Existence, uniqueness and almost surely asymptotic estimations of the solutions to neutral stochastic functional differential equations driven by pure jumps

Wei Mao[1],Quanxin Zhu[2], Xuerong Mao[3]

1.School of Mathematics and Information Technology, Jiangsu Second Normal University, Nanjing, 210013, Jiangsu, P.R.China;
2.School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, Jiangsu, P.R.China;
3.Department of Mathematics and Statistics, University of Strathclyde, Glasgow, G1 1XH, UK.

Abstract

In this paper, we are concerned with neutral stochastic functional differential equations driven by pure jumps (NSFDEwPJs). We prove the existence and uniqueness of the solution to NSFDEwPJs whose coefficients satisfying the Local Lipschitz condition. In addition, we establish the p-th exponential estimations and almost surely asymptotic estimations of the solution for NSFDEwJs.

Key words: Neutral stochastic functional differential equations, Pure jumps, Existence and uniqueness, Exponential estimations, Almost surely asymptotic estimations.

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1. Introduction

Stochastic delay differential equations (SDDEs) have come to play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. In the past few decades, qualitative theory of SDDEs have been studied intensively by many scholars. Here, we refer to S.E.A.Mohammed [1], X.Mao [2-5,9], E.Buckwar [6], U.Kuchler [7], Y.Hu [8], D.Xu [10], F.Wu [11], J.Appleby [12], I.Gyongy [13] and references therein. Recently, motivated by the theory of aeroelasticity, a class of neutral stochastic equations has also received a great deal of attention and much work has been done on neutral stochastic equations. For example, conditions of the existence and stability of the analytical solution are given in [14-20]. Various efficient computational methods are obtained and their convergence and stability have been studied in [21-25].

However, all equations of the aboved mentioned works are driven by white noise perturbations with continuous initial data and white noise perturbations are not always appropriate to interpret real datas in a reasonable way. In real phenomena, the state of neutral stochastic delay equations may be perturbed by abrupt pulses or extreme events. A more natural mathematical framework for these phenomena has been taken into account other than purely Brownian perturbations. In particular, we incorporate the Levy perturbations with jumps into neutral stochastic delay equations to model abrupt changes.

In this paper, we study the following neutral stochastic functional differential equations with pure jumps (NSFDEwPJs)

$$d[x(t) - D(x_t)] \;=\; f(x_t, t)dt + \int_U h(x_t, u)N_{\bar{p}}(dt, du), \quad t_0 \le t \le T. \tag{1}$$

To the best of our knowledge, there are no literatures concerned with the existence and asymptotic estimations of the solution to NSFDEwPJs (1). On the one hand, we prove that equation (1) has a unique solution in the sense of L^P norm. We don't use the fixed point Theorem. Instead, we get the solution of equation (1) via successive approximations. On the other hand, we study the p-th exponential estimations and almost surely asymptotic estimations of the solution to equation (1). By using the It$\widehat{o}$ formula, Taylor formula and the Burkholder Davis inequality, we have that the p-th moment of the solution will grow at most exponentially with exponent M and show that the exponential estimations implies almost surely asymptotic estimations. Although the way of analysis follows the ideas in [2], however, those results on the existence and uniqueness of the solution in [2] can not be extended to the jumps case naturally. Unlike the Brown process whose almost all sample paths are continuous, the Poisson random measure $N_{\bar{p}}(dt, du)$ is a jump process and has the sample paths which are right-continuous and have left limits. Therefore, there is a great difference between the stochastic integral with respect to the Brown process and the one with respect to the Poisson random measure. It should be pointed out that the proof for NSFDEwPJs is certainly not a straightforward generalization of that for NSFDEs without jumps and some new techniques are developed to cope with the difficulties due to the Poisson random measures.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and hypotheses concerning equation (1); In Section 3, the existence and uniqueness of the solution to equation (1) are investigated; In Section 4, we prove the p-th moment of the solution will grow at most exponentially with exponent M and show that the exponential estimations implies the almost surely asymptotic estimations.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with some filtration $(\mathcal{F}_t)_{t\geq t_0}$ satisfying the usual conditions, (i.e. it is right continuous and $(\mathcal{F}_{t_0})$ contains all P-null sets). Let $\tau > 0$, and $D([-\tau, 0]; R^n)$ denote the family of all right-continuous functions with left-hand limits φ from $[-\tau, 0] \to R^n$. The space $D([-\tau, 0]; R^n)$ is assumed to be equipped with the norm $||\varphi|| = \sup_{-\tau\leq t\leq 0} |\varphi(t)|$ and $|x| = \sqrt{x^\top x}$ for any $x \in R^n$. If A is a vector or matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^\top A)}$, while its operator norm is denoted by $||A|| = sup\{|Ax| : |x| = 1\}$. $D^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D([-\tau, 0]; R^n)$ valued random variable $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. Let $t_0 \geq 0$, $p \geq 2$, $\mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau, 0]; R^n)$ denote the family of all $\mathcal{F}_{t_0}$ measurable, $D([-\tau, 0]; R^n)$-valued random variables $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\mathrm{E} \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|^p < \infty$.

Let $(U, \mathcal{B}(U))$ be a measurable space and $\pi(du)$ a σ- finite measure on it. Let $\{\bar{p} = \bar{p}(t), t \geq t_0\}$ be a stationary $\mathcal{F}_t$-Poisson point process on U with a characteristic measure π. Then, for $A \in \mathcal{B}(U - \{0\})$, here $0 \in$ the closure of A, the Poisson counting measure $N_{\bar{p}}$ is defined by

$$N_{\bar{p}}((t_0, t] \times A) := \sharp\{t_0 < s \leq t, \bar{p}(s) \in A\} = \sum_{t_0<s\leq t} I_A(\bar{p}(s)),$$

where $\sharp$ denotes the cardinality of a set. For simplicity, we denote: $N_{\bar{p}}(t, A) := N_{\bar{p}}((t_0, t] \times A)$. It follows from [26] that there exists a σ- finite measure π satisfying

$$E[N_{\bar{p}}(t, A)] = \pi(A)t, \quad P(N_{\bar{p}}(t, A) = n) = \frac{e(-t\pi(A))(\pi(A)t)^n}{n!}.$$

This measure π is called the Levy measure. Then, the measure $\tilde{N}_{\bar{p}}$ is defined by

$$\tilde{N}_{\bar{p}}([t_0,t],A) := N_{\bar{p}}([t_0,t],A) - t\pi(A), \quad t > t_0.$$

We refer to N.Ikeda [26] for the details on Poisson point process.

