

1 Awareness programs control infectious disease - Multiple delay induced
2 mathematical model.

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5 **Abstract**

6 We propose and analyze a mathematical model to study the impact of awareness programs on an infec-
7 tious disease outbreak. These programs induce behavioral changes in the population, which divide the
8 susceptible class into two subclasses, aware susceptible and unaware susceptible. The system can have a
9 disease-free equilibrium and an endemic equilibrium. The expression of the basic reproduction number
10 and the conditions for the stability of the equilibria are derived. We further improve and study the model
11 by introducing two time-delay factors, one for the time lag in memory fading of aware people and one for
12 the delay between cases of disease occurring and mounting awareness programs. The delayed system has
13 positive bounded solutions. We study various cases for the time delays and show that in general the sys-
14 tem develops limit cycle oscillation through a Hopf bifurcation for increasing time delays. We show that
15 under certain conditions on the parameters, the system is permanent. To verify our analytical findings,
16 the numerical simulations on the model, using realistic parameters for Pneumococcus are performed.

17 **Keywords:** Epidemic model, Awareness programs, Time delay, Stability analysis, Hopf bifurcation,
18 Numerical simulation.

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19 1. Introduction

20 In developing countries more than 11 million people died each year due to infectious disease includ-
21 ing premature deaths and deaths of young children. Pneumonia, Tuberculosis (TB), Diarrheal diseases
22 (Cholera), Malaria, Measles and more recently HIV/AIDS are the major deadly infectious diseases [1].

23 The major cause of global childhood mortality is Pneumonia which is caused by a number of infectious
24 agents, including viruses, bacteria and fungi. Approximately 1.4 million children die every year because
25 of Pneumonia [2]. Diarrheal diseases (for example Cholera, Bacillary Dysentery, Typhoid, Giardia and
26 Rotavirus) are the second leading cause of death taking the lives of about 1.5 million children under five
27 every year [3]. In 2010, 8.8 million people were infected with, and 1.4 million died from, TB [4]. Malaria is
28 a life-threatening vector-borne disease caused by the bites of infected mosquitoes. In 2010, Malaria caused
29 an estimated 655,000 deaths, mostly among African children (with an uncertainty range of 537,000 to
30 907,000) [5]. In 2010, 139,300 people died worldwide due to Measles [6]. Recently, HIV/AIDS has become
31 the major concern in a global pandemic. More than 25 million people died of HIV/AIDS in the last three
32 decades. There were approximately 34.2 million people infected by HIV up to the end of 2011 [7]. Another
33 infectious disease is Influenza which causes serious public health and economic problems. Globally, these
34 annual epidemics result in about three to five million cases of severe illness, and about 250,000 to 500,000
35 deaths [8]. Other major deadly infectious diseases in humans include Dengue, Yellow Fever, Hepatitis B,
36 Avian Influenza (Bird Flu) and Chagas Disease.

37 The above description clearly indicates the severity of infectious disease. These diseases are a major
38 threat to developing and underdeveloped countries. Some diseases can be prevented through vaccina-
39 tions. However this is costly and sometimes the effect is only temporary. On the other hand sometimes
40 disease appropriate awareness in a population can control an infection most effectively. In developing
41 and underdeveloped countries, the mass media plays an important role in changing behavior related to
42 public health. The government and other health organizations should immediately make people aware
43 about the disease and relevant precautions through the media. The media not only make the population
44 acquainted with the disease but also suggest the necessary preventive practices such as social distancing,
45 wearing protective masks or vaccination. In general the people who are aware adopt these practices so
46 that their chances of becoming infected are minimized. Depending on the behavior associated with a
47 given infectious disease, improved levels of awareness may increase the use of mosquito coils, mosquito
48 nets [9], or face masks [10, 11], practice of better hygiene [12, 13], application of preventive medicine
49 or vaccination [14], voluntary quarantine [15], avoidance of places containing large numbers of people

50 [12], practice of safe sex [16], or other appropriate measures. A comprehensive review of the existing
51 mathematical literature related to the effect of media awareness programs on disease outbreaks is given in
52 Table 1. However, behavioral responses can change the transmission patterns and reduce the prevalence
53 of disease. So there is a need of epidemiological models that explicitly include the effect of awareness
54 programs and behavioral responses. It is to be noted that in general the effect of awareness can strongly
55 depend on local interactions. The individuals in the local spatial or geographical neighbourhood of an
56 outbreak may have a much stronger incentive to adopt preventive practices and this local adoption of
57 suitable preventive practices may cause a local outbreak to die out without the whole population having
58 to adopt them. It would be possible to model this using some sort of spatial model. However in this
59 paper we shall not pursue this line instead we shall study a mean field model and assume that the impact
60 of the awareness program is uniform across the whole population. This is common in the study of disease
61 awareness programs [17, 18, 19, 20] where sometimes we wish to use a relatively simple model to study
62 the effect of awareness programs applied to the whole population to reduce the disease levels in the entire
63 population rather than stop a local outbreak.

64 A comprehensive review on the impact of media awareness programmes is presented in Section 2. In
65 Section 3 the model without time delays is formulated and analyzed to observe the local stability of the
66 system around the feasible equilibria. The model with multiple time delays is proposed and analyzed in
67 Section 4. The conditions under which the system enters Hopf bifurcation and conditions for permanence
68 of the system are also worked out. In Section 5, numerical simulations are carried out to verify our
69 analytical findings and the paper ends with a brief conclusion.

70 **2. Review of media awareness program in infectious disease outbreak**

71 In this section we review the literature on the effect of media awareness programs on infectious
72 disease outbreaks. These studies are essentially of two different types. In the first type mathemati-
73 cal models are used to investigate the impact of media coverage on the spread and control of infec-
74 tious disease. The mathematical models are either compartmental models such as susceptible-infected-
75 susceptible (SIS), susceptible-infected-recovered (SIR), susceptible-exposed-infected (SEI), susceptible-
76 infected-recovered-susceptible (SIRS), exposed-infected-hospitalized (EIH), susceptible-exposed-infected-
77 hospitalized-recovered (SEIHR) and similar models, or economic or game-theoretic models. In the second
78 type of study statistical analysis is used to identify the association between media awareness and disease
79 related cases. A comprehensive summary of such studies is given in Table 1.

Table 1: Review on the impact of media awareness programs on infectious disease.

Year	References	Summary of study
2007	[21]	Cui et al. developed and analyzed an SEI model to include media influence on the spreading of a communicable disease in a given area. They concluded that if the basic reproduction number is greater than one and the media effect is high, the model shows several endemic equilibria, which causes a threat to control the disease outbreak.
	[17]	Liu et al. developed an EIH compartmental model to investigate the role of the media and its psychological impact on multiple disease outbreaks. Their model analysis reveals that this impact leads to differences in the transmission pattern.
	[22]	Using the data from the Bangladesh Demography and Health Survey (1999-2000), Rahman and Rahman identified that media and education could play a major role in controlling HIV/AIDS.
	[23]	Tai and Sun investigated media dependency amongst Chinese individuals during the SARS epidemic of 2003. Their study was mainly focused into the situation where the information was highly monitored and not easily available from the mainstream media. In those circumstances, short message service (SMS) and the Internet are the possible substitute resources of information.
2008	[24]	Cui et al. formulated and analyzed an SIS infection model to investigate the role of media coverage during an infectious disease outbreak in a given population. They concluded that increasing media coverage causes a lower infection rate, although it may not absolutely remove the infection.

Table 1 – continued from previous page

Year	References	Summary of study
	[25]	Joshi et al. investigated the effect of an information and education campaign on the HIV epidemic in Uganda. They compare their model with three types of susceptibles to a standard SIR model.
	[26]	Li et al. developed and analyzed an SIS epidemic model, including media coverage in which the susceptible population is subjected to impulsive vaccination. They showed that the disease-free solution is globally asymptotically stable.
	[27]	Liu and Cui developed a compartmental model to study the role of the media in an infectious disease outbreak. They assume a standard epidemiological model but with a reduced transmission term due to the media campaign.
	[28]	Young et al. showed that a high level of media coverage plays a crucial role in making the public aware of many diseases and influencing their perception of risk. Participants in their study often considered diseases that appeared in the media more serious, even when this was not the actual case.
2009	[29]	Chen formulated an economic game-theoretic model of epidemics incorporating self-protection of susceptible populations. He suggests that an individual makes his or her behavioral changes through the information about the disease and expanding the supply of information may decrease the likelihood of eradication.

Table 1 – continued from previous page

Year	References	Summary of study
	[30]	Funk et al. develop and study a mathematical model where the host population is less susceptible due to the spread of awareness. They reveal that change in behavioral response can reduce the size of an outbreak though the epidemic threshold will be unaffected.
	[31]	Li and Cui propose an SIS epidemic model in the presence of media coverage and analyze the model under two distinct types of vaccination strategies namely constant vaccination and pulse vaccination. They compare these two different types of vaccination policies.
2010	[32]	Kiss et al. formulated a mathematical model where the total populations are aware of the disease threat but only a certain proportion of them is responsive. They showed that the infection can be removed when the spreading of information is fast enough, otherwise information transmission can play a major role in controlling the disease.
	[33]	Mummert and Weiss proposed a modified SIR model incorporating the impact of media coverage. They conclude that the severity of the disease outbreak can be lower if the media and the public health agencies work together.
	[34]	Yoo et al. showed using a statistical analysis that there is a connection between Influenza vaccination 1999-2001 and media reporting, specifically headlines on flu-related issues. They studied three media sources: a wire service news agency, a newspaper and four television channels.

Table 1 – continued from previous page

Year	References	Summary of study
2011	[18]	Misra et al. developed and analyzed a nonlinear SIS mathematical model in the presence of a media awareness program. They suggest that an awareness program can control the diffusion of the disease but immigration of susceptibles causes the disease to be endemic.
	[35]	Misra et al. proposed and analyzed a delay induced mathematical model in the presence of an awareness program. They concluded that the awareness program plays a crucial role in controlling the spread of disease, but it cannot remove the infection completely.
	[36]	Sun et al. used the SIS model in a two patch setting with media coverage present in each patch. They analyze their model both analytically and numerically. They find that both epidemic burden and duration of the disease spread are significantly lowered by the media coverage.
	[19]	Tchuenche et al. developed a Susceptible-Infected-Vaccinated-Recovered (SIVR) epidemic model to study the effect of media broadcasting on the spread and control over an Influenza outbreak. Using optimal control theory they obtained the effect of costs due to media coverage.
2012	[37]	Olowukure et al. investigated if there is any connection between volume of newspaper reports and laboratory testing for Influenza A (H1N1) pdm09, (the swine flu Influenza A (H1N1) pandemic of 2009) in one English health region during the early phase of the pandemic. They inferred that there exists a temporal association between volume of media reporting and number of laboratory tests.

Table 1 – continued from previous page

Year	References	Summary of study
	[38]	Tchuenche and Bauch formulated an SIHR model incorporating a signal function which captures the effect of media coverage. They suggest that the disease cannot be eliminated through media coverage, but it can control the spread of the infection.
2013	[39]	Funk and Jansen studied how the interplay between the network of an awareness program and the network of infection determines the dynamics of the disease outbreak.
	[40]	Liu investigated an SIRS epidemic model with media coverage and random perturbation. The disease transmission term was reduced by media coverage as in Liu and Cui [27], Tchuenche et al. [19] and Sun et al. [36] and stochastic white noise perturbation was added. The resulting stochastic differential equation model was studied analytically and numerically.
	[20]	Samanta et al. studied an SIS epidemic model for the effect of media awareness programs on epidemic outbreaks. They concluded that although media awareness programs can have a substantial effect on controlling disease prevalence, above a threshold value of their execution rate, the system shows limit cycle oscillations.
	[41]	Wang et al. studied an SIS network model incorporating the impact of media coverage on disease transmission and suggested effective control strategies to prevent disease through media coverage and education. They find the basic reproduction number, equilibrium and global stability results for their model and explore the results by simulation.

Table 1 – continued from previous page

Year	References	Summary of study
2014	[42]	Kaur et al. proposed and analyzed an SIRS epidemic model incorporating the effects of an awareness program driven by the media. Their model is based on that of Misra et al. [18] with some significant differences in modeling the awareness programs. They conduct an equilibrium and stability analysis and use simulation to verify their results.
	[43]	Samanta and Chattopadhyay proposed and analyzed a slow-fast epidemic model in the presence of the awareness program, where a susceptible individual switches between aware and unaware states very fast, whereas the disease transmission and other biological processes are comparatively slow.
	[44]	Sharma and Misra investigated an SIR model of hepatitis B with varying population size, which couples vaccination and awareness created by the media within a single framework.
	[45]	Wang and Xiao studied an SIR Filippov epidemic model with media coverage by incorporating a piecewise continuous transmission rate to describe that the media coverage exhibits its effects once the number of infected individuals exceeds a certain critical level. The disease transmission coefficient is reduced by an exponential term as a result of a media campaign. They find that a given level of infecteds can be reached if the threshold policy and other parameters are chosen correctly.

Table 1 – continued from previous page

Year	References	Summary of study
	[46]	Zhao et al. proposed and analyzed an SIRS epidemic model incorporating media coverage with time delay. They showed that the time delay in media coverage cannot affect the stability of the disease-free equilibrium when the basic reproduction number is less than unity. However, the time delay affects the stability of the endemic equilibrium and produces limit cycle oscillations while the basic reproduction number is greater than unity.
2015	[47]	Sahu and Dhar studied the complex dynamics of an SEQIHRS epidemic model incorporating media coverage, quarantine and isolation studies in a community with pre-existing immunity. Media coverage does not alter the effective reproduction number but lowers the number of infecteds at the endemic steady state, also lowering the maximum number of infected individuals. The results of isolation and quarantine depend on the amount of transmission from isolated individuals. Higher amounts of pre-existing immunity amongst the population cause the peak infection level to happen earlier and decrease it.

80

81 The above descriptions clearly indicate that awareness programmes play a crucial role in controlling
82 the disease during an epidemic outbreak. In the next section we formulate a mathematical model to
83 capture the impact of media awareness programs in an infectious disease outbreak. The model that we
84 shall consider is a deterministic differential equation mean field SIS epidemic model for the spread of an
85 infection in the presence of awareness programs. We model the awareness programs explicitly unlike the
86 models of Cui et al. [24], Li, Ma and Cui [26] and Liu and Cui [27] who model the effect of awareness
87 through a reduction in the disease transmission term. Our work builds on the work of Misra et al. [18, 35]
88 although we allow aware people to become infected and some recovered individuals to become aware. It
89 also builds on Samanta et al. [20] After analysing the basic model we introduce and analyse two types of
90 time delays and then perform simulations based on real parameter values for Pneumococcus to verify our

91 theoretical results.

92 **3. Model with awareness program**

93 *3.1. Model Formulation*

94 To formulate the mathematical model we suppose that the whole population is divided into three
95 separate classes, the susceptible aware class, the susceptible unaware class and the infected class. We
96 assume that both susceptible classes can be infected by contact with infectives but the aware class has
97 less chance to be infected compared to the unaware class and the infection rate among aware populations
98 is dependent on the awareness programs. The unaware susceptible population becomes aware through
99 the interaction with the awareness programs [18, 35] which is considered to be a saturating function [27]
100 (Holling type-II) of the awareness programs and a proportion of infected individuals recover from the
101 infection through treatment. After recovery, a fraction of recovered people will join the aware susceptible
102 class and the remaining fraction will remain unaware susceptible. The model does not necessarily assume
103 that the transmission routes of the disease and the information are the same, indeed these may well be
104 different.

105 We consider that in the region under consideration, the total population is $N(t)$ at time t and the rate
106 of immigration of susceptibles is A , where immigrants are assumed to be unaware. The total population
107 is divided into three classes: the susceptible unaware population $X_-(t)$, the infective population $Y(t)$
108 and the susceptible aware population $X_+(t)$. Also, let $M(t)$ be the number of campaigns due to the
109 awareness programs driven by the media in that region at time t . μ denotes the implementation rate of
110 awareness programs which is proportional to the number of infective individuals in the population. We
111 assume that unaware susceptible individuals become aware under the influence of the awareness program
112 at the rate λ and the interaction between the unaware susceptible population and the awareness program
113 follows the Holling type-II functional form with half-saturating constant k . It is assumed that the disease
114 spreads only due to direct contact between susceptibles and infectives. Let β be the contact rate of
115 unaware susceptible individuals with infective individuals and it is assumed that the disease transmission
116 follows the mass action law ($\beta X_-(t)Y(t)$). However, our basic assumption is that the interaction between
117 aware susceptibles and infecteds depends on the number of campaigns due to the awareness programs.
118 Large numbers of campaigns causes less interaction between susceptible aware and infected populations,
119 a mathematical form of this assumption can be written as $\frac{\beta X_+(t)Y(t)}{1+\beta_1 M(t)}$, where β_1 is the efficacy of the
120 awareness programs - a monotonic decreasing function of the number of campaigns $M(t)$. It is also a
121 monotonic decreasing function of β_1 . We assume that aware susceptible individuals transfer to unaware

122 susceptible individuals due to fading of memory or social factors at a per capita rate λ_0 . It is also
 123 assumed that a proportion of infected individuals recover through treatment. After recovery, a fraction p
 124 of recovered people will become aware and join the aware susceptible class whereas the remaining fraction
 125 $(1 - p)$ will remain unaware susceptible.

