AN EXPONENTIALLY FAST ATTITUDE TRACKING CONTROLLER ON THE ROTATION GROUP

James D. Biggs
james.biggs@strath.ac.uk, Senior Lecturer, Department of Mechanical & Aerospace Engineering
University of Strathclyde, Glasgow, G1 1XJ, UK.
Nadjim Horri
Lecturer, Department of Aeronautical, Aviation, Electronic & Electrical,
Faculty of Engineering and Computing, Coventry University, CV1 5FB, UK

In this paper a continuous attitude control law is derived directly on the rotation group SO(3). The proposed control law is shown to reduce the closed-loop attitude dynamics to a linear oscillator description of the eigen-axis error, without the need for a small angle approximation in the case of a rest-to-rest motion. The main practical benefit of this is that the gains can be easily tuned to drive this eigen-axis error to zero exponentially fast and with a damped response without oscillations. The approach uses geodesic error metrics on the rotation group and the angular velocity to construct a Lyapunov function. The time-derivative of this Lyapunov function is control dependent and a continuous control is selected to guarantee asymptotic tracking of the reference motion. Furthermore, the closed-loop system, with this rotation-matrix based feedback control applied, is converted to its quaternion form and further reduced to an eigen-axis error description of the dynamics. This reduction reveals a simple method for tuning the control which involves only one parameter which can be selected to obtain the fastest convergence to the final prescribed state. However, this control suffers the problem related to the exponential coordinates; the control is not defined globally. This paper shows that by converting the control to quaternions and augmenting the control a globally defined exponentially fast control law can be defined. The proposed control is applied in simulation to the attitude control of a small spacecraft and shows a settling time performance enhancement, for given actuator constraints, compared to a conventional quaternion controller.

I INTRODUCTION

In this paper we present two attitude control laws with exponentially fast convergence. The main practical advantage of the proposed controls are that they are simple to tune, requiring the tuning of only a single parameter. The first control that is presented is a rotation-matrix based exponentially fast controller. Moreover, for the spacecraft attitude dynamics described by [1]:

$$J\dot{\omega} = J\omega \times \omega + \mathbf{u} \quad (1)$$

where J is the positive definite, symmetric inertia tensor, $\omega = [\omega_1, \omega_2, \omega_3]^T$ is the angular velocity vector and $\mathbf{u}$ is the applied torque with the kinematics described globally in rotation matrix form:

$$\dot{R} = R\Omega \quad (2)$$

where $R \in \mathrm{SO}(3)$ where $\mathrm{SO}(3)$ is the Special Orthogonal Group defined by:

$$\mathrm{SO}(3) \triangleq \{R \in \mathbb{R}^{3\times 3} : \; R^T R = I \quad \text{and} \quad \det(R) = 1\} \quad (3)$$

where Ω is defined by

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad (4)$$

and is related to the angular velocity vector ω via the"hat" map which maps isomorphically an element of a vector in Euclidean space to an element of the Lie algebra of SO(3), namely the vector space of skew-symmetric matrices $\hat{\cdot}$: $\mathbb{R}^3 \to \mathfrak{so}(3)$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \quad (5)$$

In addition we define the inverse "hat" map as $\cdot\vee$: $\mathfrak{so}(3) \to \mathbb{R}^3$ also via the isomorphism (5). The general aim is to design a control $\mathbf{u}$ that tracks a prescribed reference trajectory exponentially fast. Defining $\omega_d = R_e^{-1}\omega_r$ where ω_r is the desired angular velocity and where the rotation error R_e is defined by the constraint:

$$\dot{R}_e = R_e\Omega_e \quad (6)$$

where $\Omega_e = \Omega - \Omega_d$ where $\Omega_d = R_e^{-1}\Omega_r R_e$. Therefore, the aim is to asymptotically stabilize the zero-error state defined by $\omega_e = \vec{0}, R_e = Id$, where Id is the 3×3 identity matrix. The rotation-matrix based control presented in this paper is stated as:

$$\mathbf{u} = \omega \times J\omega - 2\sqrt{k}J\omega_e - kJ\left(\log R_e\right)^{\vee} + J\dot{\omega}_d \quad (7)$$

the single tuning parameter $k > 0$ can be tuned to ensure that the error expressed as the eigen-axis rotation error