The integral version of equation (1) is given by the equation

$$x(t) - D(x_t) \;=\; x_{t_0} - D(x_{t_0}) + \int_{t_0}^{t} f(x_s,s)ds + \int_{t_0}^{t}\int_U h(x_s,u)N_{\bar{p}}(ds,du), \tag{2}$$

where

$$x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$$

is regarded as a $D([-\tau,0];R^n)$-valued stochastic process. $f : D([-\tau,0];R^n)\times[t_0,T] \to R^n$ and $h : D([-\tau,0];R^n)\times U \to R^n$ are both Borel-measurable functions. The initial condition x_{t_0} is defined by

$$x_{t_0} = \xi = \{\xi(t) : -\tau \le t \le 0\} \in \mathcal{L}^p_{\mathcal{F}_{t_0}}([-\tau,0];R^n),$$

that is, ξ is an $\mathcal{F}_{t_0}$-measurable $D([-\tau,0];R^n)$-valued random variable and $E||\xi||^p < \infty$. $\tilde{N}_{\bar{p}}(dt,du)$ is the compensated Poisson random measure given by

$$\tilde{N}_{\bar{p}}(dt,du) = N_{\bar{p}}(dt,du) - \pi(du)dt,$$

here $\pi(du)$ is the Levy measure associated to $N_{\bar{p}}$.

To study the existence and asymptotic estimations of the solution to equation (1), we consider the following hypotheses.

(H1) Let $D(0) = 0$ and for all $\varphi,\psi \in D([-\tau,0];R^n)$, there exists a constant $k_0 \in (0,1)$ such that

$$|D(\varphi) - D(\psi)| \le k_0||\varphi - \psi||. \tag{3}$$

(H2) For all $\varphi,\psi \in D([-\tau,0];R^n)$, $t \in [t_0,T]$ and $u \in U$, there exist two positive constants k and L_0 such that

$$|f(\varphi,t) - f(\psi,t)|^2 \vee \int_U |h(\varphi,u) - h(\psi,u)|^2\pi(du) \le k||\varphi - \psi||^2. \tag{4}$$

$$|f(0,t)|^2 \vee |h(0,u)|^2 \le L_0. \tag{5}$$

(H3) For all $\varphi, \psi \in D([-\tau, 0]; R^n)$, $p \geq 2$ and $u \in U$, there exists a positive constant L such that

$$|h(\varphi, u) - h(\psi, u)|^p \leq L||\varphi - \psi||^p |u|^p, \tag{6}$$

where $\pi(U) < \infty$ and $\int_U |u|^p du < \infty$.

Clearly, (H2) and (H3) implies the linear growth condition

$$|f(\varphi, t)|^2 \vee \int_U |h(\varphi, u)|^2 \pi(du) \leq L_1(1 + ||\varphi||^2), \tag{7}$$

and

$$\int_U |h(\varphi, u)|^p \pi(du) \leq L_2(1 + ||\varphi||^p), \tag{8}$$

where L_1 and L_2 are two positive constants.

In fact, for any $\varphi \in D([-\tau, 0]; R^n)$ and $t \in [t_0, T]$, it follows from (4) and (5) that

$$\begin{aligned} |f(\varphi, t)|^2 &\leq 2[|f(\varphi, t) - f(0, t)|^2 + |f(0, t)|^2] \\ &\leq 2(k||\varphi||^2 + L_0) \leq L_1(1 + ||\varphi||^2). \end{aligned}$$

and

$$\begin{aligned} \int_U |h(\varphi, u)|^2 \pi(du) &\leq 2[\int_U |h(\varphi, u) - h(0, u)|^2 \pi(du) + \int_U |h(0, u)|^2 \pi(du)] \\ &\leq 2(k||\varphi||^2 + 2L_0 \pi(U)) \leq L_1(1 + ||\varphi||^2). \end{aligned}$$

where $L_1 = max\{2k, 2L_0, 2L_0\pi(U)\}$. Similarly, for any $\varphi \in D([-\tau, 0]; R^n)$ and $t \in [t_0, T]$, it follows from (5) and (6) that

$$\begin{aligned} \int_U |h(\varphi, u)|^p \pi(du) &\leq 2^{p-1} \int_U (|h(\varphi, u) - h(0, u)|^p + |h(0, u)|^p) \pi(du) \\ &\leq 2^{p-1} \int_U |u|^p du k_1 ||\varphi||^p + 2^{p-1} L_0^{\frac{p}{2}} \pi(U) \leq L_2(1 + ||\varphi||^p), \end{aligned}$$

where $L_2 = max\{2^{p-1} \int_U |u|^p du k_1, 2^{p-1} L_0^{\frac{p}{2}} \pi(U)\}$. Hence, the linear growth conditions (7) and (8) are satisfied.

Now we present the definition of the solution to equation (1).

Definition 2.1 A right continuous with left limits process $x = \{x(t),\ t \in [t_0, T]\}$ $(t_0 < T < \infty)$ is called a solution of equation (1) if

(1) $x(t)$ is $\mathcal{F}_t$-adapted and $x = \{x(t),\ t \in [t_0, T]\}$ is R^n-valued;

(2) $\int_{t_0}^T |x(t)|^2 ds < \infty$, *a.s.*;

(3) $x(t) = \xi$ and, for every $t_0 \le t \le T$,

$$x(t) - D(x_t) = x_{t_0} - D(x_{t_0}) + \int_{t_0}^t f(x_s, s)ds + \int_{t_0}^t \int_U h(x_s, u) N_{\bar{p}}(ds, du) \quad a.s.$$

A solution $x(t)$ is said to be unique if any other solution $y(t)$ is indistinguishable from it, that is,

$$P\{x(t) = y(t), t \in [t_0, T]\} = 1.$$

3. The existence and uniqueness theorem

In this section, we establish the existence and uniqueness of the solution to equation (1) under the Lipschitz condition and the local Lipschitz condition.

Define $x_{t_0}^0 = \xi$ and $x^0(t) = \xi(0)$ for $t \in [t_0, T]$. Let $x_{t_0}^n = \xi$, $n = 1, 2, \cdots$ and define the sequence of successive approximations to equation (1)

$$\begin{aligned} x^n(t) - D(x_t^n) &= \xi(0) - D(\xi) + \int_{t_0}^t f(x_s^{n-1}, s)ds \\ &\quad + \int_{t_0}^t \int_U h(x_s^{n-1}, u) N_{\bar{p}}(ds, du), n \ge 1. \end{aligned} \tag{9}$$

Theorem 3.1 Let $p \ge 2$ and suppose that the coefficients of equation (1) satisfy conditions (H1)-(H3), then equation (1) has a unique solution $x(t)$ on $[t_0, T]$ in the sense of L^p-norm.

In order to prove this theorem, let us present three useful lemmas.

Lemma 3.1 [2] Let $p \ge 2, \varepsilon > 0$ and $a, b \in R$,then

$$|a + b|^p \le [1 + \varepsilon^{\frac{1}{p-1}}]^{p-1}(|a|^p + \frac{|b|^p}{\varepsilon}). \tag{10}$$

Lemma 3.2 Under conditions (H1)-(H3), there exists a positive constant c such that

$$E \sup_{t_0 \le t \le T} |x^n(t)|^p \le c, \tag{11}$$

where $c = (c_3 + c_4(T - t_0)E||\xi||^p)e^{c_4(T-t_0)}$, c_3 and c_4 are two positive constants of (25).

