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BLOW-UP BEHAVIOR OF COLLOCATION SOLUTIONS TO HAMMERSTEIN-TYPE VOLTERRA INTEGRAL EQUATIONS*

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Abstract. We analyze the blow-up behavior of one-parameter collocation solutions for Hammerstein-type Volterra integral equations (VIEs) whose solutions may blow up in finite time. To approximate such solutions (and the corresponding blow-up time), we will introduce an adaptive stepsize strategy that guarantees the existence of collocation solutions whose blow-up behavior is the same as the one for the exact solution. Based on the local convergence of the collocation methods for VIEs, we present the convergence analysis for the numerical blow-up time. Numerical experiments illustrate the analysis.

Key words. nonlinear Volterra integral equations, finite-time blow-up, collocation methods, adaptive stepsize, convergence of numerical blow-up time

AMS subject classifications. 65R20, 45G05, 45G10

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1. Introduction.

The mathematical modeling of thermal ignition in solid combustible materials leads typically to nonlinear Volterra integral equations (VIEs) of Hammerstein-type,

$$u(t) = \phi(t) + \int_0^t k(t-s)G(s, u(s))\mathrm{d}s, \quad t \in [0, T], \tag{1.1}$$

where G is a smooth function and where the convolution kernel k may be weakly singular (see, for example, [14] and [16], as well as the references in the survey paper [17]). A particular example that arises as a mathematical model for the formation of shear bands in steel that is subjected to very high strain rates is the VIE

$$u(t) = \gamma \int_0^t (\pi(t-s))^{-1/2}(1+s)^q[u(s)+1]^p \mathrm{d}s$$

(which can be rewritten in the form (1.1); see (6.3)), where $\gamma > 0$ and $p \geqslant 0, q \geqslant 0$ are material parameters related to the constitutive law for plastic straining (cf. [16]).

While the theory of blow-up solutions of (1.1) is now well understood (cf. [4]), the design and analysis of efficient numerical schemes for such problems is not well developed (we are only aware of the paper [5] and the Ph.D. thesis [19]).

Accordingly, this paper is devoted to a systematic study of the numerical solution of nonlinear VIEs (1.1) by a class of (one-parameter) collocation methods. One of our

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key results is that these numerical methods can be used to detect finite-time blow-up (an important aspect since in many practical applications it is not known a priori whether or not the given model VIE will exhibit finite-time blow-up). For the blow-up case, we also pay attention to the convergence of the numerical blow-up time to the exact one in sections 4 and 6.

The VIE (1.1) with unknown solution $u(t)$ will be subject to the following assumptions (see also [4]). The functions $\phi : \mathbb{R} \to \mathbb{R}_+$ and $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ are continuously differentiable, and the kernel $k : (0, \infty) \to (0, \infty)$ is a locally integrable function. Moreover, we assume that the given functions in (1.1) satisfy

(G1) $G(s, 0) \equiv 0$ and $G(s_2, u_2) > G(s_1, u_1)$ for two positive vectors $(s_1, u_1), (s_2, u_2)$ with $(s_2, u_2) \geqslant (s_1, u_1)$ (interpreted componentwise) and $u_2 \neq u_1$,

(G2) $\lim_{u\to\infty} \frac{G(0,u)}{u} = \infty$;

and

(P) the function $\phi(t)$ is positive, nondecreasing,

(K) $k(z) = z^{\beta-1}k_1(z)$, where $\beta > 0$ and $k_1(z) > 0$ is bounded in any finite interval.

It was shown in [4] that the exact solution $u(t)$ of (1.1) is nondecreasing and blows up in finite time if and only if there exists a $t^* > 0$ such that

$$\phi(t^*) + F_{\min}(t^*) > 0, \tag{1.2}$$

$$\int_U^\infty \left(\frac{u}{G(t^*, u)}\right)^{1/\beta} \frac{\mathrm{d}u}{u} < \infty \text{ for all } U > 0, \tag{1.3}$$

where

$$F(t, u) := \int_0^t k(t - s)G(s, u)\mathrm{d}s - u \text{ and } F_{\min}(t) := \min_{u\in[0,\infty)} F(t, u) \leqslant 0. \tag{1.4}$$

Collocation methods for VIEs have been investigated for many years (see [3] and the references therein). In the context of VIEs with blow-up solutions, the authors of [5] apply collocation methods with one parameter and uniform mesh to the simulation of the blow-up time of

$$u(t) = \int_0^t \frac{(u(s) + \gamma)^p}{(t - s)^\alpha}\mathrm{d}s,$$

where $p > 1$, $\gamma > 1$, and $\alpha \in (0, 1)$. As in the case of ordinary differential equations (ODEs) (see [18]), it is also maintained in [5] that "a method with a fixed spacing is not well suited for blow-up problems" and that "it is not yet clear what would be an appropriate strategy for the automatic (or even a priori) computation of a variable stepsize." When the inversion formula exists, switching variables is an approach for simulating blow-up solutions. The advantage is its ability to avoid timestepping past the blow-up time in the solution and to generate timesteps that become sufficiently small near the blow-up time. The disadvantage is that this approach depends strongly on the inversion formula and the monotonicity of solutions (see [14]). Another useful technique is the so-called Sundman transformation, by which a blow-up solution is transferred to a global solution in the new variable (see [12, 18] for ODEs, [13] for PDEs, and [19] for VIEs). Both of these numerical processes employ a certain kind of adaptive stepsize strategy (in [18] it is based on time-continuous rescaling). Other stepsize strategies for computing blow-up solutions of PDEs may be found in [1, 2, 6, 7, 9].

In this paper, we adapt the approach taken in [1] to define an adaptive stepsize strategy for VIEs (1.1) so that the collocation solutions of implicit methods exist uniquely at each time level. A slightly different, but related, strategy is designed for the explicit Euler method. In section 2 the monotonicity and the dynamical behavior of the collocation solutions are discussed, and the comparison principle between collocation solutions with variable $c_1 \in [0,1]$ is investigated. In section 3 we show that the asymptotic behavior of the collocation solutions with adaptive stepsize is the same as for the exact ones, regardless of whether or not the exact solutions blow up in finite time. In section 4 we use the local convergence of collocation methods and the corresponding bounds of the numerical threshold blow-up time to establish the convergence of the numerical blow-up time. A different numerical approach to the computation of blow-up solutions, namely, implicitly linear collocation, is described in section 5. Here, we also discuss its merits when it is applied to VIEs with general Hammerstein kernels. Finally, section 6 contains numerical experiments to illustrate our main results.

2. Collocation methods.

We approximate the exact solution of (1.1) by using collocation in the piecewise constant polynomial space $S_0^{(-1)}(I_h)$, where the underlying (nonuniform) mesh $I_h := \{0 = t_0 < t_1 < t_2 < \cdots\}$, will be defined during the numerical process. The collocation solution $u_h \in S_0^{(-1)}(I_h)$ is defined by the collocation equation

$$u_h(t) = \phi(t) + \Gamma_n(t) + \int_{t_n}^{t} k(t-s)G(s,u_h(s))\mathrm{d}s, \quad t \in X_h, \tag{2.1}$$

where $h_n := t_{n+1}-t_n$ is the stepsize, $X_h := \{t_n+c_1h_n : 0 \leqslant c_1 \leqslant 1, n = 0,1,\ldots,N-1\}$ is the set of collocation points determined by I_h and the collocation parameter $c_1 \in [0,1]$, and

$$\Gamma_n(t) := \int_0^{t_n} k(t-s)G(s,u_h(s))\mathrm{d}s \text{ for } t \in [t_n,t_{n+1}] \tag{2.2}$$

is the approximate history (or lag) term.

2.1. The adaptive stepsize strategy.

In order to describe our choice of adaptive stepsizes we introduce a number of constants that will play a key role. They are

$$C_1(\beta) := \begin{cases} 1, & 0 < \beta \leqslant 1, \\ 1, & 1 < \beta \leqslant 2, \\ 2^{\beta-2}, & \beta > 2, \end{cases} \qquad C_2(\beta) := \begin{cases} 0, & 0 < \beta \leqslant 1, \\ 1, & 1 < \beta \leqslant 2, \\ 2^{\beta-2}, & \beta > 2, \end{cases}$$

and

$$\alpha := \begin{cases} C_1(\beta) + C_2(\beta) + (1 + (C_2(\beta)+1)k_1(0))\tau & \text{when } k_1(z) \text{ is nonincreasing}, \\ \frac{k^*}{k_*}(C_1(\beta)+C_2(\beta)) + (1+(C_2(\beta)+1)k^*)\tau & \text{otherwise}, \end{cases} \tag{2.3}$$

where $k^* := \max_{s\in[0,T]} k_1(s)$ and $k_* := \min_{s\in[0,T]} k_1(s)$.

2.1.1. The implicit Euler method.

Since the collocation $u_h \in S_0^{(-1)}(I_h)$ defined in (2.1) is constant on each subinterval $(t_n,t_{n+1}]$, i.e., $u_h(t) =: u_{n+1},\ t \in (t_n,t_{n+1}]$, the collocation equation (1.1) with collocation parameter $c_1 = 1$ can be written as

$$
\begin{aligned}
u_0 &= \phi(0),\\
u_{n+1} &= \phi(t_{n+1}) + \Gamma_n(t_{n+1}) + h_n \int_0^1 k((1-x)h_n) G(t_n + xh_n, u_{n+1})\mathrm{d}x,\\
\Gamma_n(t_{n+1}) &= \sum_{i=0}^{n-1} h_i \int_0^1 k(t_{n+1} - t_i - xh_i) G(t_i + xh_i, u_{i+1})\mathrm{d}x.
\end{aligned}
\tag{2.4}
$$

In the case of $G(s, u) = g(u)$, (2.4) reduces to

$$
\begin{aligned}
u_0 &= \phi(0),\\
u_{n+1} &= \phi(t_{n+1}) + \Gamma_n(t_{n+1}) + K(h_n)g(u_{n+1}), n = 0, 1, \ldots,\\
\Gamma_n(t_{n+1}) &= \sum_{i=0}^{n-1}(K(t_{n+1} - t_i) - K(t_{n+1} - t_{i+1}))g(u_{i+1}),
\end{aligned}
$$

where $K(t) := \int_0^t k(z)\mathrm{d}z$. Assume that the collocation solution $u_h(t)$ is well defined in the interval $[0, t_n]$. Following the idea in [1] on the unique existence of the collocation solution u_{n+1} of the implicit Euler method, we choose an adaptive stepsize given by

$$
\begin{aligned}
h_n \leqslant h_n^* := \min \Bigg\{ & T - t_n, \tau, \frac{\tau\phi(t_n)}{\Phi(t_n + \tau)}, \left(\frac{\beta\tau\|u_h\|_n}{G(t_n + \tau, \alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}},\\
& \left(\frac{\beta\tau}{k^* L(t_n + \tau, \alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}} \Bigg\},
\end{aligned}
\tag{2.5}
$$

where $\tau \in (0, 1)$ is an arbitrary positive number,

$$
\Phi(t) := \max_{0\leqslant s\leqslant t} \phi'(s), \quad \|u_h\|_n := \max_{t\in[0,t_n]} |u_h(t)| = \max_{0\leqslant i\leqslant n} |u_i|,
$$

and $L(t, M)$ is the local Lipschitz constant of $G(t, u)$ with respect to u in $(s, u) \in [0, t] \times [0, M]$.