θ defined by $1+2\cos\theta = trace(R_e)$ converges exponentially fast according to the equation:

$$\ddot{\theta} + 2\sqrt{k}\dot{\theta} + k\theta = 0. \tag{8}$$

on the domain $D = \{(R_e, \Omega_e) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^3 : \theta \in (-\pi,\pi), 1+2\cos\theta = trace(R)\}$. Therefore, we have the eigen-axis error on D defined by:

$$\theta = \theta_0 e^{-\sqrt{k}t} \tag{9}$$

Linear error dynamics have been realized before but through a different approach and expressed in vector form [2]. However, the problem with the controls in [2] and (7) is that they only locally stabilize the zero-error state. For example in (7) the $\log R_e$ function is not uniquely defined at $\theta = \pi$. Examples of rotation-matrix based globally stabilizing control laws can be found in [3], however, these controls do not lend themselves to simple tuning methods that yield exponentially fast tracking. This paper demonstrates that the error defined in quaternion form $\bar{q}_e = [q_{0e}, \mathbf{q}_e]^T$ with $\bar{q}_e \in \mathbb{S}^3$ with respect to the angular velocity error $\omega_e = \omega - R_e^{-1}\omega_r$ can be expressed as:

$$\begin{aligned} \frac{d\mathbf{q}_e}{dt} &= \frac{1}{2}\Omega_e \mathbf{q}_e + \frac{1}{2} q_{0e}\omega_e \\ \dot{q}_{0e} &= -\frac{1}{2}\omega_e^T \mathbf{q}_e \end{aligned} \tag{10}$$

then the following control law globally stabilizes the zero-error state exponentially fast and avoids the un-winding problem [4]:

$$\begin{aligned} \mathbf{u} &= \omega \times J\omega - 2\sqrt{k}J\omega_e + J\dot{\omega}_d \\ &-\mathrm{sgn}(q_{0e}(0))2kJ\frac{\cos^{-1}(\mathrm{sgn}(q_{0e}(0))q_{0e})}{\sqrt{1-q_{0e}^2}}\mathbf{q}_e \end{aligned} \tag{11}$$

where

$$\mathrm{sgn}(q_{0e}(0)) = \begin{cases} 1 & \text{if} \quad q_{0e}(0) \in (0,1] \\ -1 & \text{if} \quad q_{0e}(0) \in [-1,0] \end{cases} \tag{12}$$

This control also reduces the closed-loop system expressed as an eigenaxis rotation error to the form (8) which converges exponentially fast. This control only requires the tuning of a single parameter k. However, we must note that this is at the expense of defining a discontinuous term in the control [5]. This discontinuity is only dependent on the initial condition $q_{0e}(0)$ and dictates which form of the control you should use, so the control itself will be continuous during practical operation. One note of caution is that if the initial quaternion error is close to $q_{0e}(0) = 0$ and there is noise present in the system then this could cause an incorrect implementation and a hybrid control should be used if this was a possible scenario [6]. However, if the reference trajectory is well designed then it is unlikely that a tracking error close to $q_{0e}(0) = 0$ will ever occur in practise. This paper presents the development of these controls and is outlined as follows: Section II describes the preliminary derivation of a rotation-matrix based control in [7]. Section III shows that the control reduces the closed-loop system to a linear description of the error dynamics in its eigen axis angle representation. This enables a simple tuning method to be used requiring the tuning of only a single parameter. Section IV extends the control by augmenting its quaternion form to be defined globally and thus applicable to any initial error. Section V demonstrates through an application of the control to a small spacecraft, with actuator constraints, that the control algorithm in Section IV convergences faster when compared to a more conventional tracking controller. Section VI presents the concluding remarks.