Proof: For any $\varepsilon > 0$, it follows from Lemma 3.1 that

$$\begin{aligned} |x^n(t)|^p &= |D(x_t^n) + x^n(t) - D(x_t^n)| \\ &\leq [1 + \varepsilon^{\frac{1}{p-1}}]^{p-1}(|x^n(t) - D(x_t^n)|^p + \frac{|D(x_t^n)|^p}{\varepsilon}). \end{aligned} \tag{12}$$

By (H1), one gets

$$|x^n(t)|^p \leq [1 + \varepsilon^{\frac{1}{p-1}}]^{p-1}(|x^n(t) - D(x_t^n)|^p + \frac{k_0^p||x_t^n||^p}{\varepsilon}). \tag{13}$$

Letting $\varepsilon = [\frac{k_0}{1-k_0}]^{p-1}$ and taking the expectation on both sides of (13), we have

$$\begin{aligned} E\sup_{t_0 \leq s \leq t}|x^n(s)|^p &\leq k_0 E\sup_{t_0 \leq s \leq t}||x_s^n||^p \\ &\quad + \frac{1}{(1-k_0)^{p-1}}E\sup_{t_0 \leq s \leq t}|x^n(s) - D(x_s^n)|^p. \end{aligned} \tag{14}$$

On the other hand, we have

$$\begin{aligned} E\sup_{t_0 \leq s \leq t}||x_s^n||^p &\leq E\sup_{t_0 - \tau \leq s \leq t}|x^n(s)|^p \\ &\leq E||\xi||^p + E\sup_{t_0 \leq s \leq t}|x^n(s)|^p. \end{aligned} \tag{15}$$

Combing (14) and (15), we obtain

$$E\sup_{t_0 \leq s \leq t}|x^n(s)|^p \leq \frac{k_0}{1-k_0}E||\xi||^p + \frac{1}{(1-k_0)^p}E\sup_{t_0 \leq s \leq t}|x^n(s) - D(x_s^n)|^p. \tag{16}$$

By (9) and using the inequality $|a + b + c|^p \leq 3^{p-1}[|a|^p + |b|^p + |c|^p]$, we have

$$\begin{aligned} & E\sup_{t_0 \leq s \leq t}|x^n(s) - D(x_s^n)|^p \\ \leq\ & 3^{p-1}(E(\sup_{t_0 \leq s \leq t}|\xi(0) - D(\xi)|^p) + E\sup_{t_0 \leq s \leq t}|\int_{t_0}^s f(x_\sigma^{n-1}, \sigma)d\sigma|^p \\ & + E\sup_{t_0 \leq s \leq t}|\int_{t_0}^s \int_U h(x_\sigma^{n-1}, u)N_{\bar{p}}(d\sigma, du)|^p) \\ :=\ & 3^{p-1}\sum_{i=1}^3 I_i. \end{aligned} \tag{17}$$

Let us estimate the terms introduced above. Letting $\varepsilon = k_0^{p-1}$, then it follows from Lemma 3.1 that

$$I_1 \leq (1+k_0)^p E|||\xi|^p. \tag{18}$$

By using the *Hölder* inequality and the linear growth condition (7), one gets

$$\begin{aligned} I_2 &\leq (t-t_0)^{p-1} E\int_{t_0}^{t} |f(x_s^{n-1}, s)|^p ds \\ &\leq (t-t_0)^{p-1} E\int_{t_0}^{t} [L_1(1+||x_s^{n-1}||^2)^2]^{\frac{p}{2}} ds \\ &\leq (t-t_0)^{p-1}(2L_1)^{\frac{p}{2}} E\int_{t_0}^{t} (1+||x_s^{n-1}||^p) ds. \end{aligned} \tag{19}$$

For the third term I_3 in (17), we have

$$\begin{aligned} I_3 &= E\sup_{t_0\leq s\leq t} |\int_{t_0}^{s}\int_U h(x_\sigma^{n-1}, u)\tilde{N}_{\bar{p}}(d\sigma, du) + \int_{t_0}^{s}\int_U h(x_\sigma^{n-1}, u)\pi(du)d\sigma|^p \\ &\leq 2^{p-1} E\sup_{t_0\leq s\leq t} |\int_{t_0}^{s}\int_U h(x_\sigma^{n-1}, u)\tilde{N}_{\bar{p}}(d\sigma, du)|^p \\ &\quad +2^{p-1} E\sup_{t_0\leq s\leq t} |\int_{t_0}^{s}\int_U h(x_\sigma^{n-1}, u)\pi(du)d\sigma|^p, \end{aligned} \tag{20}$$

where $\tilde{N}_p(dt, du) := N_p(dt, du) - \pi(du)dt$. For the last term of (20), using the *Hölder* inequality and condition (7), we obtain

$$\begin{aligned} & E\sup_{t_0\leq s\leq t} |\int_{t_0}^{s}\int_U h(x_\sigma^{n-1}, u)\pi(du)d\sigma|^p \\ \leq\ & E[\int_{t_0}^{t} ds]^{p-1}\int_{t_0}^{t} |\int_U h(x_s^{n-1}, u)\pi(du)|^p ds \\ \leq\ & (t-t_0)^{p-1} E\int_{t_0}^{t} [\int_U \pi(du)]^{\frac{p}{2}}[\int_U |h(x_s^{n-1}, u)|^2\pi(du)]^{\frac{p}{2}} ds \\ \leq\ & (t-t_0)^{p-1}[\pi(U)]^{\frac{p}{2}} E\int_{t_0}^{t} [L_1(1+||x_s^{n-1}||^2)]^{\frac{p}{2}} ds \\ \leq\ & (t-t_0)^{p-1}(2L_1)^{\frac{p}{2}}[\pi(U)]^{\frac{p}{2}} E\int_{t_0}^{t} (1+||x_s^{n-1}||^p) ds. \end{aligned} \tag{21}$$

Now let us estimate the martingale part in (20). By the Kunita's estimates (see Kunita [27] and Applebaum [28]), conditions (7)-(8) and properties of stochastic integral with respect to a Poisson random measure, we have a positive real number c_p such that the following inequality holds:

$$\begin{aligned} & E\sup_{t_0\le s\le t}|\int_{t_0}^{s}\int_U h(x_\sigma^{n-1},u)\tilde{N}_{\bar{p}}(d\sigma,du)|^p \\ \le\ & c_p\{E\int_{t_0}^{t}[\int_U |h(x_s^{n-1},u)|^2\pi(du)]^{\frac{p}{2}}ds + E[\int_{t_0}^{t}\int_U |h(x_s^{n-1},u)|^p\pi(du)ds]\} \\ \le\ & c_p\{E\int_{t_0}^{t}[L_1(1+||x_s^{n-1}||)]^{\frac{p}{2}}ds + L_2E\int_{t_0}^{t}(1+||x_s^{n-1}||^p)ds\} \\ \le\ & c_p[(2L_1)^{\frac{p}{2}}+L_2]E\int_{t_0}^{t}(1+||x_s^{n-1}||^p)ds. \end{aligned} \tag{22}$$