Definition 2.1. *A collocation solution $u_h(t)$ for* (2.1) *with adaptive stepsize satisfying* (2.5)

(i) *exists in an interval $[0, T]$, if*

$$
t_N = \sum_{i=0}^{N-1} h_i = T \text{ for some integer } N;
$$

(ii) *exists globally, if it exists in $[0, T]$ for any given $T > 0$;*

(iii) *blows up in finite time, if*

$$
T_b(I_h) = \lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{i=0}^{n} h_i < \infty.
$$

In this case, $T_b(I_h)$ is called the numerical blow-up time.

Lemma 2.2. *Assume that conditions* (P), (K), *and* (G1) *hold and that a collocation solution $u_h(t)$ exists in the interval $t \in [0, T]$. Then $u_h(t) \geqslant \phi(0) > 0$ for all $t \in [0, T]$.*

Lemma 2.3. *Assume that conditions* (P), (K), *and* (G1) *hold and that the collocation solution* $u_h(t)$ *exists in the interval* $[0, t_n]$ *for some* $t_n \in (0, T)$. *Then for* $t_n \leqslant t < t_{n+1}$,

$$\Gamma_n(t) \leqslant \frac{k^*}{k_*}(C_1(\beta) + C_2(\beta)) \int_0^{t_n} k(t_n - s)G(s, u_h(s))\mathrm{d}s + C_2(\beta)k^* \frac{1}{\beta} h_n^\beta G(t_n, \|u_h\|_n).$$

If, in addition, $k_1(z)$ *is nonincreasing, then*

$$\Gamma_n(t) \leqslant (C_1(\beta) + C_2(\beta)) \int_0^{t_n} k(t_n - s)G(s, u_h(s))\mathrm{d}s + C_2(\beta)k_1(0) \frac{1}{\beta} h_n^\beta G(t_n, \|u_h\|_n).$$

Proof. It follow from [11] that for $t \in (t_n, t_{n+1}]$ and $s \in [0, t_n]$,

$$(t - s)^{\beta-1} \leqslant C_1(\beta)(t_n - s)^{\beta-1} + C_2(\beta)(t - t_n)^{\beta-1},$$

which together with conditions (P) and (G1) implies that

$$\begin{aligned} \Gamma_n(t) &\leqslant C_1(\beta) \int_0^{t_n} (t_n - s)^{\beta-1} k_1(t - s)G(s, u_h(s))\mathrm{d}s \\ &\quad + C_2(\beta)h_n^{\beta-1} \int_0^{t_n} k_1(t - s)G(s, u_h(s))\mathrm{d}s \\ &\leqslant (C_1(\beta) + C_2(\beta)) \int_0^{t_n} (t_n - s)^{\beta-1} k_1(t - s)G(s, u_h(s))\mathrm{d}s \\ &\quad + C_2(\beta)\frac{1}{\beta} h_n^\beta k^* G(t_n, \|u_h\|_n). \end{aligned}$$

In view of

$$k_1(t - s) \leqslant \begin{cases} k_1(t_n - s) & \text{when } k_1(z) \text{ is nonincreasing,} \\ \frac{k^*}{k_*} k_1(t_n - s) & \text{otherwise,} \end{cases}$$

the proof is complete. ∎

Theorem 2.4. *Let* $\tau \in (0, 1)$ *and conditions* (P), (K), *and* (G1) *hold. Suppose that the collocation solution* $u_h(t)$ *of the implicit Euler method exists in the interval* $[0, t_n]$ *for some* $t_n \in (0, T)$ *and that the stepsize satisfies* (2.5). *Then the collocation solution* u_{n+1} *is uniquely defined by the fixed point of* $H(v, u_h, h_n)$ *in* $[0, \alpha\|u_h\|_n]$, *where*

$$H(v, u_h, \eta) := \phi(t_n + \eta) + \Gamma_n(t_n + \eta) + \int_{t_n}^{t_n+\eta} k(t_n + \eta - s)G(s, v)\mathrm{d}s.$$

Proof. It follows from (2.5) and Lemma 2.3 that

(i) $\phi(t_n + h_n) \leqslant (1 + \tau)\phi(t_n) \leqslant \phi(t_n) + \tau\|u_h\|_n$,

(ii) $\frac{1}{\beta} h_n^\beta G(t_n + h_n, \alpha\|u_h\|_n) \leqslant \tau\|u_h\|_n$,

(iii) $H(v, u_h, h_n) \leqslant \alpha\|u_h\|_n$ for all $v \in [0, \alpha\|u_h\|_n]$.

Hence $H(v, u_h, h_n)$ is a mapping from $[0, \alpha\|u_h\|_n]$ to $[0, \alpha\|u_h\|_n]$. Moreover, for all $v_1, v_2 \in [0, \alpha\|u_h\|_n]$,

$$\begin{aligned} |H(v_1, u_h, h_n) - H(v_2, u_h, h_n)| &= \left| \int_{t_n}^{t_n+h_n} k(t_n + h_n - s)(G(s, v_1) - G(s, v_2))\mathrm{d}s \right| \\ &\leqslant \frac{1}{\beta} h_n^\beta k^* L(t_n + \tau, \alpha\|u_h\|_n)|v_1 - v_2| \leqslant \tau|v_1 - v_2|. \end{aligned}$$

Therefore $H(v, u_h, h_n)$ has a unique fixed point in $[0, \alpha\|u_h\|_n]$. The proof is complete. ∎

Remark 2.5. Let $G(t, u) = u^p$ and $k(z) = z^{\beta-1}$. Then

$$\frac{\beta\tau\|u_h\|_n}{G(t_n+\tau, \alpha\|u_h\|_n)} \leqslant \frac{\beta\tau}{\alpha^p\|u_h\|_n^{p-1}} \leqslant \frac{\beta\tau}{2\alpha^{p-1}\|u_h\|_n^{p-1}},$$
$$\frac{\beta\tau}{L(t_n+\tau, \alpha\|u_h\|_n)} \geqslant \frac{\beta\tau}{p(\alpha\|u_h\|_n)^{p-1}}.$$

Therefore,

$$\left(\frac{\beta\tau\|u_h\|_n}{G(t_n+\tau, \alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}} \leqslant \left(\frac{\beta\tau}{L(t_n+\tau, \alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}}$$

when $1 < p \leqslant 2$ and $\beta > 0$.

Remark 2.6. Assume that the adaptive stepsize is defined by $h_n = h_n^*$ and that the collocation solution exists in an interval $[0, T]$. Then

(i) $\alpha < \infty$ is a finite number,
(ii) $h^0 := \max_{0\leqslant n\leqslant N-1} h_n \leqslant \tau$,
(iii) $h^1 := \min_{0\leqslant n\leqslant N-1} \frac{\tau\phi(t_n)}{\Phi(t_n+\tau)} \geqslant \frac{\tau\phi(0)}{\Phi(T)} > 0$,
(iv) $h^2 := \min_{0\leqslant n\leqslant N-1} \left(\frac{\beta\tau\|u_h\|_n}{G(t_n+\tau,\alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}} \geqslant \left(\frac{\beta\tau\phi(0)}{G(t_n+\tau,\alpha\|u_h\|_N)}\right)^{\frac{1}{\beta}} > 0$,
(v) $h^3 := \min_{0\leqslant n\leqslant N-1} \left(\frac{\beta\tau}{k^* L(t_n+\tau,\alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}} \geqslant \left(\frac{\beta\tau}{k^* L(t_n+\tau,\alpha\|u_h\|_N)}\right)^{\frac{1}{\beta}} > 0$.

Hence $\min_{0\leqslant n\leqslant N-1} h_n \geqslant \min\{\tau, h^1, h^2, h^3\} > 0$. Thus, under conditions (K) and (G1), a blow-up collocation solution $u_h(t)$ must satisfy

$$\lim_{n\to\infty} \|u_h\|_n = \infty,$$

which together with condition (G2) implies that for sufficiently large n,

$$h_n = \left(\frac{\beta\tau\|u_h\|_n}{G(t_n+\tau, \alpha\|u_h\|_n)}\right)^{\frac{1}{\beta}} \to 0 \text{ as } n \to \infty.$$

2.1.2. The explicit Euler method. The explicit Euler method for (1.1) corresponds to $c_1 = 0$. Thus, for $t \in [t_n, t_{n+1})$ the collocation approximation $u_h(t) = u_n$ is defined by

$$\begin{aligned} u_0 &= \phi(0),\\ u_{n+1} &= \phi(t_{n+1}) + \Gamma_n^E(t_{n+1}) + h_n \int_0^1 k((1-x)h_n)G(t_n + xh_n, u_n)\mathrm{d}x,\\ \Gamma_n^E(t_{n+1}) &= \sum_{i=0}^{n-1} h_i \int_0^1 k(t_{n+1} - t_i - xh_i)G(t_i + xh_i, u_i)\mathrm{d}x. \end{aligned} \tag{2.6}$$

If $G(s, u) = g(u)$ then (2.6) reduces to

$$\begin{aligned} u_0 &= \phi(0),\\ u_{n+1} &= \phi(t_{n+1}) + \Gamma_n^E(t_{n+1}) + K(t_{n+1} - t_n)g(u_n), n = 0, 1, \ldots,\\ \Gamma_n^E(t_{n+1}) &= \sum_{i=0}^{n-1}(K(t_{n+1} - t_i) - K(t_{n+1} - t_{i+1}))g(u_i). \end{aligned}$$

In order to simulate a blow-up solution, we use again an adaptive stepsize, namely,

$$h_n \leqslant h_n^* := \min\left\{T - t_n, \tau, \frac{\tau\phi(t_n)}{\Phi(t_n+\tau)}, \left(\frac{\beta\tau\|u_h\|_n}{G(t_n+\tau, \|u_h\|_n)}\right)^{\frac{1}{\beta}}\right\}. \tag{2.7}$$

By Lemma 2.3 we obtain the following estimate.