126 Keeping the above facts in mind, the dynamics of the model is governed by the following systems of
 127 nonlinear ordinary differential equations :

$$\begin{aligned}
 \frac{dX_-}{dt} &= A - \beta X_-(t)Y(t) - \lambda X_-(t)\frac{M(t)}{k + M(t)} - dX_-(t) + \lambda_0 X_+(t) + (1 - p)\gamma Y(t), \\
 \frac{dX_+}{dt} &= \lambda X_-(t)\frac{M(t)}{k + M(t)} + p\gamma Y(t) - dX_+(t) - \lambda_0 X_+(t) - \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t), \\
 \frac{dY}{dt} &= \beta X_-(t)Y(t) + \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t) - \gamma Y(t) - \alpha Y(t) - dY(t), \\
 \frac{dM}{dt} &= \mu Y(t) - \mu_0 M(t),
 \end{aligned} \tag{3.1}$$

128 where $X_-(0) > 0$, $X_+ \geq 0$, $Y \geq 0$, $M \geq 0$.

129 Here the constants γ , α , d represent the recovery rate, disease induced death and natural death rate
 130 respectively. The constant μ_0 denotes the depletion rate of awareness programs due to ineffectiveness,
 131 social problems in the population, and similar factors. Note that p is a fraction and its value lies between
 132 0 and 1.

133 Using the fact $N = X_- + X_+ + Y$, the system (3.1) reduces to the following system:

$$\begin{aligned}
 \frac{dY}{dt} &= \beta(N(t) - X_+(t) - Y(t))Y(t) + \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t) - (\gamma + \alpha + d)Y(t), \\
 \frac{dX_+}{dt} &= \lambda(N(t) - X_+(t) - Y(t))\frac{M(t)}{k + M(t)} + p\gamma Y(t) - dX_+(t) - \lambda_0 X_+(t) \\
 &\quad - \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t), \\
 \frac{dN}{dt} &= A - dN(t) - \alpha Y(t), \\
 \frac{dM}{dt} &= \mu Y(t) - \mu_0 M(t).
 \end{aligned} \tag{3.2}$$

For the analysis of model (3.2), we need the region of attraction [48] which is given by the set:

$$\Omega = \left\{ (Y, X_+, N, M) \in \mathfrak{R}_+^4 : 0 \leq X_+ + Y \leq N \leq \frac{A}{d}, 0 \leq M \leq \frac{\mu A}{\mu_0 d} \right\}$$

and attracts all solutions initiating in the interior of the positive orthant, with $N(0) > X_+(0) + Y(0)$.

3.2. Equilibrium analysis

The above model (3.2) has two non-negative equilibria.

(i) The disease free equilibrium (DFE) $E_0(0, 0, A/d, 0)$.

(ii) The endemic equilibrium $E^*(Y^*, X_+^*, N^*, M^*)$.

Here

$$X_+^* = \frac{\mu_0}{\beta \beta_1 \mu Y^*} \left[\beta \left(\frac{A}{d} - \frac{\alpha Y^*}{d} - Y^* \right) - (\gamma + \alpha + d) \right] \left[1 + \frac{\beta_1 \mu Y^*}{\mu_0} \right],$$

$$N^* = \frac{A - \alpha Y^*}{d},$$

$$M^* = \frac{\mu Y^*}{\mu_0},$$

and Y^* satisfies the equation

$$H_1 Y^{*3} + H_2 Y^{*2} + H_3 Y^* + H_4 = 0, \quad (3.3)$$

with

$$\begin{aligned} H_1 &= \frac{\beta \beta_1 \mu^2}{\mu_0^2} \left[(d + \lambda_0) \left(\frac{\alpha}{d} + 1 \right) + p\gamma \right] + \frac{\beta^2 \mu}{\mu_0} \left(\frac{\alpha}{d} + 1 \right), \\ H_2 &= \beta \left(\frac{\alpha}{d} + 1 \right) \left(\frac{\lambda \mu}{\mu_0} + \beta k \right) - \left(\frac{\beta A}{d} - \gamma - \alpha - d \right) \left(\frac{\lambda \beta_1 \mu^2}{\mu_0^2} + \frac{\beta \mu}{\mu_0} \right) + \frac{\lambda \beta \beta_1 \mu^2 A}{\mu_0^2 d} \\ &\quad + \frac{p\gamma \beta \beta_1 \mu k}{\mu_0} - \frac{\beta_1 \mu^2}{\mu_0^2} (d + \lambda_0) \left(\frac{\beta A}{d} - \gamma - \alpha - d \right) + \frac{\beta \mu}{\mu_0} (d + \lambda_0) (1 + \beta_1 k) \left(\frac{\alpha}{d} + 1 \right), \\ H_3 &= - \left(\frac{\beta A}{d} - \gamma - \alpha - d \right) \left(\frac{\lambda \mu}{\mu_0} + \beta k \right) \\ &\quad + k\beta (d + \lambda_0) \left(\frac{\alpha}{d} + 1 \right) - \frac{\mu}{\mu_0} (d + \lambda_0) (1 + \beta_1 k) \left(\frac{\beta A}{d} - \gamma - \alpha - d \right), \\ H_4 &= -k \left(\frac{\beta A}{d} - \gamma - \alpha - d \right) (d + \lambda_0). \end{aligned} \quad (3.4)$$

An endemic equilibrium exists if

$$\frac{\beta A}{d} - (\gamma + \alpha + d) > 0. \quad (3.5)$$

Let us define $R_0 = \frac{\beta A}{d(\gamma + \alpha + d)}$, which is the basic reproduction number for system (3.2).

H_1 is always positive and H_4 is always negative if $R_0 > 1$. Hence the equation (3.3) has at least one positive root. Therefore the sufficient conditions for the existence of the interior equilibrium point of system (3.2) are as follows:

$$R_0 > 1 \text{ and } Y^* < \min \left\{ \frac{d(\gamma + \alpha + d)(R_0 - 1)}{\beta(\alpha + d)}, \frac{A}{\alpha} \right\}.$$

150 However, H_1, H_2, H_3 and H_4 are always positive if $R_0 < 1$. Hence the system (3.2) does not have any
 151 positive interior equilibrium (E^*) for $R_0 < 1$.

152 **Remark 1:** $\frac{\partial Y^*}{\partial \mu} < 0$ if $\frac{H_{1\mu}Y^{*2}+H_{2\mu}Y^*+H_{3\mu}}{3H_1Y^{*2}+2Y^*H_2+H_3} > 0$ and $\frac{\partial Y^*}{\partial \beta_1} < 0$ if $\frac{H_{1\beta_1}Y^{*2}+H_{2\beta_1}Y^*+H_{3\beta_1}}{3H_1Y^{*2}+2Y^*H_2+H_3} > 0$,
 153 which indicates that the equilibrium number of infective individuals decreases with an increase in the
 154 value of the the implementation rate of awareness programs and the efficacy of the awareness programs.
 155 Here $H_{i\bullet}$, ($i = 1, 2, 3$) denotes the partial differentiation of H_i with respect to the parameter ' \bullet '.

156 **Remark 2:** We can find the basic reproduction number of the system (3.1) in the absence of awareness
 157 program. Therefore the system (3.1) becomes

$$\begin{aligned}\frac{dS}{dt} &= A - \beta SY - dS + \gamma Y, \\ \frac{dY}{dt} &= \beta SY - \gamma Y - \alpha Y - dY,\end{aligned}\tag{3.6}$$

158 where S and Y are the number of susceptible and infected individuals and the other parameters are the
 159 same as defined in system (3.1).

160 The above model (3.6) has two non-negative equilibria:

161 (i) The disease free equilibrium (DFE) $E_0(0, A/d)$,

162 (ii) The endemic equilibrium $E^*(S^*, Y^*)$,

163 where $S^* = \frac{\gamma+\alpha+d}{\beta}$, $Y^* = \frac{\beta A-d(\gamma+\alpha+d)}{\beta(\alpha+d)}$ the basic reproduction number for the system (3.6) is $R_{01} =$
 164 $\frac{\beta A}{d(\gamma+\alpha+d)}$, which is the same as R_0 . So the awareness program cannot eradicate the infection whenever
 165 $R_0 > 1$, but it can reduce the equilibrium number of infected individuals (see Figure 2).

166 3.3. Local stability behavior

167 The roots of the characteristic equation corresponding to $E_0(0, 0, A/d, 0)$ are $\frac{\beta A}{d} - \gamma - \alpha - d$, $-d$,
 168 $-(d + \lambda_0)$, $-\mu_0$.

169 The DFE E_0 is locally asymptotically stable (LAS) if $\frac{\beta A}{d} - \gamma - \alpha - d < 0$, i.e. $R_0 < 1$.

170 The variational matrix at an endemic equilibrium $E^*(Y^*, X_+^*, N^*, M^*)$ is

$$J = \begin{pmatrix} -\Pi_1 - \xi & \Pi_2 & \Pi_3 & -\Pi_4 \\ \Pi_5 & -\Pi_6 - \xi & \Pi_7 & \Pi_8 \\ -\Pi_9 & 0 & -\Pi_{10} - \xi & 0 \\ \Pi_{11} & 0 & 0 & -\Pi_{12} - \xi \end{pmatrix}.$$

171 Here $\Pi_1 = \beta Y^*$, $\Pi_2 = -\beta Y^* + \frac{\beta Y^*}{1+\beta_1 M^*}$, $\Pi_3 = \beta Y^*$, $\Pi_4 = \frac{\beta \beta_1 X_+^* Y^*}{(1+\beta_1 M^*)^2}$, $\Pi_5 = -\frac{\lambda M^*}{k+M^*} + p\gamma - \frac{\beta X_+^*}{1+\beta_1 M^*}$,
 172 $\Pi_6 = \frac{\lambda M^*}{k+M^*} + d + \lambda_0 + \frac{\beta Y^*}{1+\beta_1 M^*}$, $\Pi_7 = \frac{\lambda M^*}{k+M^*}$, $\Pi_8 = \frac{\lambda(N^*-X_+^*-Y)k}{(k+M^*)^2} + \frac{\beta \beta_1 X_+^* Y^*}{(1+\beta_1 M^*)^2}$, $\Pi_9 = \alpha$, $\Pi_{10} = d$, $\Pi_{11} = \mu$,
 173 $\Pi_{12} = \mu_0$.

174 The characteristic equation of the system (3.2) around the interior equilibrium (E^*) is

$$\xi^4 + \sigma_1 \xi^3 + \sigma_2 \xi^2 + \sigma_3 \xi + \sigma_4 = 0. \quad (3.7)$$

175 Therefore, E^* is LAS if and only if

$$\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0, \sigma_4 > 0, \sigma_1 \sigma_2 > \sigma_3 \text{ and } \sigma_1 \sigma_2 \sigma_3 > \sigma_3^2 + \sigma_1^2 \sigma_4. \quad (3.8)$$

176 Here,

$$177 \quad \sigma_1 = \Pi_1 + \Pi_6 + \Pi_{10} + \Pi_{12},$$

$$178 \quad \sigma_2 = \Pi_1 \Pi_{10} + \Pi_1 \Pi_{12} + \Pi_{10} \Pi_{12} + \Pi_3 \Pi_9 + \Pi_4 \Pi_{11} + \Pi_6 \Pi_{10} + \Pi_6 \Pi_{12} + \Pi_1 \Pi_6 - \Pi_2 \Pi_5,$$

$$179 \quad \sigma_3 = -\Pi_2 \Pi_5 \Pi_{10} - \Pi_2 \Pi_5 \Pi_{12} + \Pi_2 \Pi_8 \Pi_{11} + \Pi_1 \Pi_{10} \Pi_{12} + \Pi_3 \Pi_9 \Pi_{12} + \Pi_4 \Pi_{10} \Pi_{11} + \Pi_6 \Pi_{10} \Pi_{12}$$

$$180 \quad + \Pi_1 \Pi_6 \Pi_{10} + \Pi_1 \Pi_6 \Pi_{12} + \Pi_3 \Pi_6 \Pi_9 + \Pi_4 \Pi_6 \Pi_{11} + \Pi_2 \Pi_7 \Pi_9,$$

$$181 \quad \sigma_4 = -\Pi_2 \Pi_5 \Pi_{10} \Pi_{12} + \Pi_2 \Pi_7 \Pi_9 \Pi_{12} - \Pi_2 \Pi_8 \Pi_{10} \Pi_{11} + \Pi_1 \Pi_6 \Pi_{10} \Pi_{12} + \Pi_3 \Pi_6 \Pi_9 \Pi_{12} + \Pi_4 \Pi_6 \Pi_{10} \Pi_{11}.$$

182 4. Model with delay

183 4.1. Model Formulation

184 In the previous section we assumed that aware susceptible individuals transfer to unaware susceptible
 185 individuals due to fading of memory or certain social factors. However, it is reasonable to consider a time
 186 lag in memory fading of aware people. Here we assume that the aware susceptible individual will become
 187 unaware susceptible at time t due to forgetting the impact of disease at time $t - \tau_1$ (for some $\tau_1 > 0$).

188 We need to consider the probability that an aware susceptible individual remains in the aware suscep-
 189 tible class throughout the interval $[t - \tau_1, t]$ which we denote by $P(t, \tau_1)$. An aware susceptible individual
 190 leaves the aware susceptible class at time ξ through death at rate d , surviving the time interval $[\xi - \tau_1, \xi]$
 191 and becoming unaware at rate $\lambda_0 P(\xi, \tau_1)$ or becoming infected at rate $\frac{\beta Y(\xi)}{1 + \beta_1 M(\xi)}$. Hence

$$P(t, \tau_1) = e^{-\int_{t-\tau_1}^t [d + \lambda_0 P(\xi, \tau_1) + \frac{\beta Y(\xi)}{1 + \beta_1 M(\xi)} d\xi]}, \quad \text{for } t \geq t_1. \quad (4.1)$$

192 Usually, the number of infective cases known to the policy makers are cases that occurred some time
 193 previously and thus the intensity of the awareness program depends on this data. So it is more plausible
 194 to consider a time delay in execution of awareness programs. We suppose that at time t the intensity of
 195 the awareness programs being executed will be in accordance with the number of infected cases reported
 196 at time $t - \tau_2$ (for some $\tau_2 > 0$).

197 Incorporating these two delays and the survival probability into the system of equations (3.1) and
 198 writing $P(t) \equiv P(t, \tau_1)$ as τ_1 is fixed we obtain the system of delay differential equations:

$$\begin{aligned}
\frac{dX_-}{dt} &= A - \beta X_-(t)Y(t) - \lambda X_-(t) \frac{M(t)}{k + M(t)} - dX_-(t) + \lambda_0 X_+(t - \tau_1)P(t) + (1 - p)\gamma Y(t), \\
\frac{dX_+}{dt} &= \lambda X_-(t) \frac{M(t)}{k + M(t)} + p\gamma Y(t) - dX_+(t) - \lambda_0 X_+(t - \tau_1)P(t) - \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t), \\
\frac{dY}{dt} &= \beta X_-(t)Y(t) + \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t) - \gamma Y(t) - \alpha Y(t) - dY(t), \\
\frac{dM}{dt} &= \mu Y(t - \tau_2) - \mu_0 M(t), \\
\frac{dP}{dt} &= \left[-\lambda_0 P(t) + \lambda_0 P(t - \tau_1) - \frac{\beta Y(t)}{1 + \beta_1 M(t)} + \frac{\beta Y(t - \tau_1)}{1 + \beta_1 M(t - \tau_1)} \right] P(t).
\end{aligned} \tag{4.2}$$

We denote by C the Banach space of continuous functions $\phi : [-\tau, 0] \rightarrow \mathbb{R}^5$ with norm

$$\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \{|\phi_1(\theta)|, |\phi_2(\theta)|, |\phi_3(\theta)|, |\phi_4(\theta)|, |\phi_5(\theta)|\}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$. As usual, the initial conditions of (4.2) are given as

$$X_-(\theta) = \phi_1(\theta), \quad X_+(\theta) = \phi_2(\theta), \quad Y(\theta) = \phi_3(\theta), \quad M(\theta) = \phi_4(\theta), \quad P(\theta) = \phi_5(\theta), \quad \theta \in [-\tau, 0], \tag{4.3}$$

where the initial function $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ belongs to the Banach space $C = C([-\tau, 0], \mathbb{R}^5)$ of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}^5 . For biological reasons, the initial functions are assumed as

$$\phi_i(\theta) \geq 0, \quad i = 1, 2, 3, 4 \quad \text{and} \quad 1 \geq \phi_5(\theta) \geq 0, \quad \theta \in [-\tau, 0]. \tag{4.4}$$

We also need the consistency condition

$$P(0) = e^{-\int_{-\tau}^0 \left[d + \lambda_0 P(\xi, \tau_1) + \frac{\beta Y(\xi)}{1 + M(\xi)} \right] d\xi}.$$

By the fundamental theory of functional differential equations [49], we know that there is a unique solution $(X_-(t), X_+(t), Y(t), M(t), P(t))$ to system (4.2) with initial conditions (4.3).