Inserting (21) and (22) into (20), we obtain that

$$I_3 \ \le\ c_1E\int_{t_0}^{t}(1+||x_s^{n-1}||^p)ds. \tag{23}$$

where $c_1 = 2^{p-1}[(2L_1)^{\frac{p}{2}}(t-t_0)^{p-1}(\pi(U))^{\frac{p}{2}} + c_p((2L_1)^{\frac{p}{2}}+L_2)]$. Therefore,

$$\begin{aligned} & E\sup_{t_0\le s\le t}|x^n(s)-D(x_s^n)|^p \\ \le\ & 3^{p-1}(1+k_0)^pE||\xi||^p + c_2E\int_{t_0}^{t}(1+||x_s^{n-1}||^p)ds, \end{aligned} \tag{24}$$

where $c_2 = 3^{p-1}[(t-t_0)^{p-1}(2L_1)^{\frac{p}{2}}+c_1]$. Combing (16) and (24) together, we have

$$E\sup_{t_0\le s\le t}|x^n(s)|^p \ \le\ c_3 + c_4E\int_{t_0}^{t}||x_s^{n-1}||^pds. \tag{25}$$

where $c_3 = [\frac{k_0}{1-k_0} + 3^{p-1}\frac{(1+k_0)^p}{(1-k_0)^p}]E||\xi||^p + \frac{c_2}{(1-k_0)^p}(T-t_0)$ and $c_4 = \frac{c_2}{(1-k_0)^p}$. For any $r\ge 1$,

$$\begin{aligned} \max_{1\le n\le r}E\sup_{t_0\le s\le t}|x^n(s)|^p \ &\le\ c_3 + c_4\int_{t_0}^{t}\max_{1\le n\le r}E\sup_{t_0\le\sigma\le s}|x^{n-1}(\sigma)|^pds \\ &\le\ c_3 + c_4\int_{t_0}^{t}(E||\xi||^p + \max_{1\le n\le r}E\sup_{t_0\le\sigma\le s}|x^n(\sigma)|^p)ds. \end{aligned}$$

From the Gronwall inequality, we derive that

$$\max_{1\le n\le r} E\sup_{t_0\le t\le T}|x^n(t)|^p \;\le\; (c_3+c_4(T-t_0)E||\xi||^p)e^{c_4(T-t_0)}.$$

Since r is arbitrary, we must have

$$E\sup_{t_0\le t\le T}|x^n(t)|^p \;\le\; (c_3+c_4(T-t_0)E||\xi||^p)e^{c_4(T-t_0)}, \tag{26}$$

which shows that the desired result holds with $c=(c_3+c_4(T-t_0)E||\xi||^p)e^{c_4(T-t_0)}$.

Lemma 3.3 Let the conditions of Theorem 3.1 hold. Then $\{x^n(t)\}$ $(n\ge 0)$ defined by (9) is a Cauchy sequence in $D([t_0,T],R^n)$.

Proof: For $n\ge 1$ and $t\in[t_0,T]$, it follows from (9) that,

$$\begin{aligned} x^{n+1}(t)-x^n(t) &= D(x_t^{n+1})-D(x_t^n)+\int_{t_0}^t[f(x_s^n,s)-f(x_s^{n-1},s)]ds \\ &\quad +\int_{t_0}^t\int_U[h(x_s^n,u)-h(x_s^{n-1},u)]N_{\bar p}(ds,du). \end{aligned} \tag{27}$$

Similar to the analysis of (14), by lemma 3.1 and taking the expectation on $|x^{n+1}(t)-x^n(t)|^p$, we have

$$\begin{aligned} &E(\sup_{t_0\le s\le t}|x^{n+1}(s)-x^n(s)|^p) \\ \le\;& k_0E(\sup_{t_0\le s\le t}|x^{n+1}(s)-x^n(s)|^p) \\ &+\frac{1}{(1-k_0)^{p-1}}E\sup_{t_0\le s\le t}|[x^{n+1}(s)-x^n(s)]-[D(x_s^{n+1})-D(x_s^n)]|^p. \end{aligned} \tag{28}$$

Consequently,

$$\begin{aligned} &E(\sup_{t_0\le s\le t}|x^{n+1}(s)-x^n(s)|^p) \\ \le\;& \frac{1}{(1-k_0)^p}E\sup_{t_0\le s\le t}|[x^{n+1}(s)-x^n(s)]-[D(x_s^{n+1})-D(x_s^n)]|^p. \end{aligned} \tag{29}$$

The basic inequality $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$ implies that

$$\begin{aligned} &E\sup_{t_0\le s\le t}|[x^{n+1}(s)-x^n(s)]-[D(x_s^{n+1})-D(x_s^n)]|^p \\ \le\;& 2^{p-1}[E\sup_{t_0\le s\le t}|\int_{t_0}^s[f(x_\sigma^n,\sigma)-f(x_\sigma^{n-1},\sigma)]d\sigma|^p \\ &+E\sup_{t_0\le s\le t}|\int_{t_0}^s\int_U[h(x_\sigma^n,u)-h(x_\sigma^{n-1},u)]N_{\bar p}(d\sigma,du)|^p]. \end{aligned} \tag{30}$$

Applying the *Hölder* inequality and (H2), we obtain

$$\begin{aligned} & E\sup_{t_0\le s\le t}|\int_{t_0}^{s}[f(x_\sigma^n,\sigma)-f(x_\sigma^{n-1},\sigma)]d\sigma|^p \\ \le\ & (t-t_0)^{p-1}E\int_{t_0}^{t}|f(x_s^n,s)-f(x_s^{n-1},s)|^p ds \\ \le\ & (t-t_0)^{p-1}k^{\frac{p}{2}}\int_{t_0}^{t}E||x_s^n-x_s^{n-1}||^p ds. \end{aligned} \tag{31}$$

By the Kunita's estimates, *Hölder* inequality and (H2)-(H3), there exists a positive constant c_5 such that

$$\begin{aligned} & E\sup_{t_0\le s\le t}|\int_{t_0}^{s}\int_U[h(x_\sigma^n,u)-h(x_\sigma^{n-1},u)]N_{\bar p}(d\sigma,du)|^p \\ \le\ & 2^{p-1}E\sup_{t_0\le s\le t}|\int_{t_0}^{s}\int_U[h(x_\sigma^n,u)-h(x_\sigma^{n-1},u)]\tilde N_{\bar p}(d\sigma,du)|^p \\ & +2^{p-1}E\sup_{t_0\le s\le t}|\int_{t_0}^{s}\int_U[h(x_\sigma^n,u)-h(x_\sigma^{n-1},u)]\pi(du)d\sigma|^p \\ \le\ & 2^{p-1}[(t-t_0)^{p-1}(\pi(U))^{\frac{p}{2}}E\int_{t_0}^{t}[\int_U|h(x_s^n,u)-h(x_s^{n-1},u)|^2\pi(du)]^{\frac{p}{2}}ds \\ & +c_p2^{p-1}\{E\int_{t_0}^{t}[\int_U|h(x_s^n,u)-h(x_s^{n-1},u)|^2\pi(du)]^{\frac{p}{2}}ds \\ & +E[\int_{t_0}^{t}\int_U|h(x_s^n,u)-h(x_s^{n-1},u)|^p\pi(du)ds]\} \\ \le\ & c_5E\int_{t_0}^{t}||x_s^n-x_s^{n-1}||^p ds, \end{aligned} \tag{32}$$