II PRELIMINARIES

Equivalent metrics on both the Euclidean vector space $\mathbb{R}^3$ and the Lie algebra $\mathfrak{so}(3)$ are used to define a Lyapunov function. The Euclidean product is defined as $\|\mathbf{x}\|_{\mathbb{R}^3} = \langle \mathbf{x}, \mathbf{x}\rangle_{\mathbb{R}^3}^{1/2}$ where $\langle \cdot,\cdot\rangle_{\mathbb{R}^3}$ is the usual dot product. Equivalently a metric on the Lie algebra $\mathfrak{so}(3)$ can be defined as:

$$\|\hat{x}\| = \langle \hat{x}, \hat{x}\rangle^{1/2} \tag{13}$$

where $\hat{x} \in \mathfrak{so}(3)$ and $\langle \cdot,\cdot\rangle$ denotes the trace form defined by:

$$\langle A, B\rangle = -\frac{1}{2}trace(AB) \tag{14}$$

where $A, B \in \mathfrak{so}(3)$. As in the paper [7] we use the geodesic metric on $SO(3)$ to define a constant velocity error curve connecting the desired totation R_d and the current rotation R whose distance is given by:

$$\|R_e\| = \langle \log R_e, \log R_e\rangle^{1/2} \tag{15}$$

where $R_e = R_r^{-1}R$. To compute $\log R_e$ in (15) the well-known formula:

$$\log R_e = \begin{cases} \frac{\theta}{2\sin\theta}(R_e - R_e^T) & \text{if} \quad \theta \in (-\pi,\pi), \theta \neq 0 \\ 0 & \qquad \theta = 0 \end{cases} \tag{16}$$

is used. Note that this expression does not uniquely define $\log R_e$ when the principal angle of rotation, θ, is an odd multiple of π radians. We define a Lyapunov function as a weighted function of the metric on the Lie algebra $\mathfrak{so}(3)$ and a metric defining a minimal geodesic on SO(3) similarly to that in [7, 8] but extending it from attitude stabilization to attitude tracking of an arbitrary time-dependent trajectory, let:

$$V(R_e, \Omega_e) = \frac{1}{2}\langle \Omega_e, \Omega_e\rangle + \frac{k}{2}\langle \log R_e, \log R_e\rangle \tag{17}$$

where k can be viewed as a weighting parameter. Clearly at the desired orientation $\Omega_e = 0$ and $R_e = Id$, where

Id is the 3×3 identity matrix, the Lyapunov function $V(Id,0)=0$ and $V(R_e,\Omega_e)>0$ on the domain $D=\{(R_e,\Omega_e)\in\mathbb{R}^{3\times 3}\times\mathbb{R}^3 : \theta\in(-\pi,\pi), 1+2\cos\theta = trace(R)\}$. Differentiating with respect to time and noting that the derivative of the distance function (15) on SO(3) is given by the equation:

$$\frac{1}{2}\frac{d}{dt}\|R_e\|^2 = \langle\Omega_e, \log R_e\rangle \tag{18}$$

yields:

$$\dot{V}(R_e,\Omega_e) = \left\langle\Omega_e,\dot{\Omega}_e\right\rangle + k\left\langle\Omega_e,\log R_e\right\rangle \tag{19}$$

using the mapping (5) and the equivalence of the distance metric on $\mathbb{R}^3$ and $\mathfrak{so}(3)$ (19) can be expressed equivalently in vector form as:

$$\dot{V}(R_e,\omega_e) = \langle\omega_e,\dot{\omega}_e\rangle_{\mathbb{R}^3} + k\left\langle\omega_e,\log R_e^{\vee}\right\rangle_{\mathbb{R}^3} \tag{20}$$

where $\vee$ denotes the inverse of the "hat" map (5). Substituting (1) into (20) yields:

$$\begin{aligned}\dot{V}(R_e,\omega_e) = &\left\langle\omega_e, J^{-1}(J\omega\times\omega - J\dot{\omega}_d + \mathbf{u})\right\rangle_{\mathbb{R}^3}\\ &+k\left\langle\omega_e,\log R_e^{\vee})\right\rangle_{\mathbb{R}^3}\end{aligned} \tag{21}$$