Theorem 2.7. *Assume that conditions* (P), (K), *and* (G1) *hold. Then the collocation solution given by the explicit Euler method satisfies*

$$u_{n+1} \leqslant \alpha\|u_h\|_n \textit{ for all } n < N,$$

where $\alpha \geqslant 1+\tau$ *is defined by* (2.3) *whenever the adaptive stepsize is such that* (2.7) *is satisfied.*

Remark 2.8. The solution of the ODE of order β,

$$u^{(\beta)}(t) = G(t, u(t)),$$
$$u^{(i)}(0) = u_i > 0, i = 0, 1, \dots, \beta-1,$$

satisfies

$$u(t) = \phi(t) + \int_0^t k(t-s)G(s, u(s))\mathrm{d}s,$$

where $\beta \geqslant 1$ is an integer, $\phi(t) = \sum_{i=0}^{\beta-1}\frac{1}{i!}u_i t^i$, and $k(z) = \frac{1}{(\beta-1)!}z^{\beta-1}$. In view of $\alpha = C_1(\beta)+C_2(\beta)+(1+\frac{C_2(\beta)+1}{(\beta-1)!})\tau$ and $\frac{\tau\phi(t)}{\Phi(t+\tau)} \geqslant 1$ for sufficiently large t, one obtains that for $1 \ll t_n < T$,

$$h_n^* = \min\left\{\tau, \left(\frac{\beta\tau\|u_h\|_n}{G(t_n+\tau, \|u_h\|_n)}\right)^{\frac{1}{\beta}}\right\}.$$

For the first-order ODE with a power function $G(t,u) = u^p$, (2.7) reduces to

$$h_n^* = \min\left\{\tau, \frac{\tau}{\|u_h\|_n^{p-1}}\right\},$$

which was used in [15].

2.2. Monotonicity of collocation solutions. We will now show that the collocation solution $u_h(t)$ with adaptive stepsize is nondecreasing whenever it exists.

Theorem 2.9. *Assume that conditions* (P), (K), *and* (G1) *hold. Then the collocation solution* $u_h(t)$ *of the explicit Euler method with adaptive stepsize such that* (2.7) *holds is nondecreasing.*

Proof. For $t \in [0, t_1)$, $u_h(t) = u_0 = \phi(0)$,

$$u_1 = \phi(t_1) + \int_0^{t_1} k(t_1 - s)G(s, u_0)\mathrm{d}s \geqslant u_0.$$

Using induction, suppose that $u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n$ for $n < N$. Then $u_h(t)$ is nondecreasing for $t \in [0, t_{n+1})$ and

$$\begin{aligned} u_{n+1} &= \phi(t_{n+1}) + \int_0^{t_{n+1}} k(t_{n+1} - s)G(s, u_h(s))\mathrm{d}s \\ &\geqslant \phi(t_n) + \int_0^{t_n} k(s)G(t_n - s, u_h(t_n - s))\mathrm{d}s = u_n. \end{aligned}$$

Hence $u_n \leqslant u_{n+1}$ and the proof is complete. ∎

THEOREM 2.10. *Assume that conditions* (P), (K), *and* (G1) *hold. Then the collocation solution* $u_h(t)$ *of the implicit Euler method with adaptive stepsize given by* (2.5) *is nondecreasing.*

Proof. For $t \in (0, t_1]$,

$$u_h(t) \equiv u_1 = \phi(h_0) + \int_0^{h_0} k(h_0 - s)G(s, u_1)\mathrm{d}s.$$

Hence $u_1 \geqslant u_0$. Suppose then that $u_h(t)$ is nondecreasing for $t \in [0, t_n]$. Define

$$v^0 := u_n, \quad v^{l+1} := H(v^l, u_h, h_n) \ \ (l = 0, 1, 2, \dots, \ ), \quad v_h^l(t) := \begin{cases} u_h(t), & t \in [0, t_n], \\ v^l, & t \in (t_n, t_{n+1}]. \end{cases}$$

Then $v_h^0(t)$ is nondecreasing and

$$\begin{aligned} v^1 &= \phi(t_{n+1}) + \int_0^{t_{n+1}} k(t_{n+1} - s)G(s, v_h^0(s))\mathrm{d}s \\ &\geqslant \phi(t_n) + \int_0^{t_n} k(s)G(t_n - s, v_h^0(t_n - s))\mathrm{d}s \geqslant u_n. \end{aligned}$$

Suppose that $v^l \geqslant u_n$ for some $l \geqslant 1$. Then $v_h^l(t)$ is nondecreasing and

$$\begin{aligned} v^{l+1} &= \phi(t_{n+1}) + \int_0^{t_{n+1}} k(t_{n+1} - s)G(s, v_h^l(s))\mathrm{d}s \\ &\geqslant \phi(t_n) + \int_0^{t_n} k(s)G(t_n - s, v_h^l(t_n - s))\mathrm{d}s \geqslant u_n. \end{aligned}$$

It follows from (2.5) and Theorem 2.4 that $H(v, u_h, h_n)$ is a contractive mapping and u_{n+1} is its fixed point. Hence $u_{n+1} = \lim_{l\to\infty} v^l \geqslant u_n$ and the proof is complete. ∎

Remark 2.11. Theorems 2.9 and 2.10 imply that $\|u_h\|_n = u_n$. Hence (2.5) and (2.7) reduce, respectively, to

(2.8)

$$h_n \leqslant h_n^* = \min\left\{T - t_n, \tau, \frac{\tau\phi(t_n)}{\Phi(t_n + \tau)}, \left(\frac{\beta\tau u_n}{G(t_n + \tau, \alpha u_n)}\right)^{\frac{1}{\beta}}, \left(\frac{\beta\tau}{k^* L(t_n + \tau, \alpha u_n)}\right)^{\frac{1}{\beta}}\right\},$$

(2.9)

$$h_n \leqslant h_n^* = \min\left\{T - t_n, \tau, \frac{\tau\phi(t_n)}{\Phi(t_n + \tau)}, \left(\frac{\beta\tau u_n}{G(t_n + \tau, u_n)}\right)^{\frac{1}{\beta}}\right\}.$$

In the remainder of this paper, we always assume that the adaptive stepsize satisfies (2.8) for implicit methods and (2.9) for the explicit Euler method.

2.3. Comparison principle.

THEOREM 2.12. *Assume that conditions* (P), (K), *and* (G1) *hold. Then the collocation solution* $u_h(t)$ *corresponding to the explicit Euler method satisfies* $u_h(t) \leqslant u(t)$ *whenever* $u(t)$ *exists.*

Proof. Since

$$u_h(t) \leqslant \phi(t) + \int_0^t k(t - s)G(s, u_h(s))\mathrm{d}s,$$

one obtains from Lemma 2.4 in [4] that $u_h(t) \leqslant u(t)$ whenever $u(t)$ exists. This verifies our assertion. ∎

THEOREM 2.13. *Assume that conditions* (P), (K)*, and* (G1) *hold. Then the collocation solution $u_h(t)$ given by the implicit Euler method satisfies $u_h(t) \geqslant u(t)$ whenever $u_h(t)$ exists.*

Proof. In view of $u_1 > \phi(0) = u(0)$, we suppose that $u(t) = u_h(t)$ and $u(s) < u_h(s)$ for $s \in [0, t)$. This implies that

$$0 < u(t) - u_h(t) \leqslant \int_0^t k(t-s)(G(s, u(s)) - G(s, u_h(s)))\mathrm{d}s < 0.$$

This contradicts our assumption, and thus the proof is complete. ∎

THEOREM 2.14. *Assume that conditions* (P), (K)*, and* (G1) *hold and that the collocation solution $u_h^1(t)$ of the implicit Euler method exists on the mesh I_h. Then for any $c_1 \in [0,1]$, the collocation solution $u_h^{c_1}(t)$ also exists on the same mesh I_h and $u_h^{c_1}(t)$ is increasing with respect to $c_1 \in [0,1]$.*

Proof. Since the collocation solution $u_h^{c_1}(t)$ with $c_1 \in (0,1)$ can be regarded as the component of the implicit Euler method applied on the subinterval $\cup_{i=0}^n [t_i, t_i + c_1 h_i]$ and the explicit Euler method applied on the subinterval $\cup_{i=0}^n [t_i + c_1 h_i, t_{i+1}]$, by resorting to Lemmas 2.15 and 2.16, the collocation solutions $u_h^{c_1}(t)$ exist on the same mesh I_h and $u_h^0(t) \leqslant u_h^{c_1}(t) \leqslant u_h^1(t)$.

Assume that $u_h^{c_1}(t)$ and $u_h^{\bar{c}_1}(t)$ are the collocation solutions corresponding to $0 < c_1 < \bar{c}_1 < 1$, respectively. Then

$$\begin{aligned} u_h^{\bar{c}_1}(t) &\geqslant v_h^1(t) \text{ by Lemma 2.15,} \\ v_h^1(t) &\geqslant v_h^2(t) \text{ by Lemma 2.16,} \\ v_h^2(t) &\geqslant u_h^{c_1}(t) \text{ by Lemma 2.17.} \end{aligned}$$

Here, $v_h^1(t)$ is the collocation solution corresponding to the implicit Euler method applied on the subinterval $\cup_{i=0}^n [t_i, t_i + c_1 h_i]$ and $\cup_{i=0}^n [t_i + c_1 h_i, t_i + \bar{c}_1 h_i]$, and the explicit Euler method applied on the subinterval $\cup_{i=0}^n [t_i + \bar{c}_1 h_i, t_{i+1}]$, while $v_h^2(t)$ is the collocation solution given by the implicit Euler method applied on the subinterval $\cup_{i=0}^n [t_i, t_i + c_1 h_i]$, and the explicit Euler method applied on the subinterval $\cup_{i=0}^n [t_i + c_1 h_i, t_i + \bar{c}_1 h_i]$ and $\cup_{i=0}^n [t_i + \bar{c}_1 h_i, t_{i+1}]$. Hence the proof is complete. ∎

LEMMA 2.15. *Assume that $c_1 = 1$, conditions* (P), (K)*, and* (G1) *hold, and the collocation solution $u_h(t)$ exists on the mesh I_h. Then the collocation solution also exists and is decreased by adding a finite number of new grid points in the mesh I_h, that is to say, $\bar{u}_h(t) \leqslant u_h(t)$ if $I_h \subseteq \bar{I}_h \subseteq [0, T]$.*

Proof. Without loss of generality, assume that only one new mesh point $\bar{t}$ is added in the subinterval $[t_n, t_{n+1}]$. Then by Theorem 2.10, the collocation solutions $u_h(t)$ and $\bar{u}_h(t)$ are increasing for all $t \in [0, T]$ and $\bar{u}_h(t) = u_h(t)$ for $t \in [0, t_n]$. Define

$$v_0 = \bar{u}_h(t_n) \text{ and } v_l = H(v_{l-1}, \bar{u}_h, \bar{t} - t_n) \text{ for all } l = 0, 1, 2, \ldots.$$