where $c_5 = 2^{p-1}\{[(t-t_0)^{p-1}(\pi(U))^{\frac{p}{2}}+c_p]k^{\frac{p}{2}}+c_pL\int_U|u|^p\pi(du)\}$. Hence, inserting (30)-(32) into (29) yields

$$E(\sup_{t_0\le s\le t}|x^{n+1}(s)-x^n(s)|^p)\le c_6\int_{t_0}^{t}E(\sup_{t_0\le\sigma\le s}|x^n(\sigma)-x^{n-1}(\sigma)|^p)ds, \tag{33}$$

where $c_6 = k^{\frac{p}{2}}2^{p-1}\frac{1}{(1-k_0)^p}[(T-t_0)^{p-1}+c_5]$.
Setting $\varphi_n(t)=E\sup_{t_0\le s\le t}|x^{n+1}(s)-x^n(s)|^p$, we have

$$\varphi_n(t)\ \le\ c_6\int_{t_0}^{t}\varphi_{n-1}(s_1)ds_1\le c_6^2\int_0^t ds_1\int_{t_0}^{s_1}\varphi_{n-2}(s_2)ds_2$$

$$\leq \cdots$$
$$\leq c_6^n \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \cdots \int_{t_0}^{s_{n-1}} \varphi_0(s_n) ds_n. \tag{34}$$

From the Kunita's estimates, *Hölder* inequality and conditions (7)-(8), we have

$$\varphi_0(t) = E \sup_{t_0 \leq s \leq t} |x^1(s) - x^0(s)|^p \leq c_0. \tag{35}$$

Substituting (35) into (34) and integrating the right hand side, we obtain

$$E(\sup_{t_0 \leq s \leq t} |x^{n+1}(s) - x^n(s)|^p) \leq c_0 \frac{(c_6(t-t_0))^n}{n!}. \tag{36}$$

Taking $t = T$ in (36), we have

$$E(\sup_{t_0 \leq t \leq T} |x^{n+1}(t) - x^n(t)|^p) \leq c_0 \frac{(c_6(T-t_0))^n}{n!}. \tag{37}$$

Then using the Chebyshev inequality, one gets

$$P(\sup_{t_0 \leq t \leq T} |x^{n+1}(t) - x^n(t)| > \frac{1}{2^n}) \leq c_0 M \frac{(c_6(T-t_0))^n}{n!}.$$

Since $\Sigma_{n=0}^{\infty} \frac{c_0 M (c_6(T-t_0))^n}{n!} < \infty$, and by the Borel-Cantelli lemma, we have

$$P(\sup_{t_0 \leq t \leq T} |x^{n+1}(t) - x^n(t)| \leq \frac{1}{2^n}) = 1. \tag{38}$$

(38) implies that for each t, $\{x^n(t)\}_{n=1,2\cdots}$ is a Cauchy sequence on $[t_0, T]$ under sup $|.|$. However, the space $D([t_0, T], R^n)$ is not a complete space under sup $|.|$ and we cannot get the limit of the sequence $\{x^n(t)\}_{n \geq 1}$. So we need to introduce a metric to make the space $D([t_0, T], R^n)$ complete. For any $x, y \in D([t_0, T], R^n)$, P.Billingsley [29] gives the following metric

$$d(x,y) = \inf_{\lambda \in \Lambda} \{ \sup_{t_0 \leq t \leq T} |x_t - y_{\lambda(t)}| + \sup_{t_0 \leq s \leq t \leq T} |log \frac{\lambda(t) - \lambda(s)}{t-s}| \},$$

where $\Lambda = \{\lambda = \lambda(t) : \lambda$ is strictly increasing, continuous on $t \in [t_0, T]$, such that $\lambda(t_0) = t_0, \lambda(T) = T\}$. So we have that $D([t_0, T], R^n)$ is a complete metric space. Taking $\lambda(t) = t$, we can see that $\{x^n(t)\}_{n \geq 1}$ is a cauchy sequence under $d(.,.)$. The proof is completed.

Proof of Theorem 3.1 Uniqueness. Let $x(t)$ and $y(t)$ be two solutions of Eq.(1). Then, for $t \in [t_0, T]$, by the Kunita's estimates, *Hölder* inequality, we have

$$E \sup_{t_0 \le s \le t} |x(s) - y(s)|^p \quad \le \quad c \int_{t_0}^{t} E \sup_{t_0 \le u \le s} |x(u) - y(u)|^p ds. \qquad (39)$$

Therefore, using the Gronwall inequality, we get

$$E \sup_{t_0 \le s \le t} |x(s) - y(s)|^p = 0, \qquad t \in [t_0, T],$$

which implies that $x(t) = y(t)$ for all $t \in [t_0, T]$. Therefore, for all $t \in [t_0, T]$, $x(t) = y(t)$ a.s.

Existence. We derive from Lemma 3.3 that $\{x^n(t)\}_{n=1,2\cdots}$ is a Cauchy sequence in $D([t_0, T], R^n)$. Hence, there exists a unique solution $x(t) \in D([t_0, T], R^n)$ such that $d(x^n(.), x(.)) \to 0$ as $n \to \infty$. For all $t \in [t_0, T]$, taking limits on both sides of (9) and letting $n \to \infty$, we then can show that $x(t)$ is the solution of equation (1). So the proof of Theorem 3.1 is completed.

Next, we relax the Lipschitz conditions (H2)-(H3) and replace them by the following the local Lipschitz conditions.

(H4) For all $\varphi, \psi \in D([-\tau, 0]; R^n)$, $t \in [t_0, T]$, $u \in U$ and $||\varphi|| \vee ||\psi|| \le n$, there exist two positive constants k_n and L_0 such that

$$|f(\varphi, t) - f(\psi, t)|^2 \vee \int_U |h(\varphi, u) - h(\psi, u)|^2 \pi(du) \le k_n ||\varphi - \psi||^2. \qquad (40)$$

(H5) For all $\varphi, \psi \in D([-\tau, 0]; R^n)$, $p \ge 2$, $u \in U$ and $||\varphi|| \vee ||\psi|| \le n$, there exists a positive constant L_n such that

$$|h(\varphi, u) - h(\psi, u)|^p \le L_n ||\varphi - \psi||^p |u|^p. \qquad (41)$$

where $\pi(U) < \infty$ and $\int_U |u|^p du < \infty$.

Then, Theorem 3.1 can be generalized as Theorem 3.2.

Theorem 3.2 Let conditions (H1),(H4) and (H5) hold. Then equation (1) has a unique solution $x(t)$ on $[t_0, T]$ in the sense of L^p-norm. Moreover, there exists a constants c such that

$$E \sup_{t_0 \le t \le T} |x(t)|^p \le c.$$

for any $t \in [t_0, T]$.