which can be simplified to:

$$\dot{V}(R_e,\omega_e) = \left\langle\omega_e, J^{-1}(J\omega\times\omega - J\dot{\omega}_d + \mathbf{u} + kJ\log R_e^{\vee}\right\rangle_{\mathbb{R}^3} \tag{22}$$

then selecting the control:

$$\mathbf{u} = \omega\times J\omega - \sigma J\omega_e - kJ\left(\log R_e\right)^{\vee} + J\dot{\omega}_d \tag{23}$$

yields:

$$\dot{V} = -\langle\omega_e,\sigma\omega_e\rangle_{\mathbb{R}^3}\leq 0 \tag{24}$$

then from La Salle's invariance principle [9], as $(Id,0)\in D$ is the largest invariant set in $S=\{(R_e,\Omega_e)\in D|\,\Omega=0\}$ then $(R_e,\Omega_e)\in D$ will approach $(Id,0)\in D$ as $t\to\infty$. This controller suffers from the same problem as thecontrol in [7] due to the use of exponential coordinates; they are not uniquely defined at odd multiples of π radians.

III SIMPLE TUNING METHOD

One of the main contributions of this paper is to derive a simple tuning algorithm for the control law (23) that guarentees exponentially fast attitude stracking. Converting the rotation error to a quaternion error using the relation:

$$R_e = \begin{pmatrix} C_1 & C_2 & C_3 \end{pmatrix} \tag{25}$$

where C_i are the column vectors:

$$C_1 = \begin{bmatrix} q_{0e}^2 + q_{1e}^2 - q_{2e}^2 - q_{3e}^2 \\ 2q_{1e}q_{2e} + 2q_{0e}q_{3e} \\ 2q_{1e}q_{3e} - 2q_{0e}q_{2e}\end{bmatrix} \tag{26}$$

$$C_2 = \begin{bmatrix} 2q_{1e}q_{2e} - 2q_{0e}q_{3e} \\ q_{0e}^2 - q_{1e}^2 + q_{2e}^2 - q_{3e}^2 \\ 2q_{2e}q_{3e} + 2q_{0e}q_{1e}\end{bmatrix} \tag{27}$$

$$C_3 = \begin{bmatrix} 2q_{1e}q_{3e} + 2q_{0e}q_{2e} \\ 2q_{2e}q_{3e} - 2q_{0e}q_{1e} \\ q_{0e}^2 - q_{1e}^2 - q_{2e}^2 + q_{3e}^2\end{bmatrix} \tag{28}$$

which on direct substitution into (16) yields $\log R_e$ in terms of quaternions:

$$\log R_e = \frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}\hat{q}_e \tag{29}$$

where $\hat{q}_e$ is the skew-symmetric form of $\mathbf{q}_e$ given by the map (5). Note that (16) and (29) are not exactly equivalent definitions for the exponential coordinate vector. Moreover, expression (29) suggests that $\log R_e$ is defined when $q_{0e}=\cos(\theta/2)=0$, where θ is the principal rotation angle tracking error which corresponds to θ being an odd multiple of π radians. But $\log R_e$ is not uniquely defined in this case. Also note that there are singularities at $q_{0e}=\pm 1$. However, the singularities are easily proved to be removable with $\lim\limits_{q_{0e}\to 1}\frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}=2$ and $\lim\limits_{q_{0e}\to -1}\frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}=-2$ and for $q_{0e}=\pm 1\Leftrightarrow \mathbf{q}_e=0$ and are, therefore, not problematic when implementing the control. We substitute (29) using the mapping (25) into the equation (23) to derive a quaternion-based tracking control:

$$\mathbf{u} = \omega\times J\omega - \sigma J\omega_e - kJ\frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}\mathbf{q}_e + J\dot{\omega}_d \tag{30}$$