4.2. Preliminaries

In this section, we will present some preliminaries, such as positive invariance, boundedness of solutions, existence of equilibria and the characteristic equation.

209 4.2.1. Positive invariance

210 **Theorem 4.1.** All the solutions of (4.2) with initial conditions (4.3) are positive.

211 **Proof :** The model (4.2) can be written in the following form:

212 $X = \text{col}(X_-(t), X_+(t), Y(t), M(t), P(t)) \in \mathbb{R}_+^5$, $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta)) \in \bar{C}_+ = ([-\tau, 0], \mathbb{R}_+^5)$,

213 $\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0) \geq 0, \phi_5(0) \geq 0$,

$$F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \\ F_4(X) \\ F_5(X) \end{pmatrix} = \begin{pmatrix} A - \beta X_-(t)Y(t) - \lambda X_-(t) \frac{M(t)}{k+M(t)} - dX_-(t) + \lambda_0 X_+(t - \tau_1)P(t) + (1-p)\gamma Y(t) \\ \lambda X_-(t) \frac{M(t)}{k+M(t)} + p\gamma Y(t) - dX_+(t) - \lambda_0 X_+(t - \tau_1)P(t) - \frac{\beta}{1+\beta_1 M(t)} X_+(t)Y(t) \\ \beta X_-(t)Y(t) + \frac{\beta}{1+\beta_1 M(t)} X_+(t)Y(t) - \gamma Y(t) - \alpha Y(t) - dY(t) \\ \mu Y(t - \tau_2) - \mu_0 M(t) \\ \left[-\lambda_0 P(t) + \lambda_0 P(t - \tau_1) - \frac{\beta Y(t)}{1+\beta_1 M(t)} + \frac{\beta Y(t - \tau_1)}{1+\beta_1 M(t - \tau_1)} \right] P(t) \end{pmatrix}.$$

214 Then the model system (4.2) becomes

$$\dot{X} = F(X) \quad (4.5)$$

with $X(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta)) \in C_+$ and $\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0), \phi_5(0) > 0$. It is easy to check in system (4.5) that whenever choosing $X(\theta) \in \mathbb{R}_+$ such that $X_- = 0, X_+ = 0, Y = 0, M = 0$ or $P = 0$ then

$$F_i(X)|_{x_i=0, X \in \mathbb{R}_+^5} \geq 0, \quad \text{for } i = 1, 2, 3, 4, 5,$$

215 with $x_1(t) = X_-(t)$, $x_2(t) = X_+(t)$, $x_3(t) = Y(t)$, $x_4(t) = M(t)$, $x_5(t) = P(t)$. Using the lemma of [50]

216 we claim that any solution of (4.5) with $X(\theta) \in C_+$, say $X(t) = X(t, X(\theta))$, is such that $X(t) \in \mathbb{R}_+^5$ for

217 all $t \geq 0$. From (4.1) we can see that $P(t) \leq 1$ for all t as well.

218 Next, we will prove the boundedness of solutions. Using the fact $N = X_- + X_+ + Y$, the system (4.2)

219 reduces to the following system:

$$\begin{aligned}
\frac{dY}{dt} &= \beta(N(t) - X_+(t) - Y(t))Y(t) + \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t) - (\gamma + \alpha + d)Y(t), \\
\frac{dX_+}{dt} &= \lambda(N(t) - X_+(t) - Y(t))\frac{M(t)}{k + M(t)} + p\gamma Y(t) - dX_+(t) - \lambda_0 X_+(t - \tau_1)P(t) \\
&\quad - \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t), \\
\frac{dN}{dt} &= A - dN(t) - \alpha Y(t), \\
\frac{dM}{dt} &= \mu Y(t - \tau_2) - \mu_0 M(t), \\
\frac{dP}{dt} &= \left[-\lambda_0 P(t) + \lambda_0 P(t - \tau_1) - \frac{\beta Y(t)}{1 + \beta_1 M(t)} + \frac{\beta Y(t - \tau_1)}{1 + \beta_1 M(t - \tau_1)} \right] P(t).
\end{aligned} \tag{4.6}$$

220 *4.2.2. Boundedness*

221 **Theorem 3.2.** *All the solutions of (4.6) with initial conditions (4.3) are ultimately bounded.*

222 **Proof :** Let, $(Y(t), X_+(t), N(t), M(t), P(t))$ be any solution of system (4.6) with initial conditions (4.3).

223 Applying the theorem of differential inequality [51] on the third equation of the system (4.6), we have

224 $N(t) \leq e^{-dt}(N(0) - \frac{A}{d}) + \frac{A}{d}$. Therefore, $\limsup_{t \rightarrow \infty} N(t) \leq \frac{A}{d}$ as $t \rightarrow \infty$. Since $N(t) = Y(t) + X_+(t) +$
225 $X_-(t)$, we can conclude that for t sufficiently large, $0 \leq Y(t), X_+(t) \leq \frac{A}{d}$.

226 Similarly, from the fourth equation of the system (4.6) we have

$$\dot{M}(t) = \mu Y(t - \tau_2) - \mu_0 M(t).$$

This implies that $\dot{M}(t) + \mu_0 M(t) = \mu Y(t - \tau_2)$.

So $\dot{M}(t) + \mu_0 M(t) \leq \mu \frac{A}{d}$, for $t \geq t_0$, for some $t_0 > 0$.

Hence $M(t) \leq M(t_0)e^{-\mu(t-t_0)} + \frac{\mu A}{\mu_0 d}$, for $t \geq t_0$,

so $\limsup_{t \rightarrow \infty} M(t) \leq \frac{\mu A}{\mu_0 d}$.

227 It is straightforward to show that if $P(t)$ is part of a solution of (4.6) then $0 \leq P(t) \leq 1$. Hence,

228 $(Y(t), X_+(t), N(t), M(t), P(t))$ is ultimately bounded above.

229 *4.2.3. Equilibrium Analysis*

230 Now the equilibrium points $(Y^*, X_+^*, N^*, M^*, P^*)$ of the delay model (4.6) satisfy

$$\begin{aligned}
\beta(N^* - X_+^* - Y^*)Y^* + \frac{\beta}{1+\beta_1 M^*} X_+^* Y^* - (\gamma + \alpha + d)Y^* &= 0, \\
\lambda(N^* - X_+^* - Y^*)\frac{M^*}{k+M^*} + p\gamma Y^* - dX_+^* - \lambda_0 X_+^* P^* - \frac{\beta}{1+\beta_1 M^*} X_+^* Y^* &= 0, \\
A - dN^* - \alpha Y^* &= 0, \\
\mu Y^* - \mu_0 M^* &= 0.
\end{aligned} \tag{4.7}$$

231 Here P^* will depend on $\tau_1 (\geq 0)$ through the following equation

$$P^* (\equiv F_1, \text{ say}) = e^{-[d\tau_1 + \lambda_0 P^* \tau_1 + \frac{\beta Y^* \tau_1}{1+\beta_1 M^*}]} \left(\equiv F_2(P^*, \tau_1), \text{ say} \right). \tag{4.8}$$

232 The expression on the righthand side (i.e. $F_2(P^*, \tau_1)$) is a decreasing function of τ_1 such that $F_2(P^*, 0) =$
233 1 , $F_2(P^*, \infty) = 0$. Note that Y^* and M^* depend on τ_1 only through $P^*(\tau_1)$. So there exists at least
234 one positive root (depending on τ_1) of the transcendental equation (4.8) as P^* lies between 0 and 1. A
235 graphical analysis to visualize this scenario is presented in Appendix B.

236 4.3. Stability analysis and local Hopf bifurcation

237 **Case (a) :** $\tau_1 = \tau_2 = 0$

238 In absence of both delays the system (4.6) reduces to the system (3.2).

239 **Case (b) :** $\tau_1 = 0, \tau_2 > 0$

240 Then the system (4.6) reduces to the following system:

$$\begin{aligned}
\frac{dY}{dt} &= \beta(N(t) - X_+(t) - Y(t))Y(t) + \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t) - (\gamma + \alpha + d)Y(t), \\
\frac{dX_+}{dt} &= \lambda(N(t) - X_+(t) - Y(t))\frac{M(t)}{k + M(t)} + p\gamma Y(t) - dX_+(t) - \lambda_0 X_+(t) \\
&\quad - \frac{\beta}{1 + \beta_1 M(t)} X_+(t)Y(t), \\
\frac{dN}{dt} &= A - dN(t) - \alpha Y(t), \\
\frac{dM}{dt} &= \mu Y(t - \tau_2) - \mu_0 M(t).
\end{aligned} \tag{4.9}$$

It has the equilibrium point $E^*(Y^*, X_+^*, N^*, M^*)$ the same as the system (3.2). The variational matrix at the endemic equilibrium $E^*(Y^*, X_+^*, N^*, M^*)$ is

$$J = \begin{pmatrix} -M_1 - \xi & M_2 & M_3 & -M_4 \\ M_5 & -M_6 - \lambda_0 - \xi & M_7 & M_8 \\ -M_9 & 0 & -M_{10} - \xi & 0 \\ \mu e^{-\xi\tau_2} & 0 & 0 & -M_{11} - \xi \end{pmatrix}.$$

241 Here $M_1 = \beta Y^*$, $M_2 = -\beta Y^* + \frac{\beta Y^*}{1+\beta_1 M^*}$, $M_3 = \beta Y^*$, $M_4 = \frac{\beta \beta_1 X_+^* Y^*}{(1+\beta_1 M^*)^2}$, $M_5 = -\frac{\lambda M^*}{k+M^*} + p\gamma - \frac{\beta X_+^*}{1+\beta_1 M^*}$,
 242 $M_6 = \frac{\lambda M^*}{k+M^*} + d + \lambda_0 + \frac{\beta Y^*}{1+\beta_1 M^*}$, $M_7 = \frac{\lambda M^*}{k+M^*}$, $M_8 = \frac{\lambda(N^* - X_+^* - Y^*)k}{(k+M^*)^2} + \frac{\beta \beta_1 X_+^* Y^*}{(1+\beta_1 M^*)^2}$, $M_9 = \alpha$, $M_{10} = d$ and
 243 $M_{11} = \mu_0$.

244 The characteristic equation is

$$\begin{aligned} & \xi^4 + (C_1 + D_1)\xi^3 + (C_2 + D_2)\xi^2 + (C_3 + D_3)\xi + (C_4 + D_4) + \\ & (E_1\xi^2 + (E_2 + F_1)\xi + (E_3 + F_2))e^{-\xi\tau_2} = 0. \end{aligned} \quad (4.10)$$

245 Here

246 $C_1 = M_1 + M_6 + M_{10} + M_{11},$

247 $C_2 = -M_2 M_5 + M_1 M_6 + M_6 M_{10} + M_1 M_{10} + M_3 M_9 + M_6 M_{11} + M_1 M_{11} + M_{10} M_{11},$

248 $C_3 = -M_2 M_5 M_{10} + M_1 M_6 M_{10} + M_3 M_6 M_9 + M_2 M_7 M_9 - M_2 M_5 M_{11} + M_1 M_6 M_{11} + M_6 M_{10} M_{11}$
 249 $+ M_1 M_{10} M_{11} + M_3 M_9 M_{11},$

250 $C_4 = -M_2 M_5 M_{10} M_{11} + M_1 M_6 M_{10} M_{11} + M_3 M_6 M_9 M_{11} + M_2 M_7 M_9 M_{11},$

251 $D_1 = \lambda_0,$

252 $D_2 = \lambda_0(M_{10} + M_{11} + M_1),$

253 $D_3 = \lambda_0(M_1 M_{10} + M_3 M_9 + M_1 M_{11} + M_{10} M_{11}),$

254 $D_4 = \lambda_0(M_3 M_9 M_{11} + M_1 M_{10} M_{11}),$

255 $E_1 = \mu M_4,$

256 $E_2 = -\mu(-M_4 M_{10} + M_2 M_8 - M_4 M_6),$

257 $E_3 = -\mu(M_2 M_8 M_{10} - M_4 M_6 M_{10}),$

258 $F_1 = \lambda_0 \mu M_4,$

259 $F_2 = \lambda_0 \mu M_4 M_{10}.$

260 **Theorem (4.1a)** : The equilibrium point E^* is locally asymptotically stable (LAS) for $\tau_2 < \tau_{2_0}$ where
 261 τ_{2_0} is the minimum positive value of

$$\bar{\tau}_{2_0} = \frac{1}{\omega_{2_0}} \arccos \left\{ \frac{(E_2+F_1)\omega_{2_0}^2[(C_1+D_1)\omega_{2_0}^2 - (C_3+D_3)] + (E_1\omega_{2_0}^2 - E_3 - F_2)[\omega_{2_0}^4 - (C_2+D_4)\omega_{2_0}^2 + (C_4+D_4)]}{(E_1\omega_{2_0}^2 - E_3 - F_2)^2 + (E_2+F_1)^2\omega_{2_0}^2} \right\}$$

for ω_{2_0} corresponding to all positive real roots of (4.12). If the coefficients A_{1i} ($i = 1, 2, 3, 4$) of equation (4.12) do not satisfy the Routh-Hurwitz conditions and $(C_4 + D_4)^2 < (E_3 + F_2)^2$ holds then the delay τ_2 will not affect the stability of the system. If the coefficients A_{1i} ($i = 1, 2, 3, 4$) of equation (4.12) satisfy the Routh-Hurwitz conditions then the system is LAS for all $\tau_2 \geq 0$, provided that it is stable in the absence of delay.

Proof : Put $\xi = i\omega$ in (4.10) and separating real and imaginary parts we get

$$\begin{aligned} (E_1\omega^2 - E_3 - F_2) \cos \omega\tau_2 - (E_2 + F_1)\omega \sin \omega\tau_2 &= \omega^4 - (C_2 + D_2)\omega^2 + (C_4 + D_4), \\ (E_1\omega^2 - E_3 - F_2) \sin \omega\tau_2 + (E_2 + F_1)\omega \cos \omega\tau_2 &= (C_1 + D_1)\omega^3 - (C_3 + D_3)\omega. \end{aligned} \quad (4.11)$$

Eliminating τ_2 from (4.11) and put $\omega^2 = \omega_1$ we get

$$\omega_1^4 + A_{11}\omega_1^3 + A_{12}\omega_1^2 + A_{13}\omega_1 + A_{14} = 0, \quad (4.12)$$

where

$$\begin{aligned} A_{11} &= (C_1 + D_1)^2 - 2(C_2 + D_2), \\ A_{12} &= (C_2 + D_2)^2 + 2(C_4 + D_4) - 2(C_1 + D_1)(C_3 + D_3) - E_1^2, \\ A_{13} &= -2(C_2 + D_2)(C_4 + D_4) + (C_3 + D_3)^2 + 2E_1(E_3 + F_2) - (E_2 + F_1)^2, \\ A_{14} &= (C_4 + D_4)^2 - (E_3 + F_2)^2. \end{aligned}$$

Case (b.1) : If the A_{1i} ($i = 1, 2, 3, 4$) satisfy the Routh-Hurwitz conditions, then (4.12) has no positive real roots. In that case E^* (if it exists) is LAS $\forall \tau_2 > 0$, provided that it is stable in the absence of delay, i.e. τ_2 will not affect the stability of the system, when equation (4.12) has no positive real root.

Case (b.2) : If the A_{1i} ($i = 1, 2, 3, 4$) do not satisfy the Routh-Hurwitz conditions, in that case $A_{14} < 0$ implies that equation (4.12) has at least one positive real root, i.e. if $(C_4 + D_4)^2 < (E_3 + F_2)^2$ then equation (4.10) has a pair of purely imaginary roots say $\pm i\omega_{2_0}$ and for this value of ω_{2_0} we can get the value of τ_{2_n} from equation (4.11) as

$$\tau_{2_n} = \frac{1}{\omega_{2_0}} \arccos \left\{ \frac{(E_2+F_1)\omega_{2_0}^2[(C_1+D_1)\omega_{2_0}^2 - (C_3+D_3)] + (E_1\omega_{2_0}^2 - E_3 - F_2)[\omega_{2_0}^4 - (C_2+D_4)\omega_{2_0}^2 + (C_4+D_4)]}{(E_1\omega_{2_0}^2 - E_3 - F_2)^2 + (E_2+F_1)^2\omega_{2_0}^2} \right\} + \frac{2n\pi}{\omega_{2_0}},$$

where $n = 0, 1, 2, \dots$

By Butler's lemma, [52] the endemic equilibrium remains stable for $\tau_2 < \bar{\tau}_{2_0}$. Without loss of generality suppose that ω_{2_0} represents the value of ω_{2_0} corresponding to τ_{2_0} .