Then, similarly to the proof of Theorem 2.10, it can be shown that v_l is an increasing sequence and bounded by $u_h(t_{n+1})$. Hence $\bar{u}_h(t)$ exists on the interval $[0, \bar{t}]$ and $\bar{u}_h(\bar{t}) \leqslant u_h(t_{n+1})$. As a consequence, the proof can be completed by an induction argument. ∎

LEMMA 2.16. *Assume that conditions* (P), (K)*, and* (G1) *hold and the collocation solution $u_h(t)$ of the implicit Euler method exists on the mesh I_h. Then the collocation solution $\bar{u}_h(t)$ corresponding to applying the explicit Euler method on some subintervals of the mesh I_h also exists and $\bar{u}_h(t) \leqslant u_h(t)$ for all $t \in [0, T]$.*

Proof. Without loss of generality, assume that the explicit Euler method is applied in only one subinterval $[t_n, t_{n+1}]$. Then $\bar{u}_h(t) = u_h(t) \leqslant u_h(t_{n+1})$ for $t \in [0, t_n]$. Hence

$$\begin{aligned}\bar{u}_h(t_{n+1}) &\leqslant \phi(t_{n+1}) + \int_0^{t_n} k(t_{n+1}-s)G(s,\bar{u}_h(s))\mathrm{d}s \\ &\quad + \int_{t_n}^{t_{n+1}} k(t_{n+1}-s)G(s,u_h(t_{n+1}))\mathrm{d}s \\ &= u_h(t_{n+1}).\end{aligned}$$

Then, similarly to the proof of Lemma 2.15, the proof is completed. ∎

LEMMA 2.17. *Assume that $c_1 = 0$, conditions* (P), (K), *and* (G1) *hold, and the collocation solution $u_h(t)$ exists on the mesh I_h. Then the collocation solution will be increased by adding a finite number of new grid points in the mesh I_h, that is to say, $\bar{u}_h(t) \geqslant u_h(t)$ if $I_h \subseteq \bar{I}_h \subseteq [0,T]$.*

Proof. Without loss of generality, assume that only one new mesh point $\bar{t}$ is added in the subinterval $[t_n, t_{n+1}]$. Then by Theorem 2.9, the collocation solutions $u_h(t)$ and $\bar{u}_h(t)$ are increasing for all $t \in [0,T]$ and $\bar{u}_h(t) \geqslant u_h(t)$ for $t \in [0, t_{n+1})$, which implies that

$$\bar{u}_h(t_{n+1}) \geqslant \phi(t_{n+1}) + \int_0^{t_{n+1}} k(s)G(t_{n+1}-s, u_h(t_{n+1}-s))\mathrm{d}s = u_h(t_{n+1}).$$

Hence by an induction argument, the proof is complete. ∎

2.4. Dynamical behavior.

THEOREM 2.18. *Assume that conditions* (P), (K), (G1), *and* (G2) *hold and that the collocation solution $u_h(t)$ of the explicit Euler method exists globally. Then $\lim_{t\to\infty} u_h(t) = \infty$, provided that there exists a $t^* \in (0,\infty)$ such that* (1.2) *holds.*

Proof. Suppose that $u_h(t)$ is bounded for all $t \in [0,\infty)$. Then by Theorem 2.12, $\lim_{t\to\infty} u_h(t) = u_\infty \in (\phi(t^*),\infty)$ exists. Thus for any given $0 < \epsilon < u_\infty$ with $\phi(t^*) - \epsilon + F_{\min}(t^*) > 0$, there exists a $T_\epsilon > t^*$ such that $u_\infty - \epsilon < u_n \leqslant u_\infty$ for all $t_n > T_\epsilon$. It therefore follows from conditions (K) and (G1) that for all $t_n > T_\epsilon + t^*$,

$$\begin{aligned}u_\infty > u_n &\geqslant \phi(t_n) + \int_0^{T_\epsilon} k(t_n-s)G(s,u_h(s))\mathrm{d}s + \int_{T_\epsilon}^{t_n} k(t_n-s)G(s,u_\infty-\epsilon)\mathrm{d}s \\ &\geqslant \phi(t_n) + \int_0^{t_n-T_\epsilon} k(s)G(t_n-s,u_\infty-\epsilon)\mathrm{d}s \\ &\geqslant \phi(t_n) + \int_0^{t_n-T_\epsilon} k(s)G(t_n-T_\epsilon-s,u_\infty-\epsilon)\mathrm{d}s.\end{aligned}$$

This implies that

$$\phi(t_n) - \epsilon + F(t_n - T_\epsilon, u_\infty - \epsilon) < 0,$$

which contradicts the hypothesis that $\phi(t^*) - \epsilon + F_{\min}(t^*) > 0$. The proof is complete. ∎

THEOREM 2.19. *Let $\tau \in (0,1)$ and conditions* (P), (K), (G1), *and* (G2) *hold. Suppose that $\phi(t) + F_{\min}(t) \leqslant 0$ for all $t \in [0,\infty)$. Then*

(i) *the analytic solution $u(t)$ exists globally;*

(ii) *the collocation solution $u_h(t)$ of the implicit Euler method with adaptive stepsize given by* (2.8) *exists globally.*

Proof. In view of Theorems 2.4 and 2.13, we only need to show that $u_n \leqslant u_F(t_n)$ for all $n \geqslant 0$, where $u_F(t) := \inf\{U : F(t,u) > F_{\min}(t) \text{ for } u \in [U,\infty)\}$ is defined in [4]. Suppose that $u_{n+1} > u_F(t_{n+1})$ and $u_\iota \leqslant u_F(t_{n+1})$ for $\iota = 0, 1, \dots, L \leqslant n$. Consider the sequence defined by

$$v_0 := u_L, \qquad v_{l+1} := H(v_l, u_h, t_{n+1} - t_L) \ \ (l = 0, 1, 2, \dots).$$

Then

$$v_1 \leqslant \phi(t_{n+1}) + \int_0^{t_L} k(t_{n+1} - s)G(s, u_h(s))\mathrm{d}s + \int_{t_L}^{t_{n+1}} k(t_{n+1} - s)G(s, u_F(t_{n+1}))\mathrm{d}s$$
$$< u_F(t_{n+1}).$$

Assuming that $v_l < u_F(t_{n+1})$ for some $l \geqslant 1$, we find that

$$v_{1+1} \leqslant \phi(t_{n+1}) + \int_0^{t_L} k(t_{n+1} - s)G(s, u_h(s))\mathrm{d}s + \int_{t_L}^{t_{n+1}} k(t_{n+1} - s)G(s, u_F(t_{n+1}))\mathrm{d}s$$
$$< u_F(t_{n+1}).$$

Therefore v_l is an increasing sequence which is bounded by $u_F(t_{n+1})$. The limiting value v_∞ of the sequence v_l satisfies

$$v_\infty = \phi(t_{n+1}) + \int_0^{t_L} k(t_{n+1} - s)G(s, u_h(s))\mathrm{d}s + \int_{t_L}^{t_{n+1}} k(t_{n+1} - s)G(s, v_\infty)\mathrm{d}s,$$

which is the collocation solution at grid point t_{n+1} with stepsize $t_{n+1} - t_L$. This contradicts the result of Lemma 2.15, and hence the proof is complete. ∎

3. Blow-up conditions for collocation solutions.

3.1. Blow-up behavior of the explicit Euler method.

LEMMA 3.1. *Let $c_1 = 0$ and $k(z) = z^{\beta-1}$, $\beta > 0$. Assume that conditions* (P), (G1)*, and* (G2) *hold and that the collocation solution $u_h(t)$ with adaptive stepsize such that* (2.9) *holds exists globally. If there is a t^* such that* (1.2) *holds, then for any given $R > 1$, there exists a sequence t_{n_i} such that $u_h(t_{n_i}) \in [\alpha^{-1}R^i, R^i)$ for all $i \geqslant \max\{1, \frac{\log(\phi(0))}{\log R}\}$, $\lim_{i\to\infty} t_{n_i} = \infty$, and $t_{n_{i+1}} - t_{n_i}$ tends to zero as $i \to \infty$.*

Proof. In fact, it follows from Theorems 2.7, 2.9, and 2.18 that there exists a sequence t_{n_i} such that $u_h(t_{n_i}) \in [\alpha^{-1}R^i, R^i)$ and $\lim_{n\to\infty} t_{n_i} = \infty$. Then it follows from conditions (P) and (G1) that

$$u_{n_{i+1}} = \phi(t_{n_{i+1}}) + \int_0^{t_{n_i}} k(t_{n_{i+1}} - s)G(s, u_h(s))\mathrm{d}s + \int_{t_{n_i}}^{t_{n_{i+1}}} k(t_{n_{i+1}} - s)G(s, u_h(s))\mathrm{d}s$$
$$\geqslant G(0, u_{n_i})K(t_{n_{i+1}} - t_{n_i}) \geqslant G(0, u_{n_i})K(t_{n_{i+1}} - t_{n_i}),$$

which together with Condition (G2) implies that $t_{n_{i+1}} - t_{n_i} \to 0$ as $i \to \infty$. Hence the proof is complete. ∎

LEMMA 3.2. *Assume that $c_1 = 0$ and conditions* (P), (G1)*, and* (G2) *hold. If*

(i) *$k(z) = z^{\beta-1}$ with $\beta > 0$,*

(ii) *there exists a $t^* > 0$ such that* (1.2) *and* (1.3) *hold,*

then the collocation solution $u_h(t)$ with adaptive stepsize given by (2.9) *blows up in finite time for any $\tau \in (0, 1)$.*

Proof. Suppose, otherwise, there exists a sequence t_{n_i} such that $u_h(t_{n_i}) \in [\alpha^{-1}r^{i\beta+1}, r^{i\beta+1})$ for $r > \alpha$ and t_{n_i} tends to ∞ as $i \to \infty$. Therefore $H_i := t_{n_{i+1}} - t_{n_i}$ tends to zero as $i \to \infty$ and $t_{n_i} \geqslant t^*$ for sufficiently large i, which together with conditions (P) and (G1) implies that $u_{n_{i+1}} \geqslant G(t^*, u_{n_i})K(H_i i)$. Thus

$$\frac{1}{\beta}H_i^\beta \leqslant \frac{r^{(i+1)\beta+1}}{G(t^*, r^{i\beta})},$$

leading to

$$H_i \leqslant (\beta)^{1/\beta} \frac{r^i - r^{i-1}}{(G(t^*, r^{i\beta}))^{1/\beta}} \frac{r^{1+1/\beta}}{1-r^{-1}} \leqslant \frac{r^{2+1/\beta}}{(r-1)\beta} (\beta)^{1/\beta} \int_{r^{(i-1)\beta}}^{r^{i\beta}} \left(\frac{u}{G(t^*, u)}\right)^{1/\beta} \frac{\mathrm{d}u}{u}.$$

As a result, (1.3) implies that for all $i > 0$,

$$t_{n_i} \leqslant \gamma_0 + \sum_{j=0}^{i-1} H_j \leqslant \gamma_0 + \sum_{j=0}^{\infty} H_j < \infty.$$

This is a contradiction and the proof is complete. ∎

THEOREM 3.3. *Assume that* $c_1 = 0$ *and conditions* (P), (K), (G1), *and* (G2) *hold. If there exists a* $t^* > 0$ *such that* (1.2) *and* (1.3) *hold, then*

(i) *the analytic solution* $u(t)$ *blows up in finite time;*

(ii) *the collocation solution* $u_h(t)$ *with adaptive stepsize such that* (2.9) *blows up in finite time for any* $\tau \in (0, 1)$.