Now assuming the error in quaternion form (10) the closed-loop equations in quaternion form can be stated as:

$$\begin{aligned}\dot{\omega}_e &= -\sigma\omega_e - k\frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}\mathbf{q}_e\\ \frac{d\,\mathbf{q}_e}{dt} &= \tfrac{1}{2}\Omega_e\mathbf{q}_e + \tfrac{1}{2}q_{0e}\omega_e\\ \dot{q}_{0e} &= -\tfrac{1}{2}\omega_e^T\mathbf{q}_e\end{aligned} \tag{31}$$

assuming the ansatz solution:

$$\begin{aligned}\mathbf{q}_e(t) &= c\,\mathbf{q}_e(0)\\ \omega_e(t) &= v\,\mathbf{q}_e(0)\end{aligned} \tag{32}$$

where c and v are scalar functions and $c(0)=1$ and substituting (32) into (31) gives:

$$\dot{v} = -\sigma v - k\frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}c \tag{33}$$

substituting again $\omega_e(t)=v(t)\mathbf{q}_e(0)$ the second equation in (31) and noting that $\Omega_e q_e\equiv 0$ then

$$\dot{c} = \frac{1}{2}q_{0e}v \tag{34}$$

substituting in the ansatz solution into the last equation in (31) yields

$$\dot{q}_{0e} = -\frac{1}{2}cv(1-q_{0e}(0)^2) \tag{35}$$

where $q_{0e}(0)$ is the initial error in the scalar part of the quaternion. Therefore, the closed-loop equations are simplified to the coupled first order ODEs:

$$\begin{aligned} \dot{v} &= -\sigma v - k\frac{\sin^{-1}[2q_{0e}(1-q_{0e}^2)^{1/2}]}{(1-q_{0e}^2)^{1/2}}c \\ \dot{c} &= \tfrac{1}{2}q_{0e}v \\ \dot{q}_{0e} &= -\tfrac{1}{2}cv(1-q_{0e}(0)^2) \end{aligned} \tag{36}$$

where $c(0)=1$. Note that we can define a conservation law from equations (34) and (35), that is:

$$\frac{dc}{dq_{0e}} = -\frac{q_{0e}}{c(1-q_{0e}^2(0))} \tag{37}$$

and therefore

$$-\int c dc = \frac{1}{(1-q_{0e}^2(0))}\int q_{0e}dq_{0e} \tag{38}$$

and with $c(0)=1$ reveals the conserved quantity that implicitly defines an ellipse:

$$1 = q_{0e}^2 + (1-q_{0e}^2(0))c^2 \tag{39}$$

this form of the conservation law suggests that a useful parameterization of this system would be:

$$q_{0e} = \cos\frac{\theta}{2}, c = \frac{\sin\theta/2}{\sin\theta_0/2} \tag{40}$$

differentiating (40) with respect to time and substituting in (36) and simplifying yields

$$\ddot{\theta} + \sigma\dot{\theta} + k\theta = 0 \tag{41}$$

where $\sigma = 2\xi\omega_n, k = \omega_n^2$ with ξ the damping ratio and ω_n the natural frequency.

The linear equations (41) yield a natural choice of the tuning parameter $\sigma^2 = 4k$ to ensure that the system is critically damped ($\xi = 1$). This gives the fastest return of the system to its reference for the control (23). Substituting $\sigma^2 = 4k$ into (23) yields (7). In contrast the magnitude of the rotation angle tracking error for other feedback controllers [3, 10, 11, 12] are described by a non-linear equation that do not have a natural choice for the tuning parameter. For example, the classical controls [10, 11, 12] (generalized to tracking controllers) an be stated as:

$$\mathbf{u} = \omega\times J\omega - \sigma J\omega_e - \operatorname{sgn}(q_{0e}(0))kJ\mathbf{q}_e + J\dot{\omega}_d \tag{42}$$

the closed-loop equations reduce to:

$$\ddot{\theta} + \sigma\dot{\theta} + k\sin(\theta/2) = 0. \tag{43}$$

The procedure to tune the control (42) is then to linearise the non-linear equations (43) and select the parameters for critical damping. However, this approximation is only suitable for small θ and for large angle manoeuvres a larger value of the parameter is required to compensate for the non-linear term.