Theorem (4.1b) : If $\Phi_1(\omega_{2_0}) > 0$, the system (4.6) undergoes a Hopf Bifurcation at the positive equilibrium as τ_2 increases through τ_{2_0} , where the expression of $\Phi_1(\omega_{2_0})$ satisfies (4.13).

287 **Proof :** Transversality condition for Hopf-bifurcation :

288 Differentiating (4.10) with respect to τ_2 we get

289
290
$$\frac{d\tau_2}{d\xi} = \frac{4\xi^3 + 3(C_1 + D_1)\xi^2 + 2(C_2 + D_2)\xi + (C_3 + D_3)}{E_1\xi^3 + (E_2 + F_1)\xi^2 + (E_3 + F_2)\xi} e^{\xi\tau_2} + \frac{2E_1\xi + (E_2 + F_1)}{E_1\xi^3 + (E_2 + F_1)\xi^2 + (E_3 + F_2)\xi} - \frac{\tau_2}{\xi},$$

291
292
$$\text{Sgn} \left[\frac{d(\text{Re}\xi)}{d\tau_2} \right]_{\tau_2 = \tau_{2_0}} = \text{Sgn} \left[\text{Re} \left(\frac{d\xi}{d\tau_2} \right)^{-1} \right]_{\xi = i\omega_{2_0}},$$

293
294
$$= \text{Sgn} \left[\text{Re} \frac{[-3(C_1 + D_1)\omega_{2_0}^2 + (C_3 + D_3)] \cos \omega_{2_0} \tau_2 - [-4\omega_{2_0}^3 + 2(C_2 + D_2)\omega_{2_0}] \sin \omega_{2_0} \tau_2}{-(E_2 + F_1)\omega_{2_0}^2 + i\omega_{2_0}[-E_1\omega_{2_0}^2 + (E_3 + F_2)]} + \right.$$

295
296
$$\text{Re} \frac{[-3(C_1 + D_1)\omega_{2_0}^2 + (C_3 + D_3)] \sin \omega_{2_0} \tau_2 + [-4\omega_{2_0}^3 + 2(C_2 + D_2)\omega_{2_0}] \cos \omega_{2_0} \tau_2}{-(E_2 + F_1)\omega_{2_0}^2 + i\omega_{2_0}[-E_1\omega_{2_0}^2 + (E_3 + F_2)]} i +$$

297
298
$$\left. \text{Re} \frac{2i\omega_{2_0} E_1 + (E_2 + F_1)}{-(E_2 + F_1)\omega_{2_0}^2 + i\omega_{2_0}[-E_1\omega_{2_0}^2 + (E_3 + F_2)]} \right],$$

299
300
$$= \text{Sgn} \left[\frac{-[-3(C_1 + D_1)\omega_{2_0}^2 + (C_3 + D_3)]\omega_{2_0} [(E_2 + F_1)\omega_{2_0} \cos \omega_{2_0} \tau_2 + (E_1\omega_{2_0}^2 - E_3 - F_2) \sin \omega_{2_0} \tau_2]}{(E_2 + F_1)^2 \omega_{2_0}^4 + \omega_{2_0}^2 [-E_1\omega_{2_0}^2 + (E_3 + F_2)]^2} \right.$$

301
302
$$+ \frac{[-4\omega_{2_0}^2 + 2(C_2 + D_2)]\omega_{2_0}^2 [(E_2 + F_1)\omega_{2_0} \sin \omega_{2_0} \tau_2 - (E_1\omega_{2_0}^2 - E_3 - F_2) \cos \omega_{2_0} \tau_2]}{(E_2 + F_1)^2 \omega_{2_0}^4 + \omega_{2_0}^2 [-E_1\omega_{2_0}^2 + (E_3 + F_2)]^2}$$

303
304
$$\left. + \frac{\omega_{2_0}^2 [-(E_2 + F_1)^2 + 2E_1(-E_1\omega_{2_0}^2 + E_3 + F_2)]}{(E_2 + F_1)^2 \omega_{2_0}^4 + \omega_{2_0}^2 [-E_1\omega_{2_0}^2 + (E_3 + F_2)]^2} \right].$$

305
306 Using relation (4.11) we get the above expression as

307
308
$$= \text{Sgn} \left[\frac{[3(C_1 + D_1)\omega_{2_0}^2 - (C_3 + D_3)][(C_1 + D_1)\omega_{2_0}^2 - (C_3 + D_3)] + [4\omega_{2_0}^2 - 2(C_2 + D_2)][\omega_{2_0}^4 - (C_2 + D_2)\omega_{2_0}^2 + C_4 + D_4]}{(E_2 + F_1)^2 \omega_{2_0}^2 + [-E_1\omega_{2_0}^2 + (E_3 + F_2)]^2} \right.$$

309
310
$$\left. + \frac{-(E_2 + F_1)^2 + 2E_1(-E_1\omega_{2_0}^2 + E_3 + F_2)}{(E_2 + F_1)^2 \omega_{2_0}^2 + [-E_1\omega_{2_0}^2 + (E_3 + F_2)]^2} \right],$$

311
312
$$= \text{Sgn} \left[\frac{4\omega_{2_0}^6 + B_1\omega_{2_0}^4 + B_2\omega_{2_0}^2 + B_3}{(E_2 + F_1)^2 \omega_{2_0}^2 + [-E_1\omega_{2_0}^2 + (E_3 + F_2)]^2} \right],$$

313
314 where

315
$$B_1 = 3(C_1 + D_1)^2 - 6(C_2 + D_2),$$

316
$$B_2 = 2(C_2 + D_2)^2 + 4(C_4 + D_4) - 4(C_1 + D_1)(C_3 + D_3) - 2E_1^2,$$

317
$$B_3 = (C_3 + D_3)^2 - 2(C_2 + D_2)(C_4 + D_4) - (E_2 + F_1)^2 + 2E_1(E_3 + F_2).$$

318 Let

$$\Phi_1(\omega_{2_0}) = 4\omega_{2_0}^6 + B_1\omega_{2_0}^4 + B_2\omega_{2_0}^2 + B_3. \quad (4.13)$$

319 If $\Phi_1(\omega_{2_0}) > 0$ then $Sgn\left[\frac{d(Re\xi)}{d\tau_2}\right]_{\tau_2=\tau_{2_0}} > 0$, i.e. the transversality condition holds and the system under-
320 goes Hopf bifurcation.

321 **Case (c) :** $\tau_1 > 0, \tau_2 = 0$

The endemic equilibrium of the model (4.6) is $E^*(Y^*, X_+^*, N^*, M^*, P^*)$ (see section 4.2.3). The variational matrix at endemic equilibrium $E^*(Y^*, X_+^*, N^*, M^*, P^*)$ is

$$J = \begin{pmatrix} -M_1 - \xi & M_2 & M_3 & -M_4 & 0 \\ M_5 & -M_6 - m_1 e^{-\xi\tau_1} - \xi & M_7 & M_8 & -M_9 \\ -M_{10} & 0 & -M_{11} - \xi & 0 & 0 \\ \bar{m} & 0 & 0 & -M_{12} - \xi & 0 \\ M_{13} - m_2 e^{-\xi\tau_1} & 0 & 0 & -M_{14} + m_3 e^{-\xi\tau_1} & -M_{15} + m_4 e^{-\xi\tau_1} - \xi \end{pmatrix}.$$

322 Here $M_1 = \beta Y^*$, $M_2 = -\beta Y^* + \frac{\beta Y^*}{1+\beta_1 M^*}$, $M_3 = \beta Y^*$, $M_4 = \frac{\beta\beta_1 X_+^* Y^*}{(1+\beta_1 M^*)^2}$, $M_5 = -\frac{\lambda M^*}{k+M^*} + p\gamma - \frac{\beta X_+^*}{1+\beta_1 M^*}$,
323 $M_6 = \frac{\lambda M^*}{k+M^*} + d + \frac{\beta Y^*}{1+\beta_1 M^*}$, $M_7 = \frac{\lambda M^*}{k+M^*}$, $M_8 = \frac{\lambda(N^* - X_+^* - Y)k}{(k+M^*)^2} + \frac{\beta\beta_1 X_+^* Y^*}{(1+\beta_1 M^*)^2}$, $M_9 = \lambda_0 X_+^*$, $M_{10} = \alpha$, $M_{11} = d$,
324 $M_{12} = \mu_0$, $M_{13} = m_2 = \frac{\beta P^*}{1+\beta_1 M^*}$, $M_{14} = m_3 = \frac{\beta\beta_1 Y^* P^*}{(1+\beta_1 M^*)^2}$, $M_{15} = m_1 = m_4 = \lambda_0 P^*$ and $\bar{m} = \mu$.

325 The characteristic equation is

$$\begin{aligned} & [\xi^5 + A_1\xi^4 + (A_2 + F_1)\xi^3 + (A_3 + F_2)\xi^2 + (A_4 + F_3)\xi + (A_5 + F_4)]e^{\xi\tau_1} + \\ & [C_1\xi^3 + C_2\xi^2 + (B_1 + C_3)\xi + (B_2 + C_4)]e^{-\xi\tau_1} + [D_1\xi^4 + D_2\xi^3 + \\ & (D_3 + E_1)\xi^2 + (D_4 + E_2)\xi + (D_5 + E_3)] = 0. \end{aligned} \quad (4.14)$$

326 Here $A_1, A_2, A_3, A_5, B_1, B_2, \dots, F_4$ are given in Appendix A.

327 **Theorem (4.2a) :** Let $(A_5 + B_2 + C_4 + F_4)^2 < (D_5 + E_3)^2$ then the equilibrium E^* is LAS for $\tau_1 \in (0, \tau_{1_0})$
328 where τ_{1_0} is the minimum positive value of

$$\bar{\tau}_{1_0} = \frac{1}{\omega_{1_0}} \left[\arccos\left(-\frac{\bar{A}_{22} \bar{A}_{26} + \bar{A}_{23} \bar{A}_{25}}{\bar{A}_{21} \bar{A}_{25} + \bar{A}_{22} \bar{A}_{24}}\right) \right]$$

329 for ω_{1_0} corresponding to all positive real roots of (4.16) and the coefficients \bar{A}_{2i} ($i = 1, 2, 3, 4, 5, 6$) are
330 described below, provided it is stable in the absence of delay.

331 **Proof :** Put $\xi = i\omega$ in (4.14) and separating real and imaginary parts we get

$$\begin{aligned} A_{21} \cos \omega\tau_1 - A_{22} \sin \omega\tau_1 + A_{23} &= 0, \\ A_{24} \cos \omega\tau_1 + A_{25} \sin \omega\tau_1 + A_{26} &= 0, \end{aligned} \quad (4.15)$$

332 where

$$333 \quad A_{21} = A_1\omega^4 - (A_3 + C_2 + F_2)\omega^2 + (A_5 + B_2 + C_4 + F_4),$$

$$334 \quad A_{22} = \omega^5 - (A_2 - C_1 + F_1)\omega^3 + (A_4 - B_1 - C_3 + F_3)\omega,$$

$$335 \quad A_{23} = D_1\omega^4 - (D_3 + E_1)\omega^2 + (D_5 + E_3),$$

$$336 \quad A_{24} = \omega^5 - (A_2 + C_1 + F_1)\omega^3 + (A_4 + B_1 + C_3 + F_3)\omega,$$

$$337 \quad A_{25} = A_1\omega^4 - (A_3 - C_2 + F_2)\omega^2 + (A_5 - B_2 - C_4 + F_4),$$

$$338 \quad A_{26} = -D_2\omega^3 + (D_4 + E_2)\omega.$$

339 Eliminating τ_1 from (4.15) we get

$$H_1(\omega) = (A_{21}A_{25} + A_{22}A_{24})^2 - (A_{22}A_{26} + A_{23}A_{25})^2 - (A_{23}A_{24} - A_{21}A_{26})^2 = 0. \quad (4.16)$$

340 If $(A_5 + B_2 + C_4 + F_4)^2 - (D_5 + E_3)^2 < 0$ then $H_1(0) < 0$ and $H_1(\infty) = +\infty$. So equation (4.16) has at
341 least one positive real root ω_{10} .

342 When $\omega = \omega_{10}$, equations (4.15) can be written as

$$\begin{aligned} \bar{A}_{21} \cos \omega_{10} \tau_1 - \bar{A}_{22} \sin \omega_{10} \tau_1 + \bar{A}_{23} &= 0, \\ \bar{A}_{24} \cos \omega_{10} \tau_1 + \bar{A}_{25} \sin \omega_{10} \tau_1 + \bar{A}_{26} &= 0. \end{aligned} \quad (4.17)$$

343 Here $\bar{A}_{21}, \bar{A}_{22}, \bar{A}_{23}, \bar{A}_{24}, \bar{A}_{25}$ and \bar{A}_{26} are obtained by substituting $\omega = \omega_{10}$ into $A_{21}, A_{22}, A_{23}, A_{24}, A_{25}$ and A_{26} .

344 Equations (4.18) are simplified to give

$$\tau'_{1n} = \frac{1}{\omega_{10}} \left[\arccos \left(-\frac{\bar{A}_{22} \bar{A}_{26} + \bar{A}_{23} \bar{A}_{25}}{\bar{A}_{21} \bar{A}_{25} + \bar{A}_{22} \bar{A}_{24}} \right) \right] + \frac{2n\pi}{\omega_{10}}, \quad n = 0, 1, 2, \dots,$$

345 here $i\omega_{10}$ is a purely imaginary root of equation (4.14).

346 If $(A_5 + B_2 + C_4 + F_4)^2 - (D_5 + E_3)^2 < 0$ then the equilibrium $E^*(Y^*, X^*, N^*, M^*, P^*)$ is LAS for
347 $\tau_1 < \tau_{10}$. Without loss of generality suppose that ω_{10} represents the value of ω_{10} corresponding to τ_{10} .

348 **Theorem (4.2b)** : If $\Phi_2(\omega_{10}) > 0$, where $\Phi_2(\omega_{10})$ satisfies (4.18) the system (4.6) undergoes a Hopf
349 Bifurcation at the positive equilibrium as τ_1 increases through τ_{10} .

350 **Proof** : Transversality condition for Hopf-bifurcation :

351 Differentiating (4.14) with respect to τ_1 , we get $\frac{d\tau_1}{d\xi} =$

$$352 \quad \frac{[5\xi^4 + 4A_1\xi^3 + 3(A_2 + F_1)\xi^2 + 2(A_3 + F_2)\xi + (A_4 + F_3)]e^{\xi\tau_1} + [4D_1\xi^3 + 3D_2\xi^2 + 2(D_3 + E_1)\xi + (D_4 + E_2)] + [3C_1\xi^2 + 2C_2\xi + (B_1 + C_3)]e^{-\xi\tau_1}}{[D_1\xi^5 + D_2\xi^4 + (D_3 + E_1)\xi^3 + (D_4 + E_2)\xi^2 + (D_5 + E_3)\xi] + 2[C_1\xi^4 + C_2\xi^3 + (B_1 + C_3)\xi^2 + (B_2 + C_4)\xi]e^{-\xi\tau_1}} - \frac{\tau_1}{\xi},$$

353

$$354 \quad Sgn \left[\frac{d(Re\xi)}{d\tau_1} \right]_{\tau_1=\tau_{10}} = Sgn \left[Re \left(\frac{d\xi}{d\tau_1} \right)^{-1} \right]_{\xi=i\omega_{10}} = Sgn \left[Re \frac{P_{11} + iP_{12}}{G_{11} + iG_{12}} + Re \frac{i\tau_1}{\omega_{10}} \right] = Sgn \left[\frac{P_{11}G_{11} + P_{12}G_{12}}{G_{11}^2 + G_{12}^2} \right].$$

355

356 P_{11}, P_{12}, G_{11} and G_{12} are given in Appendix A. Let

$$\Phi_2(\omega_{1_0}) = P_{11}G_{11} + P_{12}G_{12}. \quad (4.18)$$

357 If $\Phi_2(\omega_{1_0}) > 0$ then $Sgn\left[\frac{d(Re\xi)}{d\tau_1}\right]_{\tau_1=\tau_{1_0}} > 0$, i.e. the transversality condition holds and the system under-
358 goes Hopf bifurcation.