3.2. Blow-up behavior of the implicit Euler method.

LEMMA 3.4. *Assume that* $c_1 = 1$ *and conditions* (P), (G1) *and* (G2) *hold, and that there exists a* $t^* \in (0, \infty)$ *such that* (1.2) *holds. If*

(i) $k(z) = z^{\beta-1}$, $\beta > 0$,

(ii) *there exists a* $U > 0$ *such that*

$$\int_U^\infty \left(\frac{u}{G(t,u)}\right)^{1/\beta} \frac{\mathrm{d}u}{u} = \infty \text{ for all } t \geqslant 0, \tag{3.1}$$

then the collocation solution $u_h(t)$ *with adaptive stepsize given by* (2.8) *does not blow up in finite time.*

Proof. Suppose, otherwise, it follows from Theorem 2.4 that there exists a sequence t_{n_i} such that $u_h(t_{n_i}) \in [\alpha^{-1}r^{i\beta}, r^{i\beta}]$ for some $r > \max\{\alpha, r_0\}$, where

$$r_0 := \begin{cases} (2(1+\phi(T_b(\tau))))^{1/\beta}, & 0 < \beta < 1, \\ (2(2^\beta + \phi(T_b(\tau))))^{1/\beta}, & \beta \geqslant 1. \end{cases}$$

Hence, $H_i = t_{n_{i+1}} - t_{n_i} \to 0$ as $i \to \infty$, and there exists an $N > 0$ such that $H_i < \min\{1, t_{n_i}\}$ for all $i \geqslant N$. We claim that for $i \geqslant N$,

$$r^{(i+1)\beta-1} \leqslant \left(2^\beta + \frac{2}{\beta}\right) H_i^\beta G(T_b(\tau), r^{(i+1)\beta}). \tag{3.2}$$

Let $\beta \geqslant 1$. Then condition (G1) implies that for $n \geqslant N$,

$$\begin{aligned} u_h(t_{n_{i+1}}) &\leqslant \phi(t_{n_{i+1}}) + \int_0^{t_{n_i}} k(t_{n_{i+1}} - s)G(s, u_h(s))\mathrm{d}s \\ &\quad + \int_{t_{n_i}}^{t_{n_{i+1}}} k(t_{n_{i+1}} - s)G(s, u_h(s))\mathrm{d}s \\ &\leqslant \phi(t_{n_{i+1}}) + 2^{\beta-1} \int_0^{t_{n_i}} k(t_{n_i} - s)G(s, u_h(s))\mathrm{d}s \\ &\quad + 2^{\beta-1} H_i^{\beta-1} \int_0^{t_{n_i}} G(s, u_h(s))\mathrm{d}s + K(H_i)G(T_b(\tau), u_h(t_{n_i+1})) \\ &\leqslant \phi(T_b(\tau)) + 2^{\beta} u_h(t_{n_i}) + 2^{\beta-1} H_i^{\beta-1} \int_0^{H_i} G(s, u_h(s))\mathrm{d}s \\ &\quad + K(H_i)G(T_b(\tau), u_h(t_{n_i+1})) \\ &\leqslant \frac{1}{2} u_h(t_{n_{i+1}}) + \left(2^{\beta-1} + \frac{1}{\beta}\right) H_i^{\beta} G(T_b(\tau), u_h(t_{n_i+1})). \end{aligned}$$

This yields (3.2).

If $0 < \beta < 1$, then condition (G1) implies that for $n \geqslant N$,

$$\begin{aligned} u_h(t_{n_{i+1}}) &\leqslant \phi(t_{n_{i+1}}) + \int_0^{t_{n_i}} k(t_{n_{i+1}} - s)G(s, u_h(s))\mathrm{d}s \\ &\quad + \int_{t_{n_i}}^{t_{n_{i+1}}} k(t_{n_{i+1}} - s)G(s, u_h(s))\mathrm{d}s \\ &\leqslant \phi(t_{n_{i+1}}) + \int_0^{t_{n_i}} k(t_{n_i} - s)G(s, u_h(s))\mathrm{d}s + K(H_i)G(T_b(\tau), u_h(t_{n_i+1})) \\ &\leqslant \phi(T_b(\tau)) + u_h(t_{n_i}) + K(H_i)G(T_b(\tau), u_h(t_{n_i+1})) \\ &\leqslant \frac{1}{2} u_h(t_{n_{i+1}}) + \left(2^{\beta-1} + \frac{1}{\beta}\right) H_i^{\beta} G(T_b(\tau), u_h(t_{n_i+1})), \end{aligned}$$

which also yields (3.2).

Therefore, one obtains that

$$\begin{aligned} H_i &\geqslant (C(\beta))^{-1/\beta} \frac{r^{i+1}}{(G(T_b(\tau), r^{(i+1)\beta}))^{1/\beta}} \\ &\geqslant \frac{1}{r-1} (C(\beta))^{-1/\beta} \int_{r^{i+1}}^{r^{i+2}} \frac{1}{(G(T_b(\tau), s^{\beta}))^{1/\beta}} \mathrm{d}s, \end{aligned}$$

where $C(\beta) := 2^{\beta} r + \frac{2r}{\beta}$. Combining this with (3.1) we find

$$\lim_{n\to\infty} t_n = t_{n_0} + \lim_{i\to\infty} \sum_{i=1}^{n} H_i = \infty.$$

This is a contradiction and the proof is complete. ∎

THEOREM 3.5. *Assume that* $c_1 = 1$ *and conditions* (P), (K), (G1), *and* (G2) *hold. If there exists a* $t^* > 0$ *such that* (1.2) *and* (1.3) *hold, then*

(i) *the analytic solution* $u(t)$ *blows up in finite time;*

(ii) *the collocation solution* $u_h(t)$ *with adaptive stepsize such that* (2.8) *holds blows up in finite time for any* $\tau \in (0, 1)$.

THEOREM 3.6. *Assume that* $c_1 = 1$ *and conditions* (P), (K), (G1), *and* (G2) *hold. If there exists a* $U > 0$ *such that* (3.1) *holds, then*

(i) *the analytic solution* $u(t)$ *does not blow up in finite time;*

(ii) *the collocation solution* $u_h(t)$ *with adaptive stepsize given by* (2.8) *does not blow up in finite time for any* $\tau \in (0, 1)$.

Proof. Suppose that the collocation solution $u_h(t)$ blows up at a finite time $T_b(\tau)$. Then the collocation solution $\bar{u}_h(t)$ of

$$\bar{u}(t) = \phi(t) + \lambda \int_0^t (t-s)^{\beta-1} G(T_b(\tau), \bar{u}(s)) \mathrm{d}s,$$

also blows up in finite time, where $\lambda := \sup_{z \in [0, T_b(\tau)]} k_1(z) + T_b(\tau)$. This contradicts the result in Lemma 3.4, and thus the proof is complete. ∎

3.3. Numerical blow-up implies exact blow-up. We now state our first key result which links the blow-up behavior of the collocation solution with that of the exact solution. In particular, we can use the blow-up behavior of the collocation solution to establish finite-time blow-up for the given VIE.

THEOREM 3.7. *Assume that*

(i) *conditions* (P), (K), (G1), *and* (G2) *hold,*

(ii) $u_h(t)$ *is the collocation solution according to* $c_1 \in [0, 1]$ *and to adaptive stepsize such that* (2.8) *holds for* $c_1 \in (0, 1]$ *(or* (2.9) *holds for the explicit Euler method).*

Then the following two statements are equivalent:

(i) *for all* $\tau \in (0, 1)$, *the collocation solution* $u_h(t)$ *blows up in finite time;*

(ii) *the exact solution* $u(t)$ *blows up in finite time.*

Proof. Using somewhat intricate notations and following the proofs of Lemma 3.2 and Lemma 3.4, one may obtain these results. Indeed, in view of Theorem 2.14, a simple proof is given in the following.

Assume that the exact solution blows up in finite time and the collocation solution $u_h(t)$ does not blow up. Then it follows from Theorem 2.14 that the collocation solution of the explicit Euler method on the same grid mesh I_h does not blow up, which is a contradiction to Theorem 3.3.

On the other hand, assume that the exact solution does not blow up in finite time. Then it follows from Theorem 3.6 that the collocation solution of the implicit Euler method does not blow up, which implies by Theorem 2.14 that the collocation solution $u_h(t)$ also exists on the same mesh. ∎

4. Blow-up times. In this section we assume that the exact solution blows up at a finite time $T_b < T$ and the collocation solutions also blow up at a finite time $T_b(I_h) < T$. In actual applications, when the exact blow-up time T_b is unknown, we work with a threshold blow-up time $T_b^M := \inf\{t \in [0, T) : u(t) \geqslant M\}$ associated with a given threshold $M \gg 1$.

DEFINITION 4.1. *For a given threshold* $M \gg 1$,

$$T_b^M(I_h) := \inf\{t_n \in I_h : u_n \geqslant M\}$$

is called the numerical threshold blow-up time.

In the following we denote by $I_h^{c_1}$ the adaptive chosen mesh underlying the collocation solution corresponding to the collocation parameter c_1.

Remark 4.2. Assume that $T_b(I_h^0)$ and $T_b(I_h^1)$ are the blow-up times of the explicit Euler method and the implicit Euler method, respectively. Then it follows from

Theorems 2.12 and 2.13 that

$$T_b(I_h^1) < T_b < T_b(I_h^0). \tag{4.1}$$

Similarly, it follows from Theorem 2.14 that the numerical threshold blow-up time $T_b^M(I_h^{c_1})$ is decreasing with respect to c_1 in the sense that

$$T_b^M(I_h^{c_1}) < T_b^M(I_h^{\bar{c}_1})$$

provided $0 \leqslant \bar{c}_1 < c_1 \leqslant 1$ and $I_h^{c_1} \subseteq I_h^{\bar{c}_1}$.

For convenience, the numerical blow-up time is denoted by $T_b(\tau)$ when the adaptive stepsize is such that $h_n = h_n^*$ given by (2.8) or (2.9). Similarly, we can obtain the numerical threshold blow-up time $T_b^M(\tau)$.

