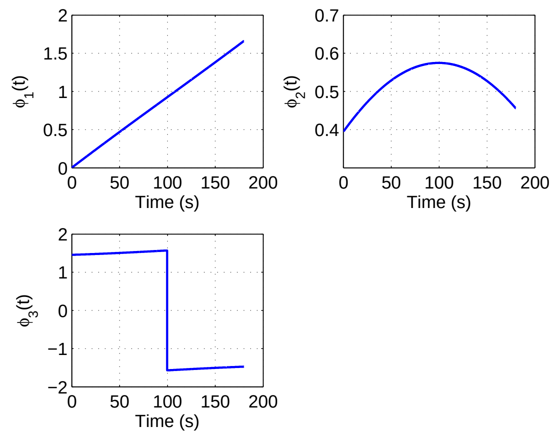

Fig. 1. Euler angles reference track generated with the classical solution

In Figure 2 the continuous and thick lines show the track flown by a spacecraft guided by a PD controller following the natural motion from a given initial condition q_0 to a prescribed target configuration q_{Target}, dashed and thin line, as a reference.

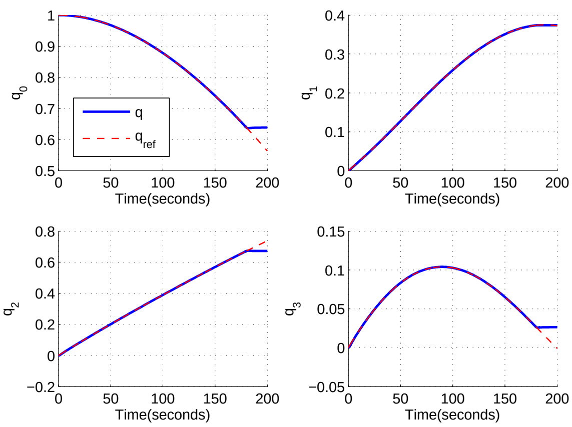

Fig. 2. Quaternion reference track generated with the solution proposed in this paper and then tracked, respectively dashed and continuous line. These quaternions describe the position of the spacecraft's Body Reference Frame w.r.t. a Geocentric Equatorial Reference Frame assumed to be inertial and fixed. Note that the dashed line differs from the continuous one because it describes the natural motion. Indeed, the natural motion track is no longer used as a reference once the target configuration is achieved.

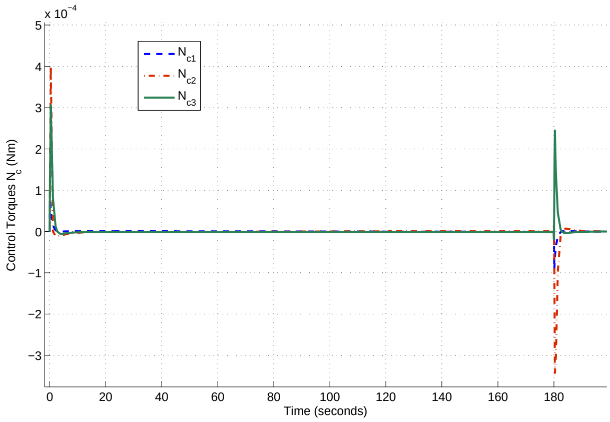

Fig. 3. This figure illustrates how a natural reference track may be tracked in terms of the control torques applied by reaction wheels. The spacecraft considered has the following principal inertias: $\{0.0109, 0.04, 0.0506\}kgm^2$ of a typical nano spacecraft. It is on a nominal circular orbit at an altitude of $600km$, with an inclination of 97.79 degrees. The following disturbances have been considered: air drag, gravity gradient, SRP and residual magnetic dipole. The spacecraft is operating with the following actuators: reaction wheels with $max\ torque = 10^{-3}Nm$, and $rate\ limit = 10^{-2}Nm/s$.

The methodology proposed consists of performing a parametric optimisation of the equations of the analytic solution of Theorem 1 in order to find the initial angular velocities the spacecraft should have in order to reach its target configuration. This may be flown actuating a pulse-like torque at the beginning and at the end of the motion, initially to reach the desired initial angular velocities, then to bring the spacecraft to rest. In theory, this is a two-burst thrust scheme, but in practice an ideal burst is not possible due to the physical constraints on the actuators. However, when the manoeuvre length is large enough relative to the time-scale of the actuation, then the actual thrust scheme is close to a two-burst thrust scheme. See Figure 3 where the torque applied to track the motion are depicted against the maneuver time. Note that the methodology suggested here is not optimal but is meant to be low computational, as the references can be obtained simply by a parametric optimisation of the solution (18). However, because the track generated exploits the free motion of the body, it can be presumed reasonably "close" to the minimum-fuel one. Therefore, a further potential application of the solution derived in this work is that it may be the used as a good initial guess for an optimisation process on-board. Future work will develop specific applications of the equations presented in this paper, providing a more detailed analysis, for example a comparison with a quaternion feedback technique. Moreover, a further issue to tackle for an efficient and feasible on-board use of the closed form solution is the analysis of the methods available to evaluate the elliptic functions and integrals, as, for instance, theta functions or Fourier series.

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