359 **Case (d)** : $\tau_1 > 0$ and τ_2 fixed in $(0, \tau_{2_0})$

The endemic equilibrium of the model (4.6) is $E^*(Y^*, X_+^*, N^*, M^*, P^*)$ (see section 4.2.3). The variational matrix at the endemic equilibrium $E^*(Y^*, X_+^*, N^*, M^*, P^*)$ is

$$J = \begin{pmatrix} -M_1 - \xi & M_2 & M_3 & -M_4 & 0 \\ M_5 & -M_6 - m_1 e^{-\xi\tau_1} - \xi & M_7 & M_8 & -M_9 \\ -M_{10} & 0 & -M_{11} - \xi & 0 & 0 \\ \bar{m}e^{-\xi\tau_2} & 0 & 0 & -M_{12} - \xi & 0 \\ M_{13} - m_2 e^{-\xi\tau_1} & 0 & 0 & -M_{14} + m_3 e^{-\xi\tau_1} & -M_{15} + m_4 e^{-\xi\tau_1} - \xi \end{pmatrix}.$$

360 The characteristic equation is

$$\begin{aligned} & [\xi^5 + A_1\xi^4 + A_2\xi^3 + A_3\xi^2 + A_4\xi + A_5]e^{\xi\tau_1} + [B_1\xi + B_2]e^{-\xi(\tau_1+\tau_2)} + \\ & [C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4]e^{-\xi\tau_1} + [D_1\xi^4 + D_2\xi^3 + D_3\xi^2 + D_4\xi + D_5] + \\ & [E_1\xi^2 + E_2\xi + E_3]e^{-\xi\tau_2} + [F_1\xi^3 + F_2\xi^2 + F_3\xi + F_4]e^{\xi(\tau_1-\tau_2)} = 0. \end{aligned} \quad (4.19)$$

361 Here M_{i_1} ($i_1 = 1-15$), m_{i_2} ($i_2 = 1-4$), \bar{m} , A_{i_3} ($i_3 = 1-5$), B_{i_4} ($i_4 = 1-2$), C_{i_5} ($i_5 = 1-4$), D_{i_6} ($i_6 = 1-5$),
362 E_{i_7} ($i_7 = 1-3$), F_{i_8} ($i_8 = 1-4$) are the same as described in Case (c).

363 **Theorem (4.3a)** : Let $(A_5 + B_2 + C_4 + F_4)^2 < (D_5 + E_3)^2$ and $\tau_2 \in [0, \tau_{2_0})$ then the equilibrium E^* is
364 LAS for $\tau_1 \in (0, \tau'_{1_0})$ where

$$\tau'_{1_0} = \frac{1}{\omega_{3_0}} \left[\arccos\left(-\frac{\bar{A}_{32}}{A_{31}} \frac{\bar{A}_{36} + \bar{A}_{33}}{A_{35} + \bar{A}_{32}} \frac{\bar{A}_{35}}{A_{34}}\right) \right]$$

365 and the coefficients \bar{A}_{3i} ($i = 1, 2, 3, 4, 5, 6$) are described below.

366 **Proof** : It is assumed that with equation (4.19), τ_2 is in its stable interval and τ_1 is considered as a
367 parameter. Put $\xi = i\omega$ in (4.19) and separating real and imaginary parts we get

$$\begin{aligned} A_{31} \cos \omega\tau_1 - A_{32} \sin \omega\tau_1 + A_{33} &= 0, \\ A_{34} \cos \omega\tau_1 + A_{35} \sin \omega\tau_1 + A_{36} &= 0. \end{aligned} \quad (4.20)$$

368 Here

$$369 \quad A_{31} = [A_1\omega^4 - C_2\omega^3 - A_3\omega^2 + (A_5 + C_4)] + [-F_2\omega^2 + (B_2 + F_4)] \cos \omega\tau_2 + [-F_1\omega^3 + (B_1 + F_3)\omega] \sin \omega\tau_2,$$

$$\begin{aligned}
370 \quad A_{32} &= [\omega^5 - (A_2 - C_1)\omega^3 + (A_4 - C_3)\omega] + [-F_1\omega^3 - (B_1 - F_3)\omega] \cos \omega\tau_2 + [F_2\omega^2 + (B_2 - F_4)] \sin \omega\tau_2, \\
371 \quad A_{33} &= [D_1\omega^4 - D_3\omega^2 + D_5] + [-E_1\omega^2 + E_3] \cos \omega\tau_2 + E_2\omega \sin \omega\tau_2, \\
372 \quad A_{34} &= [\omega^5 - (A_2 + C_1)\omega^3 + (A_4 + C_3)\omega] + [-F_1\omega^3 + (B_1 + F_3)\omega] \cos \omega\tau_2 + [F_2\omega^2 - (B_2 + F_4)] \sin \omega\tau_2, \\
373 \quad A_{35} &= [A_1\omega^4 + C_2\omega^3 - A_3\omega^2 + (A_5 - C_4)] + [-F_2\omega^2 - (B_2 - F_4)] \cos \omega\tau_2 + [-F_1\omega^3 - (B_1 - F_3)\omega] \sin \omega\tau_2, \\
374 \quad A_{36} &= [-D_2\omega^3 + D_4\omega] + E_2\omega \cos \omega\tau_2 - [-E_1\omega^2 + E_3] \sin \omega\tau_2.
\end{aligned}$$

375 Eliminating τ_1 from (4.20) we get

$$H_2(\omega) = (A_{31}A_{35} + A_{32}A_{34})^2 - (A_{32}A_{36} + A_{33}A_{35})^2 - (A_{33}A_{34} - A_{31}A_{36})^2 = 0. \quad (4.21)$$

376 Note that if $(A_5 + B_2 + C_4 + F_4)^2 - (D_5 + E_3)^2 < 0$ then $H_2(0) < 0$ and $H_2(\infty) = +\infty$.

377 Now the above equation (4.21) is a transcendental equation in ω . The form of equation (4.21) is very
378 complicated and it is difficult to predict the nature of its roots. Without going into detailed analysis with
379 (4.21) it is assumed there exists at least one real positive root ω_{3_0} .

380 When $\omega = \omega_{3_0}$, equation (4.20) can be written as

$$\begin{aligned}
&\bar{A}_{31} \cos \omega_{3_0} \tau_1 - \bar{A}_{32} \sin \omega_{3_0} \tau_1 + \bar{A}_{33} = 0, \\
&\bar{A}_{34} \cos \omega_{3_0} \tau_1 + \bar{A}_{35} \sin \omega_{3_0} \tau_1 + \bar{A}_{36} = 0,
\end{aligned} \quad (4.22)$$

381 where $\bar{A}_{31}, \bar{A}_{32}, \bar{A}_{33}, \bar{A}_{34}, \bar{A}_{35}, \bar{A}_{36}$ are obtained by substituting $\omega = \omega_{3_0}$ into $A_{31}, A_{32}, A_{33}, A_{34}, A_{35}$ and
382 A_{36} .

383 Equations (4.22) are simplified to give

$$\tau'_{1_n} = \frac{1}{\omega_{3_0}} \left[\arccos \left(-\frac{\bar{A}_{32} \bar{A}_{36} + \bar{A}_{33} \bar{A}_{35}}{\bar{A}_{31} \bar{A}_{35} + \bar{A}_{32} \bar{A}_{34}} \right) \right] + \frac{2n\pi}{\omega_{3_0}}, \quad n = 0, 1, 2, \dots$$

384 here $i\omega_{3_0}$ is a purely imaginary root of equation (4.19).

385 If $(A_5 + B_2 + C_4 + F_4)^2 < (D_5 + E_3)^2$ and $\tau_2 \in [0, \tau_{2_0})$, then the equilibrium $E^*(Y^*, X^*, N^*, M^*, P^*)$ is
386 LAS for $\tau_1 \in (0, \tau'_{1_0})$. Without loss of generality suppose that ω_{3_0} represents the value of ω_{3_0} corresponding
387 to τ'_{1_0} .

388 **Theorem (4.3b)** : If $\Phi_3(\omega_{3_0}) > 0$, the system (4.6) undergoes a Hopf Bifurcation at the positive equilib-
389 rium as τ_1 increases through τ'_{1_0} , where the expression of $\Phi_3(\omega_{3_0})$ satisfies (4.23).

390 **Proof** : Transversality condition for Hopf-bifurcation :

391 Differentiating (4.19) with respect to τ_1 we get

$$392 \quad Sgn \left[\frac{d(Re\xi)}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}} = Sgn \left[Re \left(\frac{d\xi}{d\tau_1} \right)^{-1} \right]_{\xi=i\omega_{3_0}} = Sgn \left[Re \frac{P_{21} + iP_{22}}{G_{21} + iG_{22}} + Re \frac{i\tau'_{1_0}}{\omega_{3_0}} \right] = Sgn \left[\frac{P_{21}G_{21} + P_{22}G_{22}}{G_{21}^2 + G_{22}^2} \right].$$

394

395 Here P_{21}, P_{22}, G_{21} and G_{22} are given in the Appendix. Let

$$\Phi_3(\omega_{3_0}) = P_{21}G_{21} + P_{22}G_{22}. \quad (4.23)$$

396 If $\Phi_3(\omega_{3_0}) > 0$ then $Sgn\left[\frac{d(Re\xi)}{d\tau_1}\right]_{\tau_1=\tau'_{1_0}} > 0$, i.e. the transversality condition holds and the system under-
397 goes Hopf bifurcation.

398 **Case (e)** : $\tau_2 > 0$ and τ_1 fixed in $(0, \tau_{1_0})$

399 In a similar way as in Case (d) we can find the characteristic equation as

$$\begin{aligned} & [\xi^5 + A_1\xi^4 + A_2\xi^3 + A_3\xi^2 + A_4\xi + A_5] + [B_1\xi + B_2]e^{-\xi(2\tau_1+\tau_2)} + \\ & [C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4]e^{-2\xi\tau_1} + [D_1\xi^4 + D_2\xi^3 + D_3\xi^2 + D_4\xi + D_5]e^{-\xi\tau_1} + \\ & [E_1\xi^2 + E_2\xi + E_3]e^{-\xi(\tau_1+\tau_2)} + [F_1\xi^3 + F_2\xi^2 + F_3\xi + F_4]e^{-\xi\tau_2} = 0. \end{aligned} \quad (4.24)$$

400 **Theorem (4.4a)** : Let $(A_5 + C_4 + D_5)^2 < (B_2 + E_3 + F_4)^2$ and $\tau_1 \in [0, \tau_{1_0})$ then the equilibrium E^* is

401 LAS for $\tau_2 \in (0, \tau'_{2_0})$ where τ'_{2_0} is the minimum value of

$$\tau'_{2_0} = \frac{1}{\omega_{4_0}} \left[\arccos\left(-\frac{\bar{A}_{42} \bar{A}_{46} + \bar{A}_{43} \bar{A}_{45}}{\bar{A}_{41} \bar{A}_{45} + \bar{A}_{42} \bar{A}_{44}}\right) \right]$$

402 over ω_{4_0} corresponding to all positive real roots of (4.26) and the coefficients \bar{A}_{4i} , ($i = 1, 2, 3, 4, 5, 6$) are
403 described below.

404 **Proof** : It is considered that with equation (4.24), τ_1 is in its stable interval and τ_2 is considered as a
405 parameter. Put $\xi = i\omega$ in (4.24) and separating real and imaginary parts we get

$$\begin{aligned} A_{41} \cos \omega\tau_2 - A_{42} \sin \omega\tau_2 + A_{43} &= 0, \\ A_{44} \cos \omega\tau_2 + A_{45} \sin \omega\tau_2 + A_{46} &= 0. \end{aligned} \quad (4.25)$$

406 Here

$$\begin{aligned} A_{41} &= [-F_2\omega^2 + F_4] - [E_1\omega^2 - E_3] \cos \omega\tau_1 + E_2\omega \sin \omega\tau_1 + B_2 \cos 2\omega\tau_1 + B_1\omega \sin 2\omega\tau_1, \\ A_{42} &= [F_1\omega^3 - F_3\omega] - E_2\omega \cos \omega\tau_1 - [E_1\omega^2 - E_3] \sin \omega\tau_1 - B_1\omega \cos 2\omega\tau_1 + B_2 \sin 2\omega\tau_1, \\ A_{43} &= [A_1\omega^4 - A_3\omega^2 + A_5] + [D_1\omega^4 - D_3\omega^2 + D_5] \cos \omega\tau_1 - [D_2\omega^3 - D_4\omega] \sin \omega\tau_1 \\ &\quad - [C_2\omega^2 - C_4] \cos 2\omega\tau_1 - [C_1\omega^3 - C_3\omega] \sin 2\omega\tau_1, \\ A_{44} &= [-F_1\omega^3 + F_3\omega] + E_2\omega \cos \omega\tau_1 + [E_1\omega^2 - E_3] \sin \omega\tau_1 + B_1\omega \cos 2\omega\tau_1 - B_2 \sin 2\omega\tau_1, \\ A_{45} &= [F_2\omega^2 - F_4] + [E_1\omega^2 - E_3] \cos \omega\tau_1 - E_2\omega \sin \omega\tau_1 - B_2 \cos 2\omega\tau_1 - B_1\omega \sin 2\omega\tau_1, \\ A_{46} &= [\omega^5 - A_2\omega^3 + A_4\omega] - [D_2\omega^3 - D_4\omega] \cos \omega\tau_1 - [D_1\omega^4 - D_3\omega^2 + D_5] \sin \omega\tau_1 \\ &\quad - [C_1\omega^3 - C_3\omega] \cos 2\omega\tau_1 + [C_2\omega^2 - C_4] \sin 2\omega\tau_1. \end{aligned}$$

415 Eliminating τ_1 from (4.20) we get

$$H_2(\omega) = (A_{42}A_{46} + A_{43}A_{45})^2 + (A_{43}A_{44} - A_{41}A_{46})^2 - (A_{41}A_{45} + A_{42}A_{44})^2 = 0. \quad (4.26)$$

416 Note that if $(A_5 + C_4 + D_5)^2 < (B_2 + E_3 + F_4)^2 < 0$ then $H_2(0) < 0$ and $H_2(\infty) = +\infty$.

417 Again we assume that there exists at least one real positive root ω_{40} . When $\omega = \omega_{40}$ equation (4.25)
418 can be written as

$$\begin{aligned}\bar{A}_{41} \cos \omega_{40} \tau_2 - \bar{A}_{42} \sin \omega_{40} \tau_2 + \bar{A}_{43} &= 0, \\ \bar{A}_{44} \cos \omega_{40} \tau_2 + \bar{A}_{45} \sin \omega_{40} \tau_2 + \bar{A}_{46} &= 0,\end{aligned}\tag{4.27}$$

419 where $\bar{A}_{41}, \bar{A}_{42}, \dots, \bar{A}_{46}$ are obtained by substituting $\omega = \omega_{40}$ into $A_{41}, A_{42}, \dots, A_{46}$.

420 Equations (4.27) are simplified to give

$$\tau'_{2n} = \frac{1}{\omega_{40}} \left[\arccos \left(-\frac{\bar{A}_{42} \bar{A}_{46} + \bar{A}_{43} \bar{A}_{45}}{\bar{A}_{41} \bar{A}_{45} + \bar{A}_{42} \bar{A}_{44}} \right) \right] + \frac{2n\pi}{\omega_{40}}; \quad n = 0, 1, 2, \dots$$

421 here $i\omega_{40}$ is a purely imaginary root of equation (4.24).

422 If $(A_5 + C_4 + D_5)^2 < (B_2 + E_3 + F_4)^2$ and $\tau_1 \in [0, \tau_{10})$, then the equilibrium $E^*(Y^*, X_+^*, N^*, M^*, P^*)$ is
423 LAS for $\tau_2 \in (0, \tau'_{20})$. Without loss of generality suppose that ω_{40} represents the value of ω_{40} corresponding
424 to τ'_{20} .

425 **Theorem (4.4b)** : If $\Phi_4(\omega_{40}) > 0$, the system (4.6) undergoes a Hopf Bifurcation at the positive equilib-
426 rium as τ_2 increases through τ'_{20} , where $\Phi_4(\omega_{40})$ satisfies (4.28).

427 **Proof** : Transversality condition for Hopf-bifurcation :

428 Differentiating (4.24) with respect to τ_2 we get

$$429 \quad Sgn \left[\frac{d(Re\xi)}{d\tau_2} \right]_{\tau_2=\tau'_{20}} = Sgn \left[Re \left(\frac{d\xi}{d\tau_2} \right)^{-1} \right]_{\xi=i\omega_{40}} = Sgn \left[Re \frac{P_{31} + iP_{32}}{G_{31} + iG_{32}} + Re \frac{i\tau'_{20}}{\omega_{40}} \right] = Sgn \left[\frac{P_{31}G_{31} + P_{32}G_{32}}{G_{31}^2 + G_{32}^2} \right],$$

430 where P_{31}, P_{32}, G_{31} and G_{32} are given in the Appendix. Let

$$431 \quad \Phi_4(\omega_{40}) = P_{31}G_{31} + P_{32}G_{32}.\tag{4.28}$$

432 If $\Phi_4(\omega_{40}) > 0$ then $Sgn \left[\frac{d(Re\xi)}{d\tau_2} \right]_{\tau_2=\tau'_{20}} > 0$, i.e. the transversality condition holds and the system under-
433 goes Hopf bifurcation.