Definition 4.3. *The numerical blow-up time $T_b(\tau)$ is said to converge to the exact blow-up time T_b if*

$$\lim_{\tau \to 0} T_b(\tau) = T_b.$$

Based on the local convergence of the collocation solution for (1.1), we first derive upper and lower bounds of the numerical threshold blow-up time $T_b^M(\tau)$. In order to do so we shall need the following result from [3].

Lemma 4.4. *Let $\beta > 0$, $\phi(t)$ and $G(t,u)$ be C^1-smooth functions, and $k(z) = z^{\beta-1}k_1(z) \in C^1((0,\infty))$. Then the collocation solutions $u_h(t)$ converge to the exact solution $u(t)$ with at least order 1, that is, there exist constants $C(M) > 0$ and $\tau^*(M)$ such that*

$$|u_h(t_n) - u(t_n)| \leqslant C(M)\tau \text{ for } \tau < \tau^*(M)$$

whenever $|u_h(t_n)|$ and $|u(t_n)|$ are bounded by M.

Theorem 4.5. *Assume that the conditions in Lemma 4.4 hold. Then for any given $M > 0$,*

$$T_b^{M-1} \leqslant T_b^M(\tau) \leqslant T_b^{\alpha M} \text{ for all } 0 < \tau < \min\left\{T_b^{\alpha M} - T_b^{M+1}, \frac{\tau^*(\alpha M)}{C(\alpha M)}\right\},$$

where α is defined by (2.3).

Proof. Suppose that $T_b^M(\tau) > T_b^{\alpha M}$ for some $\tau < \min\{T_b^{\alpha M} - T_b^{M+1}, \frac{\tau^*(\alpha M)}{C(\alpha M)}\}$. Then one obtains that

$$|u(t)| \leqslant \alpha M \text{ for } t \in [T_b^{M+1}, T_b^{\alpha M}],$$
$$|u_h(t_n)| \leqslant M \text{ for } t_n \in I_h \cap [T_b^{M+1}, T_b^{\alpha M}].$$

On the other hand, Lemma 4.4 states that

$$1 \leqslant |u_h(t_n) - u(t_n)| \leqslant C(\alpha M)\tau < 1 \text{ for } t_n \in I_h \cap [T_b^{M+1}, T_b^{\alpha M}].$$

This is a contradiction. In a similar way, it can be shown that $T_b^{M-1} \leqslant T_b^M(\tau)$ and the proof is complete. ∎

Lemma 4.6. *Assume that the conditions* (P), (K), *and* (G1) *hold. Then*

$$|T_b^M(\tau) - T(M,\tau)| \leqslant \begin{cases} \left(\frac{\beta|u(T_b^M(\tau)) - u(T(M,\tau))|}{k_* G(0,\phi(0))}\right)^{1/\beta} & \text{for } \beta > 1, \\ \frac{|u(T_b^M(\tau)) - u(T(M,\tau))|}{k_* T_b^{\beta-1} G(0,\phi(0))} & \text{for } \beta \in (0,1], \end{cases}$$

where $T(M,\tau) := \inf\{t : u(t) \geqslant u_h(T_b^M(\tau))\}$.

Proof. Without loss of generality, assume that $T(M,\tau) < T_b^M(\tau)$. Then

$$\begin{aligned} u(T_b^M(\tau)) &\geqslant \phi(T(M,\tau)) + \int_0^{T(M,\tau)} k(s)G(T(M,\tau)-s, u(T(M,\tau)-s))\mathrm{d}s \\ &\quad + \int_{T(M,\tau)}^{T_b^M(\tau)} k(s)G(T_b^M(\tau)-s, u(T_b^M(\tau)-s))\mathrm{d}s \\ &\geqslant u(T(M,\tau)) + k_* G(0,\phi(0)) \int_{T(M,\tau)}^{T_b^M(\tau)} s^{\beta-1}\mathrm{d}s. \end{aligned}$$

Hence the proof is complete. ∎

LEMMA 4.7. *Let $c_1 \in [0,1]$ and $\tau \in (0,1)$. Assume that the conditions* (P), (K), (G1)*, and* (1.3) *hold. Then for any given $\epsilon > 0$ there exists an $M_0 > 0$ such that*

$$|T_b - T_b^M| \leqslant \epsilon \textit{ for all } M \geqslant M_0,$$
$$|T_b(\tau) - T_b^M(\tau)| \leqslant \epsilon \textit{ for all } M \geqslant M_0 \textit{ and } \tau > 0.$$

Proof. Similarly to the proof of Lemma 3.2, we can show that for sufficiently large M,

$$|T_b - T_b^M| \leqslant \frac{4}{\beta}\left(\frac{\beta}{k_*}\right)^{1/\beta} \int_M^\infty \left(\frac{u}{G(t^*,u)}\right)^{\frac{1}{\beta}} \frac{\mathrm{d}u}{u},$$
$$|T_b(\tau) - T_b^M(\tau)| \leqslant \frac{2^{2+1/\beta}}{\beta}\left(\frac{\beta}{k_*}\right)^{1/\beta} \int_M^\infty \left(\frac{u}{G(t^*,u)}\right)^{\frac{1}{\beta}} \frac{\mathrm{d}u}{u}.$$

The desired results are now verified by recalling (1.3). ∎

Using the local convergence of the collocation solutions and the upper bound of the numerical threshold blow-up time, we are ready to show the convergence of the numerical blow-up time.

THEOREM 4.8. *Under the conditions in Lemmas* 4.4 *and* 4.7*, numerical blow-up times converge to the exact one.*

Proof. For any given $\epsilon > 0$, let $M_0 = M_0(\epsilon)$ be as in Lemma 4.7 and $\tau < \frac{\tau^*(\alpha M_0+1)}{C(\alpha M_0+1)}$. Then Theorem 4.5 yields

$$T_b^{M_0}(\tau), T(M_0,\tau) \leqslant T_b^{\alpha M_0+1},$$

which together with the result of Lemma 4.4 implies that

$$|u(T(M_0,\tau)) - u(T_b^{M_0}(\tau))| = |u_h(T_b^{M_0}(\tau)) - u(T_b^{M_0}(\tau))| \leqslant C(\alpha M_0+1)\tau.$$

Thus it follows from Lemma 4.6 that there exists a $\tau_0(\epsilon, M_0) > 0$ such that

$$|T(M_0,\tau) - T_b^{M_0}(\tau)| \leqslant \epsilon \text{ for } 0 < \tau < \tau_0.$$

Hence, by Lemma 4.7,

$$|T_b(\tau) - T_b| \leqslant 3\epsilon \text{ for } 0 < \tau < \tau_0,$$

and the proof is complete. ∎

5. Different numerical approaches. In some application models such as thermal ignition in a diffusive medium, $G(s,u)$ is dependent on s (see [16]). Hence, it is computationally more convenient to apply the implicitly linear collocation methods or the fully discretized collocation methods to Hammerstein-type VIEs (1.1) (see detailed discussion in [3]).

5.1. Implicitly linear collocation methods. In (1.1), let $z(t) := G(t, u(t))$. Then

$$z(t) = G\left(t, \phi(t) + \int_0^t k(t-s)z(s)\mathrm{d}s\right), \tag{5.1}$$

$$u(t) = \phi(t) + \int_0^t k(t-s)z(s)\mathrm{d}s. \tag{5.2}$$

The collocation solution $z_h(t) \in S_0^{-1}(I_h)$ of (5.1) is defined by

$$z_h(t) = G\left(t, \phi(t) + \int_0^t k(t-s)z_h(s)\mathrm{d}s\right), t \in X_h, \tag{5.3}$$

and the corresponding implicitly linear collocation solution $u_h^{IL}(t)$ of (1.1) is given by

$$u_h^{IL}(t) = \phi(t) + \int_0^t k(t-s)z_h(s)\mathrm{d}s, t \in I. \tag{5.4}$$

Remark 5.1. Since, in general, $z_h(t)$ is a piecewise polynomial, the exact integrals in (5.3) comparing with (2.1) are available even for a general nonlinear function $G(s, u)$. Moreover, in Examples 5.2, 5.3, and Example 6.3, $z_h(t)$ is piecewise constant, so the integrals in (5.3) are only concerned with the kernel.

Example 5.2. For implicitly linear collocation methods with $c_1 \in (0, 1]$, $z_h(t) \equiv z_{n+1}$ for $t \in (t_n, t_{n+1}]$ satisfies the implicit nonlinear algebraical equation

$$z_{n+1} = G\left(t_n + c_1 h_n, \phi(t_n + c_1 h_n) + \Gamma_n^I(t_n + c_1 h_n) + K(c_1 h_n)z_{n+1}\right),$$

and $u_h^{IL}(t)$ for $t \in (t_n, t_{n+1}]$ is defined by

$$u_h^{IL}(t) = \phi(t) + \Gamma_n^I(t) + K(t - t_n)z_{n+1},$$

where

$$\Gamma_n^I(t) = \sum_{i=0}^{n-1}(K(t-t_i) - K(t-t_{i+1}))z_{i+1}.$$

Example 5.3. The implicitly linear collocation method with $c_1 = 0$ reads

$$z_{n+1} = G\left(t_{n+1}, \phi(t_{n+1}) + \Gamma_n^E(t_{n+1}) + K(h_n)z_n\right),$$
$$u_h^{IL}(t) = \phi(t) + \Gamma_n^E(t) + K(t-t_n)z_n,$$

where, in analogy to section 2.1.2,

$$\Gamma_n^E(t) = \sum_{i=0}^{n-1}(K(t-t_i) - K(t-t_{i+1}))z_i.$$

5.2. Fully discretized collocation methods. In the collocation equation (2.1), the integrals cannot, in general, be found exactly but have to be approximated by some numerical quadrature formulas. The fully discretized version of (2.1) has the form

$$\hat{u}_h(t_n+c_1h_n) = \phi(t_n+c_1h_n)+\hat{\Gamma}_n(t_n+c_1h_n)+c_1h_nk(c_1^2h_n)G(t_n+c_1h_n, \hat{u}_h(t_n+c_1h_n)),$$

and its iteration collocation solution is

$$\hat{u}_h^{it}(t_n + vh_n) = \phi(t_n + vh_n) + \hat{\Gamma}_n(t_n + vh_n) + vh_n k(vc_1h_n)G(t_n + c_1h_n, \hat{u}_h(t_n + c_1h_n)), v \in [0,1],$$

where $\hat{\Gamma}_n(t)$ is the approximation history term

$$\hat{\Gamma}_n(t_n + vh_n) = \sum_{i=0}^{n} h_i k(t_n + vh_n - t_i - c_1h_i)G(t_i + c_1h_i, \hat{u}_h(t_i + c_1h_i)).$$

Remark 5.4. The integrals of the kernel and $G(s,u)$ are replaced by a numerical quadrature formula and the fully discretized collocation solution $\hat{u}_h(t)$ also belongs to $S_0^{-1}(I_h)$, but in general, $\hat{u}_h(t) \neq u_h(t)$ even when the nonlinear function $G(s,u)$ is independent of s.