Proof: For each $n \geq 1$, define the truncation function

$$f_n(t,x) = \begin{cases} f(t,x), & if \ \ ||x|| \leq n, \\ f(t, \dfrac{nx}{||x||}), & if \ ||x|| \geq n, \end{cases} \tag{42}$$

and

$$h_n(x,u) = \begin{cases} h(x,u), & if \ \ ||x|| \leq n, \\ h(\dfrac{nx}{||x||}, u), & if \ ||x|| \geq n. \end{cases} \tag{43}$$

Then f_n and h_n satisfy the conditions (H1)-(H3). From Theorem 3.1, we have that the following equation

$$\begin{aligned} x_n(t) &= \xi(0) + D((x_n)_t) - D(\xi) + \int_{t_0}^t f_n((x_n)_s, s)ds \\ &\quad + \int_{t_0}^t \int_U h_n((x_n)_s, u) N_{\bar{p}}(ds, du) \end{aligned} \tag{44}$$

has a unique solution $x_n(t)$. Moreover, $x_{n+1}(t)$ is the unique solution of the equation

$$\begin{aligned} x_{n+1}(t) &= \xi(0) + D((x_{n+1})_t) - D(\xi) + \int_{t_0}^t f_{n+1}((x_{n+1})_s, s)ds \\ &\quad + \int_{t_0}^t \int_U h_{n+1}((x_{n+1})_s, u) N_{\bar{p}}(ds, du). \end{aligned} \tag{45}$$

By (44) and (45), we have

$$\begin{aligned} & x_{n+1}(t) - x_n(t) \\ = \ & D((x_{n+1})_t) - D((x_n)_t) + \int_{t_0}^t [f_{n+1}((x_{n+1})_s, s) - f_n((x_n)_s, s)]ds \\ & + \int_{t_0}^t \int_U [h_{n+1}((x_{n+1})_s, u) - h_n((x_n)_s, u)] N_{\bar{p}}(ds, du). \end{aligned} \tag{46}$$

For any fixed $n \geq 1$, define the stopping time

$$\tau_n = T \wedge inf\{t \in [t_0, T] : |(x_n)_t| \geq n\}.$$

Taking the expectation on $|x_{n+1}(t) - x_n(t)|^p$ and by lemma 3.1, it deduces that

$$\begin{aligned} & E \sup_{t_0 \le s \le t \wedge \tau_n} |x_{n+1}(s) - x_n(s)|^p \\ \le \; & \frac{1}{(1-k_0)^{p-1}} E \sup_{t_0 \le s \le t \wedge \tau_n} |[(x_{n+1})_s - (x_n)_s] - [D((x_{n+1})_s) - D((x_n)_s)]|^p \\ & + k_0 E(\sup_{t_0 \le s \le t \wedge \tau_n} |(x_{n+1})_s - (x_n)_s|^p). \end{aligned} \tag{47}$$

Therefore,

$$\begin{aligned} & E \sup_{t_0 \le s \le t \wedge \tau_n} |x_{n+1}(s) - x_n(s)|^p \\ \le \; & \frac{1}{(1-k_0)^p} 2^{p-1} \{ E(\sup_{t_0 \le s \le t \wedge \tau_n} | \int_{t_0}^{s} [f_{n+1}((x_{n+1})_\sigma, \sigma) - f_n((x_n)_\sigma, \sigma)] d\sigma |^p \\ & + E(\sup_{t_0 \le s \le t \wedge \tau_n} | \int_{t_0}^{s} \int_U [h_{n+1}((x_{n+1})_\sigma, u) - h_n((x_n)_\sigma, u)] N_{\bar{p}}(d\sigma, du)|^p \} \\ = \; & \frac{1}{(1-k_0)^p} 2^{p-1} (J_1 + J_2). \end{aligned} \tag{48}$$

By the *Hölder* inequality and rearranging the terms on the right-hand side by plus and minus technique, we have

$$\begin{aligned} J_1 \; \le \; & (t \wedge \tau_n - t_0)^{p-1} E \int_{t_0}^{t \wedge \tau_n} |f_{n+1}((x_{n+1})_s, s) - f_n((x_n)_s, s)|^p ds \\ \le \; & (t \wedge \tau_n - t_0)^{p-1} E \int_{t_0}^{t \wedge \tau_n} \{ 2^{p-1} |f_{n+1}((x_{n+1})_s, s) - f_{n+1}((x_n)_s, s)|^p \\ & + 2^{p-1} |f_{n+1}((x_n)_s, s) - f_n((x_n)_s, s)|^p \} ds. \end{aligned} \tag{49}$$

The Kunita's estimates implies that

$$\begin{aligned} J_2 \; \le \; & c_7 E \int_{t_0}^{t \wedge \tau_n} [\int_U |h_{n+1}((x_{n+1})_s, u) - h_n((x_n)_s, u)|^2 \pi(du)]^{\frac{p}{2}} ds \\ & + c_p 2^{p-1} E \int_{t_0}^{t \wedge \tau_n} \int_U |h_{n+1}((x_{n+1})_s, u) - h_n((x_n)_s, u)|^p \pi(du) ds \\ \le \; & c_7 E \int_{t_0}^{t \wedge \tau_n} \{ \int_U [2 |h_{n+1}((x_{n+1})_s, u) - h_{n+1}((x_n)_s, u)|^2 \end{aligned}$$

$$
\begin{aligned}
&+2|h_{n+1}((x_n)_s,u)-h_n((x_n)_s,u)|^2]\pi(du)\}^{\frac{p}{2}}ds\\
&+c_p2^{2p-2}\{E\int_{t_0}^{t\wedge\tau_n}\int_U|h_{n+1}((x_{n+1})_s,u)-h_{n+1}((x_{n+1})_s,u)|^p\pi(du)ds\\
&+E\int_{t_0}^{t\wedge\tau_n}\int_U|h_{n+1}((x_{n+1})_s,u)-h_n((x_n)_s,u)|^p\pi(du)ds\},
\end{aligned}
\tag{50}
$$

where $c_7=2^{p-1}[(t\wedge\tau_n-t_0)^{p-1}(\pi(U))^{\frac{p}{2}}+c_p]$. Combing (48)-(50) together, it follows that

$$
\begin{aligned}
&E\sup_{t_0\le s\le t\wedge\tau_n}|x_{n+1}(s)-x_n(s)|^p\\
\le\;&\frac{1}{(1-k_0)^p}2^{p-1}(T-t_0)^{p-1}E\int_{t_0}^{t\wedge\tau_n}\{2^{p-1}|f_{n+1}((x_{n+1})_s,s)\\
&-f_{n+1}((x_n)_s,s)|^p+2^{p-1}|f_{n+1}((x_n)_s,s)-f_n((x_n)_s,s)|^p\}ds\\
&+\frac{1}{(1-k_0)^p}2^{p-1}\{c_7E\int_{t_0}^{t\wedge\tau_n}\{\int_U[2|h_{n+1}((x_{n+1})_s,u)-h_{n+1}((x_n)_s,u)|^2\\
&+2|h_{n+1}((x_n)_s,u)-h_n((x_n)_s,u)|^2]\pi(du)\}^{\frac{p}{2}}ds\\
&+c_p2^{2p-2}\{E\int_{t_0}^{t\wedge\tau_n}\int_U|h_{n+1}((x_{n+1})_s,u)-h_{n+1}((x_{n+1})_s,u)|^p\pi(du)ds\\
&+E\int_{t_0}^{t\wedge\tau_n}\int_U|h_{n+1}((x_{n+1})_s,u)-h_n((x_n)_s,u)|^p\pi(du)ds\}\}.
\end{aligned}
\tag{51}
$$