434 4.4. Permanence

435 Biologically, persistence of a system means the survival of all populations of the system in future time.
436 Mathematically, persistence of a system means that strictly positive solutions do not have omega limit
437 points on the boundary of the non-negative cone. Butler, Freedman and Waltman [53], [54] developed
438 the following definition of persistence:
439

Definition 4.4.1. System (4.6) is said to be permanent if there are positive constants l, L such that each positive solution $(Y(t), X_+(t), N(t), M(t), P(t))$ of system (4.6) with initial conditions corresponding to (4.3) satisfies

$$\begin{aligned} l &\leq \liminf_{t \rightarrow +\infty} Y(t) \leq \limsup_{t \rightarrow +\infty} Y(t) \leq L, \\ l &\leq \liminf_{t \rightarrow +\infty} X_+(t) \leq \limsup_{t \rightarrow +\infty} X_+(t) \leq L, \\ l &\leq \liminf_{t \rightarrow +\infty} N(t) \leq \limsup_{t \rightarrow +\infty} N(t) \leq L, \\ l &\leq \liminf_{t \rightarrow +\infty} M(t) \leq \limsup_{t \rightarrow +\infty} M(t) \leq L, \\ l &\leq \liminf_{t \rightarrow +\infty} P(t) \leq \limsup_{t \rightarrow +\infty} P(t) \leq L. \end{aligned}$$

440 In order to prove permanence of system (4.6), we present the theory of permanence of infinite dimen-
441 sional systems from Theorem 4.1 of Hale and Waltman [55]. Let X be a complete metric space. Suppose
442 that $X^0 \in X, X_0 \in X, X^0 \cap X_0 = \emptyset$. Assume that $T(t)$ is a C_0 semigroup on X satisfying

$$\begin{aligned} T(t) : X^0 &\rightarrow X^0, \\ T(t) : X_0 &\rightarrow X_0. \end{aligned} \tag{4.29}$$

443 Let $T_b(t) = T(t)|_{X_0}$ and let A_b be the global attractor for $T_b(t)$.

444 **Lemma 4.4.1** [55]. Suppose that $T(t)$ satisfies (4.29) and we have the following

445 (i) there is a $t_0 \geq 0$ such that $T(t)$ is compact for $t > t_0$;

446 (ii) $T(t)$ is point dissipative in X ;

(iii) $\bar{A}_b = \bigcup_{x \in A_b} w(x)$ is isolated and has an acyclic covering L , where

$$L = \{L_1, L_2, \dots, L_n\};$$

447 (iv) $W^s(L_i) \cap X^0 = \emptyset$ for $i = 1, 2, \dots, n$.

448 Then X_0 is a uniform repeller with respect to X^0 , i.e., there is an $\epsilon_0 > 0$ such that, for any $x \in X^0$,
449 $\liminf_{t \rightarrow +\infty} \tilde{d}(T(t)x, X_0) \geq \epsilon$, where \tilde{d} is the distance of $T(t)x$ from X_0 .

450 **Theorem 4.4.1.** If $\frac{\beta\epsilon_0}{(\gamma+\alpha+d)} + 1 < R_0 < \frac{\beta\epsilon_0+p\gamma+d+\lambda_0}{(\gamma+\alpha+d)} + 1$, then the system (4.6) is permanent.

Proof : We begin by showing that the boundary planes of \mathbb{R}_+^5 repel the positive solutions to system (4.2) uniformly. Let us define C_0 to be

$$\{(\psi_1, \psi_2, \psi_3, \psi_4) \in C([-\tau, 0], \mathbb{R}_+^4 \times [0, 1]) : \psi_1(\theta_1) \neq 0, \psi_2(\theta_1) = 0, \psi_3(\theta_1) = 0, \psi_4(\theta_1) = 0 \text{ and } \psi_5(\theta_1) = 0\}.$$

451 If $C^0 = \text{int}C([- \tau, 0], \mathbb{R}_+^4 \times [0, 1])$, it suffices to show that there exists an ϵ_0 such that for all solutions u_t of
452 system (4.2) initiating from C^0 , $\liminf_{t \rightarrow +\infty} \tilde{d}(u_t, C_0) \geq \epsilon_0$. To this end we verify below that the conditions
453 of Lemma 4.4.1 are satisfied. It is easy to see that C_0 and C^0 are positive invariant. Moreover, conditions
454 (i) and (ii) of Lemma 4.4.1 are clearly satisfied. Thus, we only need to verify conditions (iii) and (iv).

455 There is a constant solution E_0 in C_0 . That is $X_-(t) = \frac{A}{d}$, $X_+(t) = 0$, $Y(t) = M(t) = P(t) = 0$. If
456 $(X_-(t), X_+(t), Y(t), M(t), P(t))$ is a solution of system (4.2) initiating in C_0 , then $X_-(t) \rightarrow \frac{A}{d}$, $X_+(t) \rightarrow$
457 0 , $Y(t) \rightarrow 0$, $M(t) \rightarrow 0$ and $P(t) \rightarrow 0$ as $t \rightarrow \infty$. It is obvious that E_0 is isolated invariant.

458 We now show that $W^s(E_0) \cap C^0 = \emptyset$. Assuming the contrary, i.e. $W^s(E_0) \cap C^0 \neq \emptyset$, then there exists
459 a positive solution $(X_-(t), X_+(t), Y(t), M(t), P(t))$ of the system (4.2) such that $(Y(t), X_+(t), N(t), M(t),$
460 $P(t)) \rightarrow (0, 0, \frac{A}{d}, 0, 0)$ as $t \rightarrow +\infty$. Let us choose $\epsilon_0 > 0$ small enough such that $R_0 > 1 + \epsilon_0$. Let $t_0 > 0$
461 be sufficiently large such that $\frac{A}{d} - \epsilon_0 < X_-(t) < \frac{A}{d} + \epsilon_0$ for $t > t_0 - \tau$. Then we have, for $t > t_0$,

$$\frac{dY}{dt} \geq \beta \left(\frac{A}{d} - \epsilon_0 - X_+(t) - Y(t) \right) Y(t) + \frac{\beta}{1 + \beta_1 M(t)} X_+(t) Y(t) - (\gamma + \alpha + d) Y(t). \quad (4.30)$$

462

$$\text{Hence} \quad \frac{dY}{dt} \geq \beta \left(\frac{A}{d} - \epsilon_0 - X_+(t) - Y(t) \right) Y(t) - (\gamma + \alpha + d) Y(t), \quad (4.31)$$

463

$$\text{or} \quad \frac{1}{Y} \frac{dY}{dt} \geq \beta \left[\left(\frac{A}{d} - \epsilon_0 \right) - X_+(t) - Y(t) \right] - (\gamma + \alpha + d). \quad (4.32)$$

464 For X_+, Y sufficiently small and $R_0 > 1 + \frac{\beta \epsilon_0}{(\gamma + \alpha + d)}$, $\frac{1}{Y} \frac{dY}{dt} \geq \epsilon_1 > 0$ for some $\epsilon_1 > 0$. Hence $\exists t_1 \geq t_0$
465 such that $\frac{1}{Y} \frac{dY}{dt} \geq \epsilon_1 > 0$ for $T \geq t_1$. So $Y(t) \geq Y(t_1) e^{\epsilon_1(t-t_1)}$ for $t \geq t_1$ and $Y(t_1) > 0$. This contradicts
466 $Y(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore $(Y(t), X_+(t), N(t), M(t), P(t)) \rightarrow (0, 0, \frac{A}{d}, 0, 0)$, which is a contradiction.
467 Hence $W^s(E_0) \cap C^0 = \emptyset$. At this time, we are able to conclude from Lemma 4.4.1 that C_0 repels the
468 positive solutions of the system (4.2) uniformly, then the conclusion of Theorem 4.4.1 follows.

469 5. Numerical simulations

470 To observe the dynamics of the system, numerical experiments were carried out using Matlab. We
471 base our parameters on the spread of Pneumococcus amongst children under two in Scotland [56]. Pneu-
472 mococcus is a bacterial disease which has no permanent immunity. Hence an SIS model is suitable. We
473 try to illustrate the analytical results of this paper with realistic parameter values although the objective
474 is more to illustrate the analytical results rather than obtain accurate predictions.

475 Lamb et al. estimate the size of the population at risk as $N = 150,000$ and the per capita death rate
476 as $d = 1.3736 \times 10^{-3} \text{ day}^{-1}$ giving $A = dN = 206.04 \text{ day}^{-1}$. The infectious period $\frac{1}{\gamma}$ is given by Weir
477 [57] as $\frac{1}{\gamma} = 7.1$ weeks giving $\gamma = 0.1408 \text{ week}^{-1} = 0.02011 \text{ day}^{-1}$. There is extremely low disease-related

478 mortality from Pneumococcus carriage so we take $\alpha = 0.0 \text{ day}^{-1}$. A Pneumococcus study by Zhang
479 et al. [58] gives the basic reproduction number R_0 to be in the range 1.8-2.2. We take $R_0 = 2$ which
480 then implies that $\beta = 2.865 \times 10^{-7} \text{ day}^{-1}$. The remaining parameters are concerned with the disease
481 awareness program and as we do not have the data on this these are estimated hypothetically as follows:
482 $\lambda = 0.9 \text{ day}^{-1}$, $\lambda_0 = 0.3 \text{ day}^{-1}$, $\mu = 1.3736 \times 10^{-3} \text{ day}^{-1}$, $\mu_0 = 0.01 \text{ day}^{-1}$, $k = 500$, $\beta_1 = 1$ and $p = 0.6$.

483 For the above set of parameter values we obtain $E^* = (1787.4, 73524, 150000, 245.51)$, $\sigma_1 = 0.6097 > 0$,
484 $\sigma_2 = 0.0068 > 0$, $\sigma_3 = 3.1364 \times 10^{-5} > 0$, $\sigma_4 = 3.1425 \times 10^{-8} > 0$, $\sigma_1\sigma_2 - \sigma_3 = 0.0041 > 0$ and
485 $\sigma_1\sigma_2\sigma_3 - \sigma_3^2 - \sigma_1^2\sigma_4 = 1.179 \times 10^{-7} > 0$. Hence this clearly indicates that for the above set of parameter
486 values the system is LAS around the positive interior equilibrium. Figure 1 illustrates that, as expected,
487 simulations carried out for a long time appear to converge to this equilibrium. For the above parameter
488 values and initial conditions we observe that the solutions converge to the steady state in approximately
489 three years. We repeated the simulations with the same parameters and other starting values and found
490 similar behaviour and convergence times. Note that including environmental or demographic stochasticity,
491 and seasonal forcing (or more than one of these together) might change the behaviour of the system.

492 Next, we find the values of $\partial Y^*/\partial\mu$, Y^* and $\partial Y^*/\partial\beta_1$, Y^* and plot them with respect to μ , β_1 in
493 Figure 2, 3 respectively. It is clear from Figure 2 and Figure 3 that if we increase either μ or β_1 or both,
494 the equilibrium number of infected individuals decreases, which confirms the result given in Remark 1.

495 To study the impact of delays in system (4.2) we first fix $\tau_1 = 0$ days, and increase the value of τ_2
496 gradually. We observed that the system is LAS below a critical value τ_{2_0} (≈ 146 days, see Theorem 4.1) of
497 τ_2 and undergoes Hopf bifurcation as τ_2 increases through τ_{2_0} (see Figure 4). For $\tau_2 \leq \tau_{2_0}$ there is a unique
498 LAS endemic equilibrium whose components are plotted on the y -axes in Figure 4. For $\tau > \tau_{2_0}$ a stable
499 limit cycle arises by Hopf bifurcation from this endemic equilibrium and Figure 4 plots the minimum and
500 maximum values of these long-term stable limit cycle oscillations. Then we fixed $\tau_1 = 120$ days and drew
501 the bifurcation diagram of the system (4.2) with respect to τ_2 , we observe that the system enters into
502 limit cycle oscillation from a stable equilibrium as we increase the value of τ_2 (see Figure 5). The system
503 undergoes a Hopf bifurcation at $\tau_2 \approx 90$ days (i.e. $\tau'_{2_0} \approx 90$ days, see Theorem 4.4). Similarly, the system
504 (4.2) loses its stability and enters into limit cycle oscillations through Hopf bifurcation at $\tau_{1_0} \approx 128.4$ days,
505 when the second delay is absent ($\tau_2 \approx 0$). In a similar way, keeping τ_2 fixed at 60 days we observe that
506 the system (4.2) undergoes a Hopf bifurcation at $\tau_1 \approx 134.7$ days (i.e. $\tau'_{1_0} \approx 134.7$ days, see Theorem
507 4.3). In Figure 6 we have drawn the domain of the stability region with respect to τ_1 and τ_2 to visualize
508 the impact of delays in the stability of the system (4.2).

509 It is worth mentioning here that the interior equilibrium point of the system (4.6) depends on τ_1 ,

510 which is very different from traditional delay models. In traditional delay models the equilibrium points
511 of the delay model and the non-delay model are the same. However in the present investigation, we have
512 considered the survival probability (P) in the interval of the time lag τ_1 corresponding to aware people
513 forgetting the impact of disease after this time lag. The equilibrium value of P depends on τ_1 explicitly
514 (see Appendix B). Consequently, the value of τ_1 directly influences equilibrium population numbers. In
515 Figure 7 we have plotted the equilibrium number of infected individuals, Y^* , and the value of the survival
516 probability, P^* , against τ_1 . We observe that as τ_1 increases the equilibrium number of infected individuals
517 decreases.

518 Our numerical computation also shows that for $\tau_1 = 0$ days, $P^* = 1$ and $Y^* = 1787.4$ and for $\tau_1 = 180$
519 days, $P^* = 0.051$ and $Y^* = 81.9$. Therefore, it is clear that if the susceptible individuals become aware
520 and remain aware for a long time then the equilibrium number of infected individuals decreases. However,
521 we have also observed that for $\tau_1 > \tau_{10}$ ($\tau_{10} \approx 128.4$ days), the system shows limit cycle oscillation, which
522 poses a challenge for controlling the epidemic outbreak.

523 6. Conclusion

524 In this paper we have considered the effect of disease awareness programs on disease dynamics where
525 the susceptible population is divided into two different classes, aware susceptible and unaware susceptible.
526 The model was considered first without any time delay and then with two time delays. The first time
527 delay was due to people forgetting the impact of the disease after a time lag τ_1 . The second time delay was
528 due to the media mounting a disease awareness campaign because of cases that had previously occurred
529 after a time lag τ_2 .

530 A differential equation model was used to examine the disease spread firstly with no time delay and
531 then with a time delay. For the model with no time delay an expression for the basic reproduction number
532 R_0 was calculated. The DFE is LAS if and only if $R_0 < 1$. For $R_0 > 1$ the DFE becomes unstable and
533 an endemic equilibrium exists.

534 For the model with no time delay sufficient conditions for the endemic equilibrium to be LAS were
535 derived. For the model with two time delays sufficient conditions for the stability of the endemic equi-
536 librium and the existence of Hopf bifurcations were obtained for four different sets of values of the delay
537 parameters, i.e. when $\tau_1 = 0$, $\tau_2 > 0$; $\tau_2 = 0$, $\tau_1 > 0$ and the two cases when $\tau_1 > 0$ and $\tau_2 > 0$ (see
538 Theorems 4.1, 4.2, 4.3 and 4.4).

539 Numerical simulations were performed to investigate the behavior of the system. They indicated that
540 the system was LAS with realistic parameter values. We used the numerical simulations to visualize the

541 effect of increasing time delays on the dynamics of the system.

542 We observed that in our model if we increase the number of campaigns due to the awareness program
543 then the disease transmission rate amongst the susceptible population declines. The numerical simulations
544 also indicate that if the implementation rate of the awareness program increases then the equilibrium
545 number of infected individuals decreases. We have also observed that if the time lag (τ_1) in rejoining the
546 unaware class of aware individuals increases, i.e. the susceptible individuals remain aware for a longer
547 time, then the equilibrium number of infected individuals reduces. However, sustained oscillation may
548 arise if the the time lag increases over a threshold value which could possibly pose a challenge in controlling
549 the epidemic.

550 However, the restrictions on the rate of immigration could have the ability to control the epidemic. It
551 might be possible to control oscillations by controlling the rate of immigration [20]. Restricting immigra-
552 tion might have a stabilizing effect on disease dynamics.

553 In the present study we have considered the impact of an awareness campaign that acts on the
554 whole population uniformly. This is a commonly made assumption in the literature on modeling media
555 awareness campaigns. It would be appropriate for control of a disease that is established over a wide
556 area. However it would not be appropriate for controlling a local outbreak of disease where an awareness
557 campaign would have to be much more geographically focussed and act mostly on the local population. In
558 those circumstances we would expect the impact of an awareness campaign to decrease as we move away
559 from the epidemic outbreak or the number of infected individuals reduces. This would require a more
560 sophisticated model and is a possible direction for future research. Note also that although the functional
561 forms of the disease transmission term and the spread of information term have similarities we are not
562 necessarily assuming the same transmission routes. Some other possible information transfer mechanisms
563 could require fundamentally different information transmission terms [32]. This is also another potential
564 direction for future work.