Remark 5.5. In applications, if the integrals of the kernel can be found analytically, then the fully discretized collocation methods can be alternated by

$$\hat{u}_h(t) = \phi(t) + \hat{\Gamma}_n(t) + K(t - t_n)G(t, \hat{u}_h(t)), t \in X_h,$$

and its iteration collocation solution is

$$\hat{u}_h^{it}(t) = \phi(t) + \hat{\Gamma}_n(t) + K(t - t_n)G(t_n + c_1h_n, \hat{u}_h(t)), t \in I,$$

where $\hat{\Gamma}_n(t)$ is the approximation history term

$$\hat{\Gamma}_n(t) = \sum_{i=0}^{n}(K(t - t_i) - K(t - t_{i+1}))G(t_i + c_1h_i, \hat{u}_h(t_i + c_1h_i)).$$

In this case, the iterated collocation solution $\hat{u}_h^{it}(t)$ is just same as the implicitly linear collocation solutions $u_h^{IL}(t)$ (see [3]).

Similarly to the discussions in sections 2.1.1, 2.2, and 2.3, one may obtain the following results.

THEOREM 5.6. *Assume that* $c_1 \in (0,1]$ *and the adaptive stepsize is defined similarly to* (2.5). *Then there exists a unique solution of both the implicitly linear collocation method and the fully discretized collocation method.*

THEOREM 5.7. *Let* $\tau \in (0,1]$, $c_1 \in [0,1]$, *and conditions* (P), (K), *and* (G1) *hold. Then* $\hat{u}_h(t)$ *and* $u_h^{IL}(t)$ *are nondecreasing.*

Remark 5.8. With the adaptive stepsize similar to (2.8) for the case of $c_1 \in (0,1]$, the results in Theorems 2.12, 2.13, and 2.14 are also true for $\hat{u}_h(t)$ and $u_h^{IL}(t)$.

Remark 5.9. Different from the collocation solution $u_h(t)$ with exact integrals, the dynamical behavior of $\hat{u}_h(t)$ will be influenced by

$$\hat{F}(t_n + c_1h_n, u) = \sum_{i=0}^{n} h_i k(t_n + vh_n - t_i - c_1h_i)G(t_i + c_1h_i, u) + c_1h_n k(c_1^2h_n)G(t_n + c_1h_n, u) - u,$$

(recall (1.4)), while the dynamical behavior of $u_h^{IL}(t)$ is determined by

$$F^{IL}(t,u) = \int_0^t k(t-s)G_h(s,u)\mathrm{d}s - u,$$

where $G_h(t,u) \equiv G(t_i + c_1 h_i, u)$ for $t \in [t_i, t_{i+1}]$. Therefore, the results in Theorems 2.18 and 2.19 are true for $\hat{u}_h(t)$ (or $u^{IL}(t)$), if we replace F by $\hat{F}$ (or F^{IL}).

Since $F(t,u)$, $F^{IL}(t,u)$, and $\hat{F}(t,u)$ may be different, for example $F^{IL}(t,u) > F(t,u)$ when $c_1 = 1$, the blow-up behaviors of $\hat{u}_h(t)$, $u_h^{IL}(t)$, and $u_h(t)$ may also differ. But, in any case, under the condition $\lim_{t\to\infty}\phi(t) = \infty$, the blow-up behaviors of $u(t)$, $u_h(t)$, $\hat{u}_h(t)$, and $u_h^{IL}(t)$ are same.

6. Numerical experiments. We now present some numerical experiments to illustrate the blow-up behavior of collocation solutions.

Example 6.1. As a special case, consider first an ODE of order $\beta \geq 1$,

$$\begin{aligned} &u^{(\beta)}(t) = u(t)^p, \\ &u^{(i)}(0) = u_i > 0, i = 0, 1, \ldots, \beta - 1. \end{aligned} \tag{6.1}$$

This initial-value problem is equivalent to the VIE

$$u(t) = \phi(t) + \int_0^t k(t-s)u(s)^p \mathrm{d}s,$$

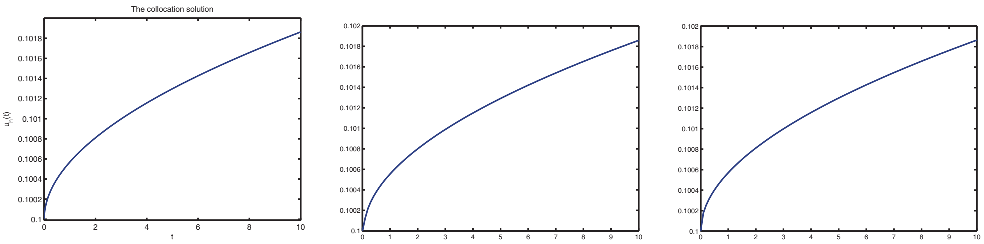
where $\phi(t) = \sum_{i=0}^{\beta-1} \frac{1}{i!} u_i t^i$ and $k(z) = \frac{1}{(\beta-1)!} z^{\beta-1}$. In Table 6.1, the errors, $|T_b^M(\tau) - T(M,\tau)|$, are listed for $\beta = 1, p = 1.5, 1.8, 2.0, 2.5, 3.0$, and $c_1 = 0, 0.5, 1$. In Table 6.2, $u_0 = 1, u_1 = \frac{\sqrt{6}}{3}, \beta = 2, p = 2.0$, and $c_1 = 0, 0.5, 1$, and the numerical threshold blow-up times for $M = 1E8$ are listed (the exact threshold blow-up time is 2.4495; cf. [10]). The numerical results suggest that for (6.1) with $\beta = 1$, the numerical threshold

Table 6.1
The errors between $T_b^M(\tau)$ and $T(M,\tau)$ of (6.1) *when $\beta = 1$.*

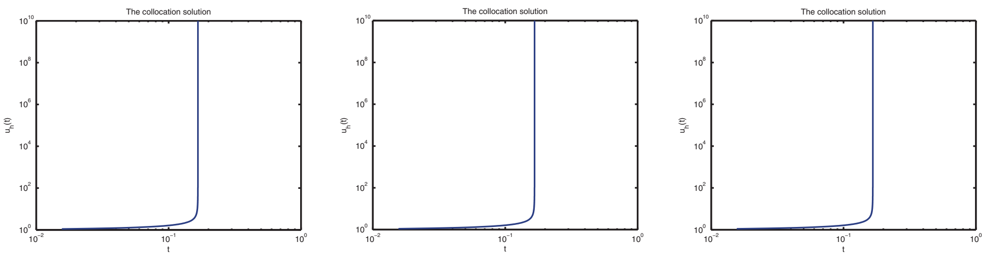
p	c_1	$\tau = 0.1$			$\tau = 0.05$		
		$M = 1E5$	$M = 1E6$	$M = 1E8$	$M = 1E5$	$M = 1E6$	$M = 1E8$
1.5	0	$1.4834E-1$	$1.4866E-1$	$1.4879E-1$	$7.4459E-2$	$7.4622E-2$	$7.4688E-2$
	0.5	$3.4813E-4$	$3.8692E-4$	$4.0305E-4$	$7.1087E-5$	$9.0524E-5$	$9.8517E-5$
	1.0	$5.4008E-2$	$5.4126E-2$	$5.4174E-2$	$2.6712E-2$	$2.6770E-2$	$2.6794E-2$
1.8	0	$1.1213E-1$	$1.1214E-1$	$1.1214E-1$	$5.6153E-2$	$5.6158E-2$	$5.6159E-2$
	0.5	$2.2535E-4$	$2.2656E-4$	$2.2678E-4$	$5.4667E-5$	$5.5271E-5$	$5.5383E-5$
	1.0	$3.3118E-2$	$3.3120E-2$	$3.3121E-2$	$1.6350E-2$	$1.6352E-2$	$1.6352E-2$
2.0	0	$9.9999E-2$	$1.0000E-1$	$1.0000E-1$	$5.0000E-2$	$5.0000E-2$	$5.0000E-2$
	0.5	$1.6424E-4$	$1.6436E-4$	$1.6437E-4$	$3.9994E-5$	$4.0051E-5$	$4.0058E-5$
	1.0	$2.5658E-2$	$2.5658E-2$	$2.5658E-2$	$1.2660E-2$	$1.2660E-2$	$1.2660E-2$
2.5	0	$8.3995E-2$	$8.3995E-2$	$8.3995E-2$	$4.1836E-2$	$4.1836E-2$	$4.1836E-2$
	0.5	$8.0149E-5$	$8.0149E-5$	$8.0149E-5$	$1.9502E-5$	$1.9502E-5$	$1.9502E-5$
	1.0	$1.5119E-2$	$1.5119E-2$	$1.5119E-2$	$7.4602E-3$	$7.4602E-3$	$7.4602E-3$
3.0	0	$7.6190E-2$	$7.6190E-2$	$7.6190E-2$	$3.7805E-2$	$3.7805E-2$	$3.7805E-2$
	0.5	$4.1091E-5$	$4.1091E-5$	$4.1091E-5$	$1.0014E-5$	$1.0014E-5$	$1.0014E-5$
	1.0	$9.5995E-3$	$9.5995E-3$	$9.5995E-3$	$4.7424E-3$	$4.7424E-3$	$4.7424E-3$

Table 6.2
The numerical threshold blow-up time of (6.1) *when $M = 1E8$ and $\beta = 2$.*

τ	$c_1 = 0$	$c_1 = 0.5$	$c_1 = 1$
$\frac{1}{10}$	2.6481	2.4492	2.4055
$\frac{1}{20}$	2.5652	2.4493	2.4186
$\frac{1}{100}$	2.4801	2.4494	2.4369



(a) $\phi(t) \equiv 0.1$ and $c_1 = 0$ (b) $\phi(t) \equiv 0.1$ and $c_1 = 0.5$ (c) $\phi(t) \equiv 0.1$ and $c_1 = 1$



(d) $\phi(t) \equiv 1$, $c_1 = 0$, $T_b^M(\tau) = 0.1667$ (e) $\phi(t) \equiv 1$, $c_1 = 0.5$, $T_b^M(\tau) = 0.1655$ (f) $\phi(t) \equiv 1$, $c_1 = 1$, $T_b^M(\tau) = 0.1650$

FIG. 6.1. *The collocation solutions of* (6.2) *when* $\tau = 0.1$ *and* $M = 1E10$.

blow-up times not only converge to $T(M, \tau)$, but also converge with the local order of the collocation methods. The numerical threshold blow-up times of (6.1) with $\beta = 2$ also converge to the exact one, but the convergence order is not the same as the order of the corresponding methods. This also happens for wave equations (see [8]). In any case, the upper bound of $T_b^M(\tau)$ given by Theorem 4.5 also encourages us to investigate the convergence order of $T_b^M(\tau)$ by the local convergence and convergence order of a numerical method.