For $t_0\le t\le\tau_n$, we have

$$
\begin{aligned}
f_{n+1}((x_n)_t,t)&=f_n((x_n)_t,t)=f((x_n)_t,t),\\
h_{n+1}((x_n)_t,u)&=h_n((x_n)_t,u)=h((x_n)_t,u).
\end{aligned}
\tag{52}
$$

By (52), we get from (51),

$$
\begin{aligned}
&E\sup_{t_0\le s\le t\wedge\tau_n}|x_{n+1}(s)-x_n(s)|^p\\
\le\;&\frac{1}{(1-k_0)^p}2^{2p-2}(T-t_0)^{p-1}E\int_{t_0}^{t\wedge\tau_n}|f_{n+1}((x_{n+1})_s,s)-f_{n+1}((x_n)_s,s)|^pds\\
&+\frac{1}{(1-k_0)^p}2^{p-1}\{c_7E\int_{t_0}^{t\wedge\tau_n}\{\int_U[2|h_{n+1}((x_{n+1})_s,u)\\
&-h_{n+1}((x_n)_s,u)|^2]\pi(du)\}^{\frac{p}{2}}ds
\end{aligned}
$$

$$+c_p 2^{2p-2} E\int_{t_0}^{t\wedge\tau_n}\int_U |h_{n+1}((x_{n+1})_s,u)-h_{n+1}((x_{n+1})_s,u)|^p\pi(du)ds\}. \tag{53}$$

By the local Lipschitz conditions (H4) and (H5), we have

$$\begin{aligned} E\sup_{t_0\le s\le t\wedge\tau_n}|x_{n+1}(s)-x_n(s)|^p &\le c_8 E\int_{t_0}^{t\wedge\tau_n}||(x_{n+1})_s-(x_n)_s||^p ds \\ &\le c_8\int_{t_0}^{t} E\sup_{t_0\le\sigma\le s\wedge\tau_n}|x_{n+1}(\sigma)-x_n(\sigma)|^p ds, \end{aligned} \tag{54}$$

where $c_8=\frac{1}{(1-k_0)^p}\{k_{n+1}^{\frac{p}{2}}[2^{2p-2}(T-t_0)^{p-1}+2^{\frac{3}{2}p-2}c_7]+c_p 2^{3p-3}L_{n+1}\int_U|u|^p\pi(du)\}$. From (54) and the Gronwall inequality, we get

$$E\sup_{t_0\le s\le t\wedge\tau_n}|x_{n+1}(s)-x_n(s)|^p=0, \tag{55}$$

which yields

$$x_{n+1}(t)=x_n(t),\quad for\quad t\in[t_0,\tau_n]. \tag{56}$$

It then deduced that τ_n is increasing, that is as $n\to\infty$, $\tau_n\uparrow T$ a.s. By the linear growth condition (7) and (8), for almost all $\omega\in\Omega$, there exists an integer $n_0=n_0(\omega)$ such that $\tau_n=T$ as $n\ge n_0$. Now define $x(t)$ by $x(t)=x_{n_0}(t)$ for $t\in[t_0,T]$. Next to verify that $x(t)$ is the solution of equation (1). By (56), $x(t\wedge\tau_n)=x_n(t\wedge\tau_n)$, and it follows from (44) that

$$\begin{aligned} & x(t\wedge\tau_n)-D(x_{t\wedge\tau_n}) \\ =\ & \xi(0)-D(\xi)+\int_{t_0}^{t\wedge\tau_n} f_n(x_s,s)ds+\int_{t_0}^{t\wedge\tau_n}\int_U h_n(x_s,u)N_{\bar p}(ds,du) \\ =\ & \xi(0)-D(\xi)+\int_{t_0}^{t\wedge\tau_n} f(x_s,s)ds+\int_{t_0}^{t\wedge\tau_n}\int_U h(x_s,u)N_{\bar p}(ds,du). \end{aligned} \tag{57}$$

Letting $n\to\infty$ on both sides of (57), we obtain

$$\begin{aligned} & x(t\wedge T)-D(x_{t\wedge T}) \\ =\ & \xi(0)-D(\xi)+\int_{t_0}^{t\wedge T} f(x_s,s)ds+\int_{t_0}^{t\wedge T}\int_U h(x_s,u)N_{\bar p}(ds,du). \end{aligned}$$

that is

$$x(t) - D(x_t) = \xi(0) - D(\xi) + \int_{t_0}^{t} f(x_s, s)ds + \int_{t_0}^{t}\int_U h(x_s, u)N_{\bar{p}}(ds, du),$$

which implies that $x(t)$ is the solution of equation (1). By stopping our process $x(t)$, uniqueness of the solution to equation (1) is obtained. Moreover, by the proof of Theorem 3.1, we can easily obtain that $E \sup_{t_0\le t\le T} |x(t)|^p \le c$. The proof is completed.

4. Asymptotic estimations for solutions

In this section, we will give the exponential estimate of the solution to equation (1).

According to the definition of $\tilde{N}_{\bar{p}}(dt, du) := N_{\bar{p}}(dt, du) - \pi(du)dt$, we can rewrite equation (1) as the following equation

$$d[x(t) - D(x_t)] = F(t, x_t)dt + \int_U h(x_t, u)\tilde{N}_{\bar{p}}(dt, du), \tag{58}$$

where $F(t, x_t) = f(x_t, t) + \int_U h(x_t, u)\pi(du)$.

Let $C^{2,1}(R^n \times [t_0 - \tau, T), R_+)$ denote the family of all nonnegative functions $V(x, t)$ on $R^n \times [t_0 - \tau, T)$ which are continuously twice differentiable with respect to x and continuously once differentiable with respect to t. For a $V \in C^{2,1}(R^n \times [t_0 - \tau, T), R_+)$, one can define the Kolmogorov operator LV as follows:

$$\begin{aligned} LV(x, y, t) \equiv\ & V_t(x - D(y), t) + V_x(x - D(y), t)F(t, y) \\ & + \int_U [V(x - D(y) + h(y, u), t) - V(x - D(y), t) \\ & - V_x(x - D(y), t)h(y, u)]\pi(du), \end{aligned} \tag{59}$$

where

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t}, \quad V_x(x, t) = (\frac{\partial V(x, t)}{\partial x_1}, \cdots, \frac{\partial V(x, t)}{\partial x_n}).$$

First, we establish the p-th exponential estimations of the solution to equation (1).

Theorem 4.1 Let $\{x(t), t_0 \leq t \leq T\}$ be a solution of equation (1) whose coefficients satisfy conditions (H1) and (H2). For a given integer $p \geq 2$ and any $2 \leq q \leq p$, there exists a positive constant K such that

$$\int_U |h(\varphi, u)|^q \pi(du) \leq K||\varphi||^q. \tag{60}$$

Then, for any $t_0 \leq t \leq T$,

$$E \sup_{t_0-\tau \leq s \leq t} |x(s)|^p \leq [1 + (1 + c_{12})E||\xi||^p]e^{M(t-t_0)}, \tag{61}$$

where $M = \frac{2(c_9+c_{10}+c_{11})}{(1-k_0)^p}$. c_9, c_{10}, c_{11}, c_{12} are four positive constants of (68), (74), (80), (82).