IV TRACKING CONTROL FOR ANY INITIAL ERROR

Note here that although the controls in rotation-matrix form (23) and quaternion form (30) can be tuned to give exponentially fast convergence, they are only locally defined. However, the following observation leads to an intuitive extension to a globally stabilizing control law in its quaternion form. Moreover, on the domain $q_{0e} \in (\frac{1}{\sqrt{2}}, 1]$ we can write $q_{0e} = \cos\theta/2$ and substituting this into (30) then the control simplifies to:

$$\mathbf{u} = \omega\times J\omega - \sigma J\omega_e - 2kJ\frac{\cos^{-1}q_{0e}}{(1-q_{0e}^2)^{1/2}}\mathbf{q}_e + J\dot{\omega}_d \tag{44}$$

It is straighforward to prove this control asymptotically stabilizes $\omega_e = \vec{0}, q_{0e} = 1$ by using the Lyapunov function:

$$V \equiv \frac{1}{2}\langle\omega_e,\omega_e\rangle_{\mathbb{R}^3} + 2k\arccos^2 q_{0e} \tag{45}$$

in an analogous way on the domain $q_{0e} \in [-1, -\frac{1}{\sqrt{2}})$ we can write $\theta = 2\cos^{-1}(-q_{0e})$ which gives

$$\mathbf{u} = \omega\times J\omega - \sigma J\omega_e + 2kJ\frac{\cos^{-1}(-q_{0e})}{(1-q_{0e}^2)^{1/2}}\mathbf{q}_e + J\dot{\omega}_d \tag{46}$$

and which can be shown to asymptotically stabilize $\omega_e = \vec{0}, q_{0e} = -1$ by using the Lyapunov function:

$$V \equiv \frac{1}{2}\langle\omega_e,\omega_e\rangle_{\mathbb{R}^3} + 2k\arccos^2(-q_{0e}) \tag{47}$$

Therefore, we can intuitively define a globally stabilizing feedback law by defining a piecewise control on the entire domain:

$$\begin{aligned} \mathbf{u} &= \omega\times J\omega - \sigma J\omega_e + J\dot{\omega}_d \\ &-\operatorname{sgn}(q_{0e}(0))2kJ\frac{\cos^{-1}(\operatorname{sgn}(q_{0e}(0))q_{0e})}{(1-q_{0e}^2)^{1/2}}\mathbf{q}_e \end{aligned} \tag{48}$$

$$\operatorname{sgn}(q_{0e}(0)) = \begin{cases} 1 & \text{if} \quad q_{0e}(0) \in (0,1] \\ -1 & \text{if} \quad q_{0e}(0) \in [-1,0] \end{cases} \tag{49}$$

to prove that the control (48) globally stabilizes the zero-error set $\omega_e = \omega_d = \vec{0}, q_{0e} = \pm 1$ we use the discontinuous Lyapunov function:

$$V \equiv \frac{1}{2}\langle\omega_e,\omega_e\rangle_{\mathbb{R}^3} + 2k\arccos^2(\operatorname{sgn}(q_{0e}(0))q_{0e}) \tag{50}$$

differentiating (50) with respect to time yields:

$$\dot{V} = \langle\omega_e,\dot{\omega}_e\rangle_{\mathbb{R}^3} - \operatorname{sgn}(q_{0e}(0))\frac{4k\cos^{-1}(\operatorname{sgn}(q_{0e}(0))q_{0e})}{\sqrt{1-q_{0e}^2}}\dot{q}_{0e} \tag{51}$$

then substituting in $\dot{q}_{0e} = -\frac{1}{2}\omega_e^T \mathbf{q}_e$ and (1) which on simplification gives:

$$\begin{aligned} \dot{V} &= \left\langle \omega_e, J^{-1}(J\omega \times \omega - J\dot{\omega}_d + u + f_1) \right\rangle_{\mathbb{R}^3} \\ f_1 &= -\text{sgn}(q_{0e}(0)) 2kJ \frac{\cos^{-1}(\text{sgn}(q_{0e}(0))q_{0e})}{\sqrt{1-q_{0e}^2}} \mathbf{q}_e \end{aligned} \quad (52)$$

it follows that the control (48) gives

$$\dot{V} = -\sigma \left\langle \omega_e, \omega_e \right\rangle_{\mathbb{R}^3} \quad (53)$$

It follows from La Salle's invariance principle that the zero-error state is asymptotically stable.