Example 6.2. Consider a VIE with smooth memory kernel,

$$u(t) = \phi(t) + \int_0^t (t-s)\exp(-(t-s))u(s)(\log(1+u(s)))^p \mathrm{d}s, \tag{6.2}$$

where $\phi(t) > 0$. In Figure 6.1, we exhibit the collocation solutions with $p = 2.5$, $c_1 = 0, 0.5, 1$, and $\tau = 0.1$. It follows from Figure 6.1 that the collocation solutions blow up in finite time for $\phi(t) \equiv 1$ but do not blow up in finite time for $\phi(t) \equiv 0.1$. It detects from Theorem 3.7 that the exact solutions blow up in finite time for $\phi(t) \equiv 1$ and do not blow up in finite time for $\phi(t) \equiv 0.1$. On the other hand, since

$$I(t) = \int_0^t z\exp(-z)\mathrm{d}z = 1 - (1+t)e^{-t} \to 1 \text{ as } t \to \infty,$$

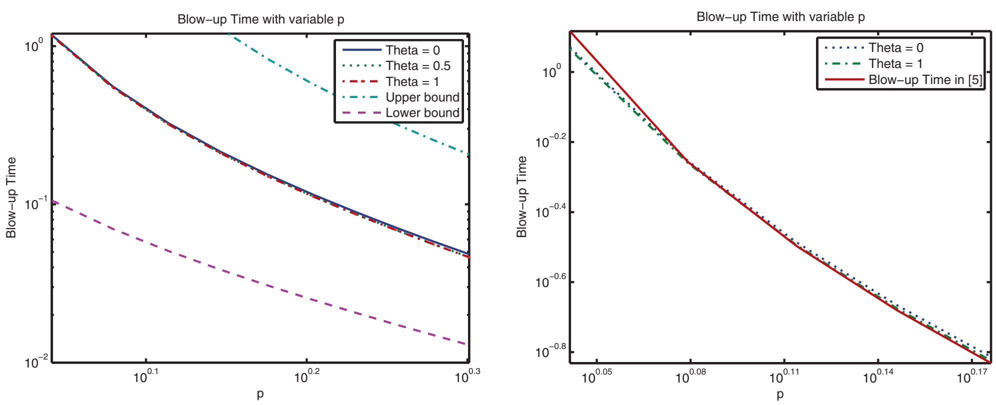
the exact solution in the case of $p = 2.5$ blows up in finite time if and only if

$$\phi(t^*) + F_{\min} > 0 \text{ for some } t^* > 0,$$

where $F_{\min} = \min_{u\in[0,\infty)}(u(\log(1+u))^{2.5} - u) \approx -0.60399$.

TABLE 6.3
The numerical threshold times $T_b^M(\tau)$ of (6.3) *when $M = 1E5$ and $\tau = 0.1$.*

γ / p	1.1	1.2	1.3	1.4	1.5
1.1	1.1635	1.1408	1.1203	1.1016	1.0844
1.2	0.55051	0.53143	0.51446	0.49922	0.48544
1.3	0.32108	0.30472	0.29041	0.27776	0.26648
1.4	0.21159	0.19736	0.18511	0.17446	0.16509
1.5	0.15028	0.13776	0.12716	0.11808	0.11021



(a) the numerical threshold blow-up times and the upper and lower bounds

(b) the numerical threshold blow-up times and the simulating blow-up time in [5]

FIG. 6.2. *Numerical threshold blow-up times of* (6.3) *with $M = 1E8$ and $\tau = 0.1$.*

Example 6.3. The shear band model arising in the formation of steel (cf. [16]) is described by the nonlinear VIE

$$u(t) = 1 + \gamma \int_0^t \frac{(1+s)^q u^p(s)}{\sqrt{t-s}} \mathrm{d}s, \tag{6.3}$$

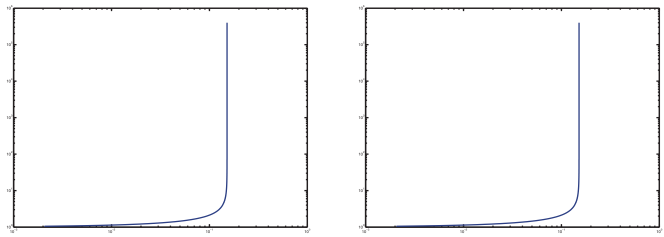
where $\gamma > 0, q \geqslant 0$, and $p > 1$ are constants.

For the special case of $q = 0$, it follows from [16] that the exact solution blows up at a finite time T_b satisfying

$$T_b \in \left[\frac{(1-p^{-1})^{2p}}{4\gamma^2(p-1)^2}, \frac{1}{4\gamma^2(p-1)^2} \right]. \tag{6.4}$$

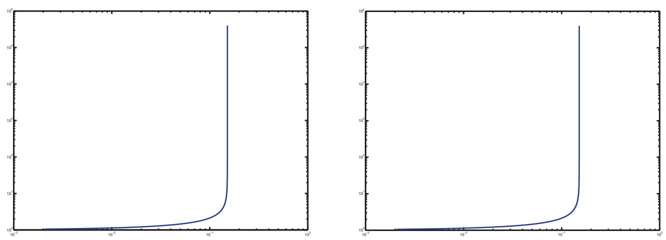
In Table 6.3, we list the numerical threshold blow-up time with variable p and γ when $M = 1E5$, $\tau = 0.1$. In Figure 6.2, we plot the numerical threshold blow-up times with $M = 1E8$ and $\tau = 0.1$, the simulated blow-up time in [5], and the upper and lower bounds with fixed $\gamma = 1.1$ and various p. Figure 6.2(a) shows that the numerical threshold blow-up times also satisfy (6.4). The numerical threshold blow-up times of collocation methods with adaptive stepsize are more precise than the simulating blow-up time in [5] (see Figure 6.2(b)), since the exact blow-up time must be larger than the threshold blow-up times of the implicit Euler method.

Consider the case of $q = 1$, i.e., the nonlinear function $G(s, u) = (1 + s)u^p$ depends on s. In addition to the collocation method, we also employ the implicitly



(a) $c_1 = 0.5$ and $T_b^M(\tau) = 0.1513$ (b) $c_1 = 1$ and $T_b^M(\tau) = 0.1487$

FIG. 6.3. *The collocation solutions of* (6.3) *with* $q = 1, p = 2, \gamma = 1$, *and* $M = 1E6$.



(a) $c_1 = 0.5$ and $T_b^M(\tau) = 0.1513$ (b) $c_1 = 1$ and $T_b^M(\tau) = 0.1498$

FIG. 6.4. *The implicitly linear collocation solutions of* (6.3) *with* $q = 1, p = 2, \gamma = 1$, *and* $M = 1E6$.

linear collocation method. We draw both $u_h(t)$ and $u_h^{IL}(t)$ with $p = 2, \gamma = 1$, $\tau = 0.1$, and $c_1 = 0.5, 1$ in Figures 6.3 and 6.4, respectively. These figures corresponding to $M = 1E6$ show that the solutions $u_h(t)$ and $u_h^{IL}(t)$ blow up in finite time, which is in agreement with the exact solution.

REFERENCES

[1] C. Bandle and H. Brunner, *Numerical analysis of semilinear parabolic problems with blow-up solutions*, Rev. R. Acad. Cienc. Exactas. Fis. Nat. (Esp.), 88 (1994), pp. 203–222.

[2] C. Bandle and H. Brunner, *Blow-up in diffusion equations: A survey*, J. Comput. Appl. Math., 97 (1998), pp. 3–22.

[3] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge Monogr. Appl. Comput. Math. 15, Cambridge University Press, Cambridge, 2004.

[4] H. Brunner and Z.W. Yang, *Blow-up behavior of Hammerstein-type Volterra integral equations*, J. Integral Equations Appl., 24 (2012), pp. 487–512.

[5] F. Calabró and G. Capobianco, *Blowing up behavior for a class of nonlinear VIEs connected with parabolic PDEs*, J. Comput. Appl. Math., 228 (2009), pp. 580–588.

[6] Y.G. Chen, *Asymptic behaviours of blowing-up solutions for finite difference analogue of* $u_t = u_{xx} + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 33 (1986), pp. 541–574.

[7] C.-H. Cho, S. Hamada, and H. Okamoto, *On the finite difference approximation for a parabolic blow-up problem*, Japan J. Indust. Appl. Math., 24 (2007), pp. 105–134.

[8] C.-H. Cho, *A finite difference scheme for blow-up solutions of nonlinear wave equations*, Numer. Math. Theory Method Appl., 3 (2010), pp. 475–498.

[9] C.-H. Cho, *On the convergence of numerical blow-up time for a second order nonlinear ordinary differential equation*, Appl. Math. Lett., 24 (2011), pp. 49–54.

[10] A. Goriely and C. Hyde, *Finite-time blow-up in dynamical systems*, Phys. Lett. A, 250 (1998), pp. 311–318.

[11] A. Gut, *Probability: A Graduate Course*, Springer, New York, 2005, p. 127.

[12] R.D. Haynes and C. Turner, *A numerical and theoretical study of blow-up for a system of ordinary differential equations using the Sundman transformation*, Atlantic Electron. J. Math., 2 (2007), pp. 1–13.

[13] C. Hirota and K. Ozawa, *Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations–an application to the blowup problems of partial differential equations*, J. Comput. Appl. Math., 193 (2006), pp. 614–637.

[14] D.G. Lasseigne and W.E. Olmstead, *Ignition of a combustible solid with reactant consumption*, SIAM J. Appl. Math., 47 (1987), pp. 332–342.

[15] T. Nakagawa, *Blowing up of a finite difference solution to* $u_t = u_{xx} + u^2$, Appl. Math. Optim., 2 (1976), pp. 337–350.

[16] C.A. Roberts, D.G. Lasseigne, and W.E. Olmstead, *Volterra equations which model explosion in a diffusive medium*, J. Integral Equations Appl., 5 (1993), pp. 531–546.

[17] C.A. Roberts, *Recent results on blow-up and quenching for nonlinear Volterra equations*, J. Comput. Appl. Math., 205 (2007), pp. 736–743.

[18] A.M. Stuart and M.S. Floater, *On the computation of blow-up*, European J. Appl. Math., 1 (1990), pp. 47–71.

[19] C.F. Touron, *An Adaptive Method for Calculating Blow-up Solutions*, Ph.D. Thesis, Old Dominion University, Norfolk, VA, 2009.