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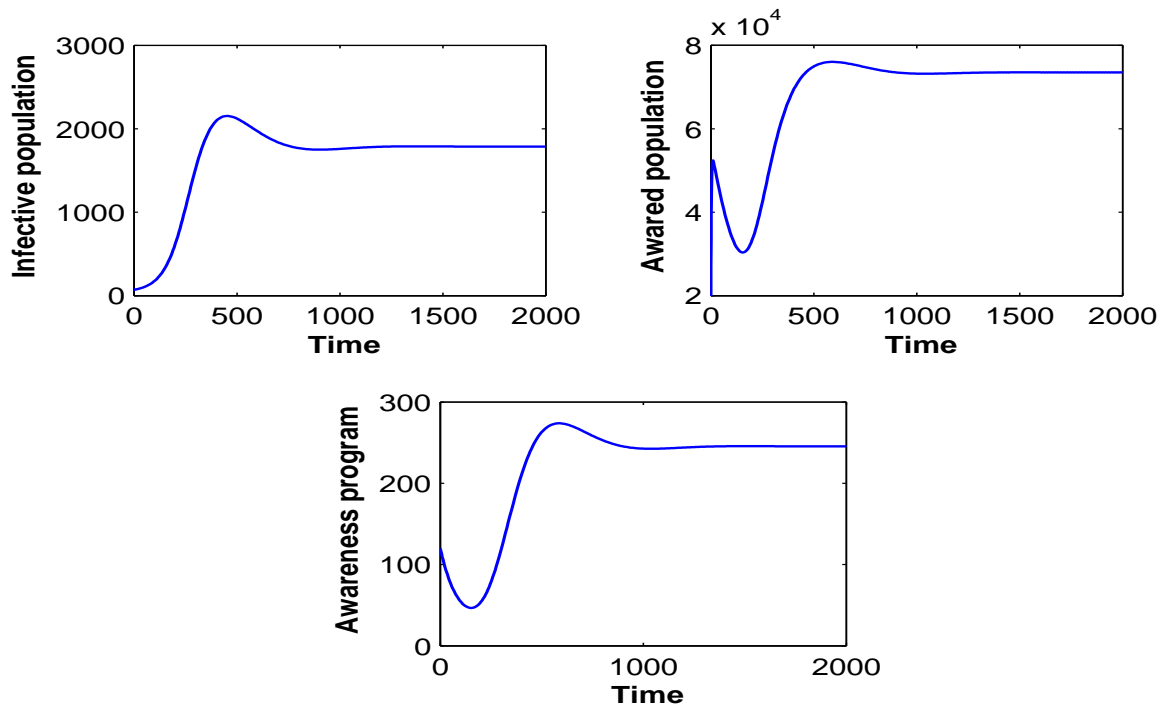


Figure 1: Stable population distribution of (3.2) in absence of both delays ($\tau_1 = \tau_2 = 0$ days). Other parameter values are $\beta = 2.8650 \times 10^{-7} \text{ day}^{-1}$, $\lambda = 0.9 \text{ day}^{-1}$, $\lambda_0 = 0.3 \text{ day}^{-1}$, $\gamma = 0.02011 \text{ day}^{-1}$, $d = 1.3736 \times 10^{-3} \text{ day}^{-1}$, $\mu = 1.3736 \times 10^{-3} \text{ day}^{-1}$, $\mu_0 = 0.01 \text{ day}^{-1}$, $\alpha = 0$, $k = 500$, $\beta_1 = 1$, $A = 206.04 \text{ day}^{-1}$, $p = 0.6$.

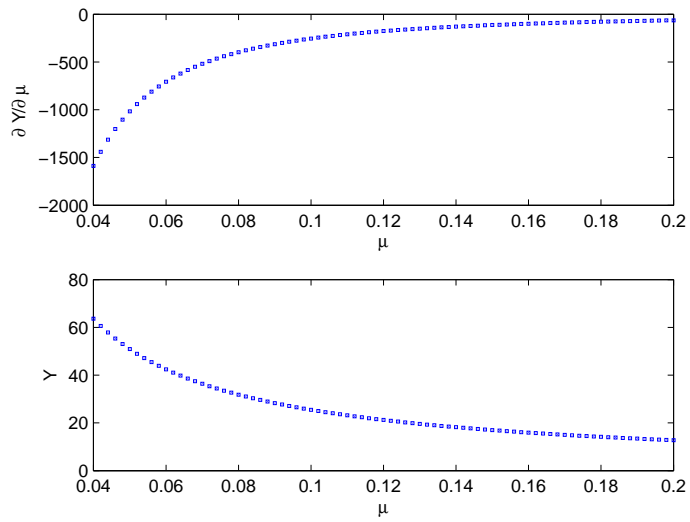


Figure 2: The figure depicts that the equilibrium number of infected individuals reduces with increasing μ (day^{-1}) where other parameter values are kept the same as in Figure 1.

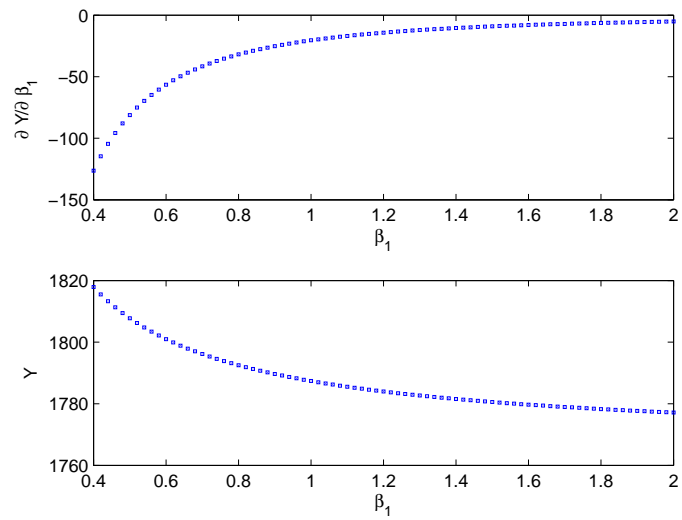


Figure 3: The figure depicts that the equilibrium number of infected individuals reduces with increasing β_1 .

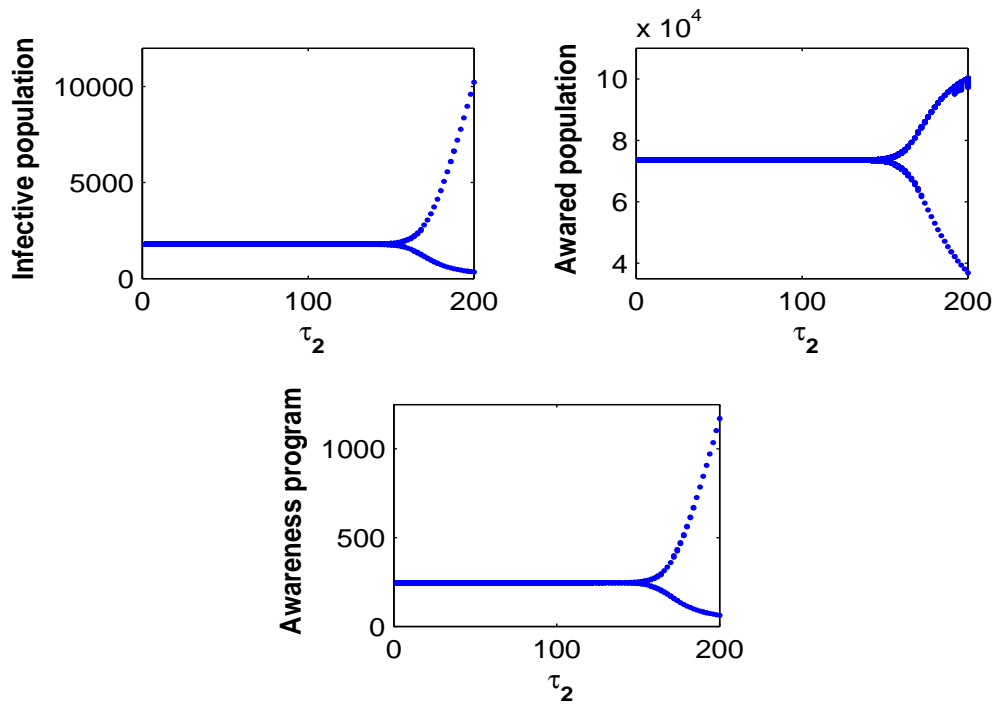


Figure 4: Diagram showing Hopf bifurcation of system (4.2) with respect to τ_2 (days) when $\tau_1 = 0$ days.

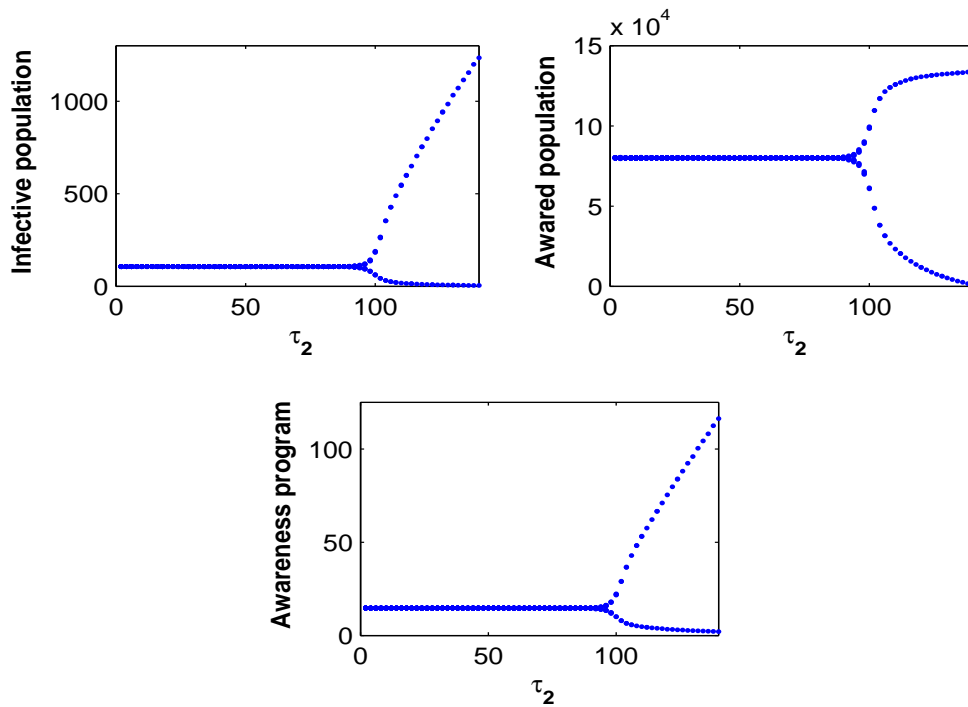


Figure 5: Diagram showing Hopf bifurcation of system (4.2) with respect to τ_2 (days) when $\tau_1 = 120$ days.

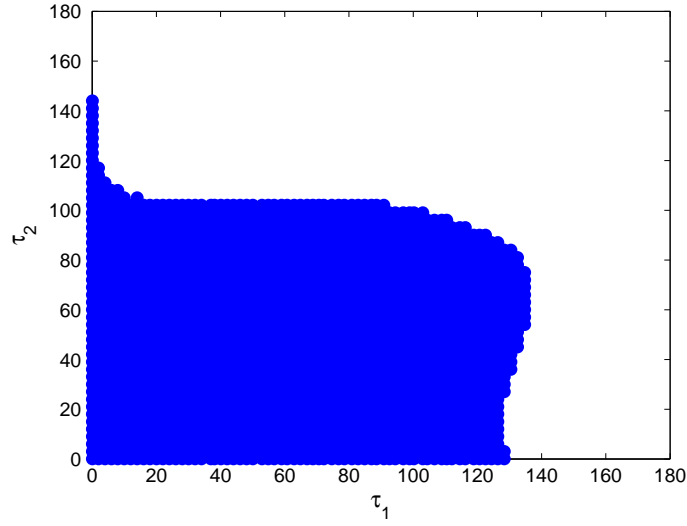


Figure 6: Domain of stability region with respect to τ_1 (days) and τ_2 (days) for the model (4.2). Other parameter values are kept the same as in Figure 1.

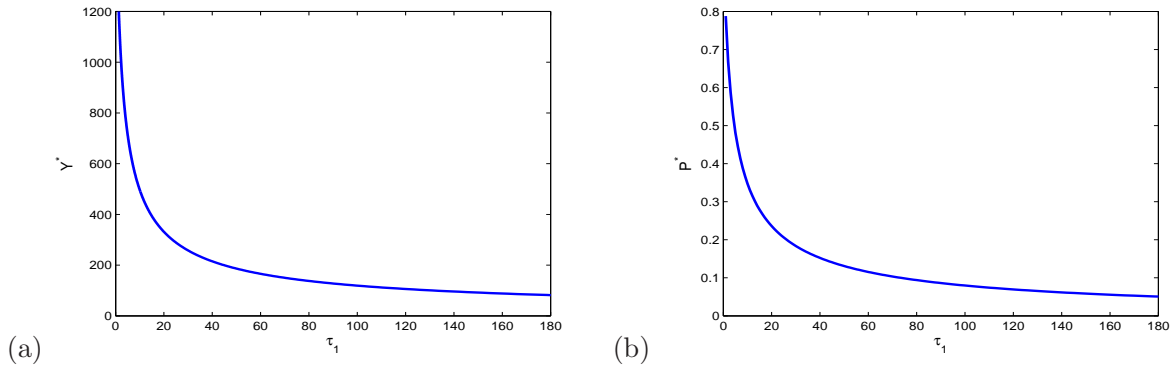


Figure 7: Figures 7(a) and 7(b) show that the equilibrium number of infected individuals (Y^*) and survival probability (P^*) decrease for increase in τ_1 .

704 **Appendix A**

705 **Detailed mathematical expansions of terms in the paper.**

706

707 **A.1 Terms in characteristic equation (4.14).**

708 $A_1 = M_1 + M_6 + M_{11} + M_{12} + M_{15},$

709 $A_2 = M_1 M_{11} + M_1 M_{12} + M_1 M_{15} + M_{11} M_{12} + M_{11} M_{15} + M_{12} M_{15} + M_3 M_{10} + M_1 M_6 - M_2 M_5 + M_6 M_{11}$
 710 $+ M_6 M_{12} + M_6 M_{15},$

711 $A_3 = M_1 M_{11} M_{12} + M_1 M_{11} M_{15} + M_1 M_{12} M_{15} + M_{11} M_{12} M_{15} + M_3 M_{10} M_{12} + M_3 M_{10} M_{15} + M_1 M_6 M_{11}$
 712 $- M_2 M_5 M_{11} + M_1 M_6 M_{12} - M_2 M_5 M_{12} + M_1 M_6 M_{15} - M_2 M_5 M_{15} + M_6 M_{11} M_{12} + M_6 M_{11} M_{15}$
 713 $+ M_6 M_{12} M_{15} + M_2 M_7 M_{10} + M_3 M_6 M_{10} + M_2 M_9 M_{13},$

714 $A_4 = M_1 M_{11} M_{12} M_{15} + M_3 M_{10} M_{12} M_{15} + M_1 M_6 M_{11} M_{12} - M_2 M_5 M_{11} M_{12} + M_1 M_6 M_{11} M_{15} - M_2 M_5 M_{11} M_{15}$
 715 $+ M_1 M_6 M_{12} M_{15} - M_2 M_5 M_{12} M_{15} + M_6 M_{11} M_{12} M_{15} + M_2 M_7 M_{10} M_{12} + M_3 M_6 M_{10} M_{12} + M_2 M_7 M_{10} M_{15}$
 716 $+ M_3 M_6 M_{10} M_{15} + M_2 M_9 M_{11} M_{13} + M_2 M_9 M_{12} M_{13},$

717 $A_5 = M_1 M_6 M_{11} M_{12} M_{15} - M_2 M_5 M_{11} M_{12} M_{15} + M_2 M_7 M_{10} M_{12} M_{15} + M_3 M_6 M_{10} M_{12} M_{15} + M_2 M_9 M_{11} M_{12} M_{13},$

718 $B_1 = -M_4 \bar{m} m_1 m_4,$

719 $B_2 = -M_{11} M_4 \bar{m} m_1 m_4,$

720 $C_1 = -m_1 m_4,$

721 $C_2 = -(M_1 + M_{11} + M_{12}) m_1 m_4,$

722 $C_3 = -(M_1 M_{11} + M_1 M_{12} + M_{11} M_{12} + M_3 M_{10}) m_1 m_4,$

723 $C_4 = -(M_1 M_{11} M_{12} + M_3 M_{10} M_{12}) m_1 m_4,$

724 $D_1 = m_1 - m_4,$

725 $D_2 = (M_1 + M_{11} + M_{12} + M_{15}) m_1 - (M_1 + M_{11} + M_{12} + M_6) m_4,$

726 $D_3 = (M_1 M_{11} + M_1 M_{12} + M_1 M_{15} + M_{11} M_{12} + M_{11} M_{15} + M_{12} M_{15} + M_3 M_{10}) m_1$
 727 $- (M_1 M_{11} + M_1 M_{12} + M_{11} M_{12} + M_3 M_{10} + M_1 M_6 - M_2 M_5 + M_6 M_{11} + M_6 M_{12}) m_4 - M_2 M_9 m_2,$