V NUMERICAL SIMULATIONS

The purpose of this simulation study is to assess the presented controls in terms of tuning simplicity. In other words the simple tuning method $\sigma^2 = 4k$ is used while varying k to establish the best convergence time while respecting the constraints on the maximum torque. This tuning law was proposed in [10] for small angle rotations despite the closed-loop rotation angle error dynamics being nonlinear. This simple tuning algorithm has recently been used to design the attitude control system for real spacecraft [13, 14], again, despite the fact that the closed-loop rotation angle error dynamics are nonlinear. The performance of the proposed control, against the more conventional tracking control (42) is demonstrated in simulation of a micro-spacecraft with moments of inertia (kg/m^2);

$$J = \begin{pmatrix} 19 & 0.41 & 0.44 \\ 0.41 & 19.5 & -0.46 \\ 0.44 & -0.46 & 12.6 \end{pmatrix} \quad (54)$$

which is typical of a $100kg$ class satellite. This class of micro-satellite can realistically be equipped with reaction wheels having up to a 100 mNm torque capability and we include this actuator constraint in the simulation. Simulations are undertaken for a constant reference motion of $\bar{q}_d = [1,0,0,0]^T$ and $\omega_d = [0,0,0]^T$ with initial conditions $\bar{q}_e(0) = [0, 0.57735, 0.57735, 0.57735]^T$ and $\omega_e = [0,0,0]^T$.The convergence time is taken to be complete when all components of the angular velocity and quaternion error are of the order 10^{-5} for the first time. Figure 1 demonstrates the convergence times between the presented tracking control and the conventional tracking control for various tuning parameter values. It is clearly seen that using a simple tuning law, commonly used in practise, that the tracking law in this paper improves the convergence time. We acknowledge that the performance of the conventional control could be improved using a non-intuitive tuning law. However, with extensive manual tuning of control B it was not possible to better the convergence time of control A for this example.

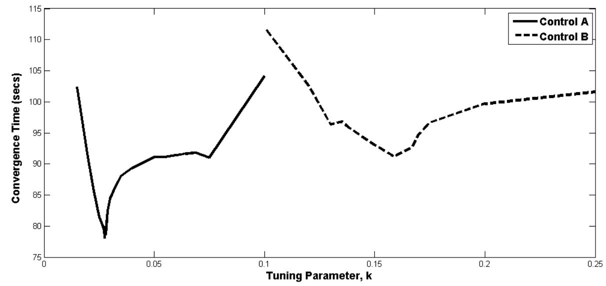

Figure 1: Convergence times for Control A defined by (48) and Control B defined by (42)

VI CONCLUSION

A Lyapunov function was defined in terms of the rotation and angular velocity error using a weighted function of a geodesic metric on SO(3) and the trace form on its Lie algebra $\mathfrak{so}(3)$. This Lyapunov function was then used to derive a control law on SO(3) that asymptotically stabilizes the zero-error of the closed-loop error dynamics. It has been shown by parameterizing the closed-loop error dynamics by its eigenaxis angle representation that the closed-loop error dynamics reduce to a linear description of the error. This means that the control law can be easily tuned using a single parameter to guarentee exponentially fast convergence. However, this control law suffers from the problem of exponential coordinates and only locally asymptotically stabilizes the zero-error state. It was shown that by converting this control to an augmented quaternion form a globally defined control law can be defined. This quaternion-based control can then be tuned to drive the error to zero exponentially fast without oscillation for any initial error.

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