728 $D_4 = (M_1 M_{11} M_{12} + M_1 M_{11} M_{15} + M_1 M_{12} M_{15} + M_{11} M_{12} M_{15} + M_3 M_{10} M_{12} + M_3 M_{10} M_{15}) m_1$
 729 $- (M_1 M_{11} M_{12} + M_3 M_{10} M_{12} + M_1 M_6 M_{11} - M_2 M_5 M_{11} + M_1 M_6 M_{12} - M_2 M_5 M_{12} + M_6 M_{11} M_{12}$
 730 $+ M_2 M_7 M_{10} + M_3 M_6 M_{10}) m_4 - (M_2 M_9 M_{11} + M_2 M_9 M_{12}) m_2,$

731 $D_5 = (M_1 M_{11} M_{12} M_{15} + M_3 M_{10} M_{12} M_{15}) m_1$
 732 $- (M_1 M_6 M_{11} M_{12} - M_2 M_5 M_{11} M_{12} + M_2 M_7 M_{10} M_{12} + M_3 M_6 M_{10} M_{12}) m_4 - M_2 M_9 M_{11} M_{12} m_2,$

$$733 \quad E_1 = M_4 \bar{m} (m_1 - m_4),$$

$$734 \quad E_2 = (M_4 M_{11} m_1 + M_4 M_{15} m_1 - M_4 M_{11} m_4 + M_2 M_8 m_4 + M_2 M_9 m_3 - M_4 M_6 m_4) \bar{m},$$

$$735 \quad E_3 = (M_4 M_{15} m_1 + M_2 M_8 m_4 + M_2 M_9 m_3 - M_4 M_6 m_4) M_{11} \bar{m},$$

$$736 \quad F_1 = M_4 \bar{m},$$

$$737 \quad F_2 = (M_4 M_{11} + M_4 M_{15} - M_2 M_8 + M_4 M_6) \bar{m},$$

$$738 \quad F_3 = (M_4 M_{11} M_{15} - M_2 M_8 M_{11} + M_4 M_6 M_{11} - M_2 M_8 M_{15} + M_4 M_6 M_{15} - M_2 M_9 M_{14}) \bar{m},$$

$$739 \quad F_4 = -(M_2 M_8 M_{15} - M_4 M_6 M_{15} + M_2 M_9 M_{14}) M_{11} \bar{m}.$$

740 **A.2 Terms in the transversality condition of Theorem 4.2b.**

$$741 \quad P_{11} = [5\omega_{10}^4 - 3(A_2 + F_1 + C_1)\omega_{10}^2 + (A_4 + F_3 + B_1 + C_3)] \cos \omega_{10} \tau_{10}$$

$$742 \quad + [4A_1\omega_{10}^3 - 2(A_2 + F_2 - C_2)\omega_{10}] \sin \omega_{10} \tau_{10} + [-3D_2\omega_{10}^2 + (D_4 + E_2)],$$

$$743 \quad P_{12} = -[4A_1\omega_{10}^3 - 2(A_3 + F_2 + C_2)\omega_{10}] \cos \omega_{10} \tau_{10}$$

$$744 \quad + [5\omega_{10}^4 - 3(A_2 + F_1 - C_1)\omega_{10}^2 + (A_4 + F_3 - B_1 - C_3)] \sin \omega_{10} \tau_{10} + [-4D_1\omega_{10}^3 + 2(D_3 + E_1)\omega_{10}],$$

$$745 \quad G_{11} = 2[C_1\omega_{10}^4 - (B_1 + C_3)\omega_{10}^2] \cos \omega_{10} \tau_{10} + 2[-C_2\omega_{10}^3 + (B_2 + C_4)\omega_{10}] \sin \omega_{10} \tau_{10}$$

$$746 \quad + [D_2\omega_{10}^4 - (D_4 + E_2)\omega_{10}^2],$$

$$747 \quad G_{12} = 2[-C_2\omega_{10}^3 + (B_2 + C_4)\omega_{10}] \cos \omega_{10} \tau_{10} - 2[C_1\omega_{10}^4 - (B_1 + C_3)\omega_{10}^2] \sin \omega_{10} \tau_{10}$$

$$748 \quad + [D_1\omega_{10}^5 - (D_4 + E_1)\omega_{10}^3 + (D_5 + E_3)\omega_{10}].$$

749 **A.3 Terms in the transversality condition of Theorem 4.3b.**

$$750 \quad P_{21} = [5\omega_{30}^4 - 3A_2\omega_{30}^2 + A_4] + [B_1 - \tau_2 B_2] \cos \omega_{30} (2\tau'_{10} + \tau_2) - \tau_2 B_1 \omega_{30} \sin \omega_{30} (2\tau'_{10} + \tau_2)$$

$$751 \quad + [-3C_1\omega_{30}^2 + C_3] \cos 2\omega_{30} \tau'_{10} + 2C_2 \sin 2\omega_{30} \tau'_{10} + [-3D_2\omega_{30}^2 + D_4] \cos \omega_{30} \tau'_{10} + [-4D_1\omega_{30}^3 + 2D_3\omega_{30}] \sin \omega_{30} \tau'_{10}$$

$$752 \quad + [E_2 - \tau_2(-E_1\omega_{30}^2 + E_3)] \cos \omega_{30} (\tau'_{10} + \tau_2) + [2E_1\omega_{30} - \tau_2 E_2 \omega_{30}] \sin \omega_{30} (\tau'_{10} + \tau_2)$$

$$753 \quad + [(-3F_1\omega_{30}^2 + F_3) - \tau_2(-F_2\omega_{30}^2 + F_4)] \cos \omega_{30} \tau_2 + 2[F_2\omega_{30} - \tau_2(-F_1\omega_{30}^3 + F_3\omega_{30})] \sin \omega_{30} \tau_2,$$

$$754 \quad P_{22} = [-4A_1\omega_{30}^3 + 2A_3\omega_{30}] - \tau_2 B_1 \omega_{30} \cos \omega_{30} (2\tau'_{10} + \tau_2) - [B_1 - \tau_2 B_2] \sin \omega_{30} (2\tau'_{10} + \tau_2) + 2C_2 \cos 2\omega_{30} \tau'_{10}$$

$$755 \quad + [3C_1\omega_{30}^2 - C_3] \sin 2\omega_{30} \tau'_{10} + [-4D_1\omega_{30}^3 + 2D_3\omega_{30}] \cos \omega_{30} \tau'_{10} + [3D_2\omega_{30}^2 - D_4] \sin \omega_{30} \tau'_{10}$$

$$756 \quad + [2E_1\omega_{30} - \tau_2 E_2 \omega_{30}] \cos \omega_{30} (\tau'_{10} + \tau_2) - [E_2 + \tau_2(E_1\omega_{30}^2 - E_3)] \sin \omega_{30} (\tau'_{10} + \tau_2)$$

$$757 \quad + [2F_2\omega_{30} + \tau_2(F_1\omega_{30}^3 - F_3\omega_{30})] \cos \omega_{30} \tau_2 + [(3F_1\omega_{30}^2 - F_3) + \tau_2(-F_2\omega_{30}^2 + F_4)] \sin \omega_{30} \tau_2,$$

$$758 \quad G_{21} = -2B_1\omega_{30}^2 \cos \omega_{30} (2\tau'_{10} + \tau_2) + 2B_2\omega_{30} \sin \omega_{30} (2\tau'_{10} + \tau_2) + 2[C_1\omega_{30}^4 - C_3\omega_{30}^2] \cos 2\omega_{30} \tau'_{10}$$

$$759 \quad - 2\omega_{30} [C_2\omega_{30}^2 - C_4] \sin 2\omega_{30} \tau'_{10} + \omega_{30}^2 [D_2\omega_{30}^2 - D_4] \cos \omega_{30} \tau'_{10} + \omega_{30} [D_1\omega_{30}^4 - D_3\omega_{30}^2 + D_5] \sin \omega_{30} \tau'_{10}$$

$$\begin{aligned}
& -E_2\omega_{3_0}^2 \cos \omega_{3_0}(\tau'_{1_0} + \tau_2) - \omega_{3_0}[E_1\omega_{3_0}^2 - E_3] \sin \omega_{3_0}(\tau'_{1_0} + \tau_2), \\
761 \quad G_{22} &= 2B_2\omega_{3_0} \cos \omega_{3_0}(2\tau'_{1_0} + \tau_2) + 2B_1\omega_{3_0}^2 \sin \omega_{3_0}(2\tau'_{1_0} + \tau_2) - 2\omega_{3_0}[C_2\omega_{3_0}^2 - C_4] \cos 2\omega_{3_0}\tau'_{1_0} \\
762 & -2\omega_{3_0}^2[C_1\omega_{3_0}^2 - C_3] \sin 2\omega_{3_0}\tau'_{1_0} + \omega_{3_0}[D_1\omega_{3_0}^4 - D_3\omega_{3_0}^2 + D_5] \cos \omega_{3_0}\tau'_{1_0} - \omega_{3_0}^2[D_2\omega_{3_0}^2 - D_4] \sin \omega_{3_0}\tau'_{1_0} \\
763 & -\omega_{3_0}[E_1\omega_{3_0}^2 - E_3] \cos \omega_{3_0}(\tau'_{1_0} + \tau_2) + E_2\omega_{3_0}^2 \sin \omega_{3_0}(\tau'_{1_0} + \tau_2).
\end{aligned}$$

764 A.4 Terms in the transversality condition of Theorem 4.4b.

$$\begin{aligned}
765 \quad P_{31} &= [5\omega_{4_0}^4 - 3A_2\omega_{4_0}^2 + A_4] + [B_1 - 2\tau_1 B_2] \cos \omega_{4_0}(2\tau_1 + \tau'_{2_0}) - 2\tau_1 B_1 \omega_{4_0} \sin \omega_{4_0}(2\tau_1 + \tau'_{2_0}) \\
766 & + [-3C_1\omega_{4_0}^2 + C_3 + 2\tau_1(C_2\omega_{4_0}^2 - C_4)] \cos 2\omega_{4_0}\tau_1 + [2C_2\omega_{4_0} + 2\tau_1(C_1\omega_{4_0}^3 - C_3\omega_{4_0})] \sin 2\omega_{4_0}\tau_1 \\
767 & + [-3D_2\omega_{4_0}^2 + D_4 - \tau_1(D_1\omega_{4_0}^4 - D_3\omega_{4_0}^2 + D_5)] \cos \omega_{4_0}\tau_1 + [-4D_1\omega_{4_0}^3 + 2D_3\omega_{4_0} + \tau_1(D_2\omega_{4_0}^3 - D_4\omega_{4_0})] \sin \omega_{4_0}\tau_1 \\
768 & + [E_2 + \tau_1(E_1\omega_{4_0}^2 - E_3)] \cos \omega_{4_0}(\tau_1 + \tau'_{2_0}) + [2E_1\omega_{4_0} - \tau_1 E_2\omega_{4_0}] \sin \omega_{4_0}(\tau_1 + \tau'_{2_0}) - [3F_1\omega_{4_0}^2 - F_3] \cos \omega_{4_0}\tau'_{2_0} \\
769 & + 2F_2\omega_{4_0} \sin \omega_{4_0}\tau'_{2_0}, \\
770 \quad P_{32} &= [-4A_1\omega_{4_0}^3 + 2A_3\omega_{4_0}] - 2\tau_1 B_1 \omega_{4_0} \cos \omega_{4_0}(2\tau_1 + \tau'_{2_0}) - [B_1 - 2\tau_1 B_2] \sin \omega_{4_0}(2\tau_1 + \tau'_{2_0}) \\
771 & + [2C_2\omega_{4_0} + 2\tau_1(C_1\omega_{4_0}^3 - C_3\omega_{4_0})] \cos 2\omega_{4_0}\tau_1 + [3C_1\omega_{4_0}^2 - C_3 - 2\tau_1(C_2\omega_{4_0}^2 - C_4)] \sin 2\omega_{4_0}\tau_1 \\
772 & + [-4D_1\omega_{4_0}^3 + 2D_3\omega_{4_0} + \tau_1(D_2\omega_{4_0}^3 - D_4\omega_{4_0})] \cos \omega_{4_0}\tau_1 + [3D_2\omega_{4_0}^2 - D_4 + \tau_1(D_1\omega_{4_0}^4 - D_3\omega_{4_0}^2 + D_5)] \sin \omega_{4_0}\tau_1 \\
773 & + [2E_1\omega_{4_0} - \tau_1 E_2\omega_{4_0}] \cos \omega_{4_0}(\tau_1 + \tau'_{2_0}) - [E_2 + \tau_1(E_1\omega_{4_0}^2 - E_3)] \sin \omega_{4_0}(\tau_1 + \tau'_{2_0}) + 2F_2\omega_{4_0} \cos \omega_{4_0}\tau'_{2_0} \\
774 & + [3F_1\omega_{4_0}^2 - F_3] \sin \omega_{4_0}\tau'_{2_0}, \\
775 \quad G_{31} &= -B_1\omega_{4_0}^2 \cos \omega_{4_0}(2\tau_1 + \tau'_{2_0}) + B_2\omega_{4_0} \sin \omega_{4_0}(2\tau_1 + \tau'_{2_0}) - E_2\omega_{4_0}^2 \cos \omega_{4_0}(\tau_1 + \tau'_{2_0}) \\
776 & - [E_1\omega_{4_0}^3 - E_3\omega_{4_0}] \sin \omega_{4_0}(\tau_1 + \tau'_{2_0}) + [F_1\omega_{4_0}^4 - F_3\omega_{4_0}^2] \cos \omega_{4_0}\tau'_{2_0} - [F_2\omega_{4_0}^3 - F_4\omega_{4_0}] \sin \omega_{4_0}\tau'_{2_0}, \\
777 \quad G_{32} &= B_2\omega_{4_0} \cos \omega_{4_0}(2\tau_1 + \tau'_{2_0}) + B_1\omega_{4_0}^2 \sin \omega_{4_0}(2\tau_1 + \tau'_{2_0}) - [E_1\omega_{4_0}^3 - E_3\omega_{4_0}] \cos \omega_{4_0}(\tau_1 + \tau'_{2_0}) \\
778 & + E_2\omega_{4_0}^2 \sin \omega_{4_0}(\tau_1 + \tau'_{2_0}) - [F_2\omega_{4_0}^3 - F_4\omega_{4_0}] \cos \omega_{4_0}\tau'_{2_0} - [F_1\omega_{4_0}^4 - F_3\omega_{4_0}^2] \sin \omega_{4_0}\tau'_{2_0}.
\end{aligned}$$

779 **Appendix B**

780 **B.1 Numerical simulations to find the equilibria.**

781

782 In this appendix we obtain the equilibrium point of the equation (4.6) using the equations (4.7) and
 783 (4.8).

784 First we fix parameters of the system (4.7) to be the same as in Figure 1 and vary P^* in the entire
 785 range within 0 and 1 to find (Y^*, X_+^*, N^*, M^*) for each value of P^* . Now we use equation (4.8) which is
 786 a transcendental equation in P^* to draw Figure 8. Let us consider the right hand side of the equation
 787 (4.8) as $F_2(P^*, \tau_1) = e^{-[d\tau_1 + \lambda_0 P^* \tau_1 + \frac{\beta Y^* \tau_1}{1 + \beta_1 M^*}]}$. We fix τ_1 and plot $F_2(P^*, \tau_1)$ for P^* lying between 0 and
 788 1. In Figure 8, we have taken some values of τ_1 and plot F_2 , here blue, red, black and green solid curves
 789 correspond to the value of F_2 at τ_1 equal to 25, 50, 100 and 150 respectively. Lastly we plot the left hand
 790 side of the equation (4.8), i.e. $F_1(P^*) = P^*$ (the dashed blue line). The intersection between F_1 and F_2
 791 is the equilibrium value of P^* for different values of τ_1 .

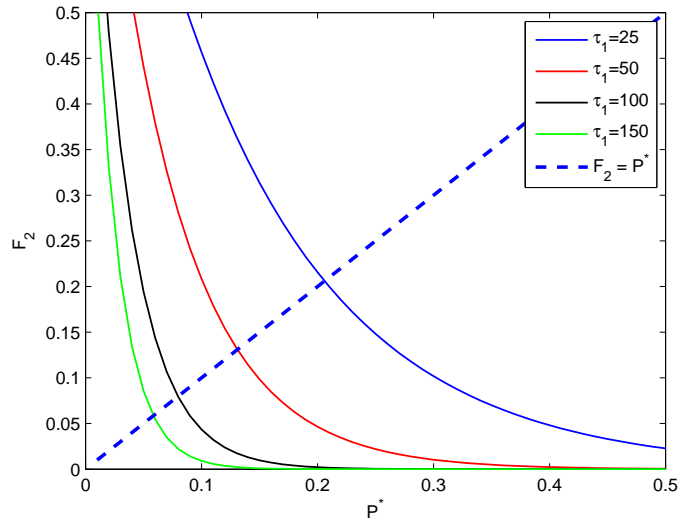


Figure 8: Graphical representation of equation (4.8) to find P^* for different τ_1 .