Proof: Let $V(x(t) - D(x_t), t) = 1 + |x(t) - D(x_t)|^p$, then $V_t(x(t) - D(x_t), t) = 0$. Applying the It$\widehat{o}$ formula to $V(x(t) - D(x_t), t)$, we obtain that

$$\begin{aligned} V(x(t) - D(x_t), t) &= V(x(t_0 - D(x_{t_0})), t_0) + \int_{t_0}^t LV(x(s), x_s, s)ds \\ &\quad + \int_{t_0}^t \int_U [V(x(s) - D(x_s) + h(x_s, u), s) \\ &\quad - V(x(s) - D(x_s), s)]\tilde{N}_{\bar{p}}(ds, du). \end{aligned} \tag{62}$$

By (59), we have

$$\begin{aligned} &1 + |x(t) - D(x_t)|^p = 1 + |x(t_0) - D(x_{t_0})|^p \\ &+p \int_{t_0}^t |x(s) - D(x_s)|^{p-2}[x(s) - D(x_s)]^\top F(s, x_s)ds \\ &+ \int_{t_0}^t \int_U \{(1 + |x(s) - D(x_s) + h(x_s, u)|^p) - (1 + |x(s) - D(x_s)|^p) \\ &-p|x(s) - D(x_s)|^{p-2}[x(s) - D(x_s)]^\top h(x_s, u)\}\pi(du)ds \\ &+ \int_{t_0}^t \int_U \{(1 + |x(s) - D(x_s) + h(x_s, u)|^p) \\ &-(1 + |x(s) - D(x_s)|^p)\}\tilde{N}_{\bar{p}}(ds, du). \end{aligned} \tag{63}$$

Taking the expectation on both sides of (63), one gets

$$E \sup_{t_0 \leq s \leq t} (1 + |x(s) - D(x_s)|^p)$$

$$
\begin{aligned}
&\le 1 + E\sup_{t_0\le s\le t}|\xi + D(\xi)|^p + pE\int_{t_0}^{t}|x(s)-D(x_s)|^{p-1}|F(s,x_s)|ds \\
&\quad +pE\sup_{t_0\le s\le t}\int_{t_0}^{s}\int_U |x(\sigma)-D(x_\sigma)|^{p-2}[x(\sigma)-D(x_\sigma)]^\top h(x_\sigma,u)\tilde{N}_{\bar{p}}(d\sigma,du) \\
&\quad +E\sup_{t_0\le s\le t}\int_{t_0}^{s}\int_U \{|x(\sigma)-D(x_\sigma)+h(x_\sigma,u)|^p - |x(\sigma)-D(x_\sigma)|^p \\
&\quad -p|x(\sigma)-D(x_\sigma)|^{p-2}[x(\sigma)-D(x_\sigma)]^\top h(x_\sigma,u)\}N_{\bar{p}}(d\sigma,du) \\
&\le 1 + (1+k_0)^p E||\xi||^p + \sum_{i=1}^{3} Q_i, \qquad (64)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1 &= pE\int_{t_0}^{t}|x(s)-D(x_s)|^{p-1}|F(s,x_s)|ds, \\
Q_2 &= pE\sup_{t_0\le s\le t}\int_{t_0}^{s}\int_U |x(\sigma)-D(x_\sigma)|^{p-2}[x(\sigma)-D(x_\sigma)]^\top h(x_\sigma,u)\tilde{N}_{\bar{p}}(d\sigma,du), \\
Q_3 &= E\sup_{t_0\le s\le t}\int_{t_0}^{s}\int_U \{|x(\sigma)-D(x_\sigma)+h(x_\sigma,u)|^p - |x(\sigma)-D(x_\sigma)|^p \\
&\quad -p|x(\sigma)-D(x_\sigma)|^{p-2}[x(\sigma)-D(x_\sigma)]^\top h(x_\sigma,u)\}N_{\bar{p}}(d\sigma,du).
\end{aligned}
$$

Let us estimate Q_1. By the basic inequality

$$
a^r b^{1-r} \le ra + (1-r)b, \qquad r\in[0,1],
$$

we derive that

$$
a^{p-1}b \le \frac{\varepsilon_1(p-1)}{p}a^p + \frac{1}{p\varepsilon_1^{p-1}}b^p,
$$

where $a, b, \varepsilon_1 > 0$. Hence,

$$
\begin{aligned}
Q_1 &\le pE\int_{t_0}^{t}[\frac{\varepsilon_1(p-1)}{p}|x(s)-D(x_s)|^p + \frac{1}{p\varepsilon_1^{p-1}}|F(s,x_s)|^p]ds \\
&\le pE\int_{t_0}^{t}[\frac{\varepsilon_1(p-1)}{p}(1+k_0)^p||x_s|^p + \frac{1}{p\varepsilon_1^{p-1}}|F(s,x_s)|^p]ds. \qquad (65)
\end{aligned}
$$

By using lemma 3.1, we have

$$
E\int_{t_0}^{t}|F(s,x_s)|^p ds
$$

$$\leq E\int_{t_0}^{t}[1+\varepsilon^{\frac{1}{p-1}}]^{p-1}[|\int_U h(x_s,u)\pi(du)|^p+\frac{|f(x_s,s)|^p}{\varepsilon}]ds.$$
$$\leq (2L)^{\frac{p}{2}}E\int_{t_0}^{t}[1+\varepsilon^{\frac{1}{p-1}}]^{p-1}[((\pi(U))^{\frac{p}{2}}+\frac{1}{\varepsilon})(1+||x_s||^p)]ds. \quad (66)$$

Letting $\varepsilon=(2L)^{p-1}$, then we get

$$E\int_{t_0}^{t}|F(s,x_s)|^p ds \leq (1+2L)^{\frac{3p}{2}-1}(\pi(U))^{\frac{p}{2}}E\int_{t_0}^{t}(1+||x_s||^p)ds. \quad (67)$$

Inserting (67) into (65) and letting $\varepsilon_1=\frac{1+2L}{1+k_0}$, we obtain that

$$\begin{aligned} Q_1 &\leq pE\int_{t_0}^{t}[\frac{\varepsilon_1(p-1)(1+k_0)^p}{p}||x_s||^p \\ &\quad +\frac{(1+2L)^{\frac{3p}{2}-1}(\pi(U))^{\frac{p}{2}}}{p\varepsilon_1^{\frac{p-2}{2}}}(1+||x_s||^p)]ds \\ &\leq c_9E\int_{t_0}^{t}(1+||x_s||^p)ds, \end{aligned} \quad (68)$$

where $c_9=(1+2L)^p(1+k_0)^{p-1}[p+(\pi(U))^{\frac{p}{2}}]$. For the estimation of Q_2. By using the Burkholder-Davis inequality, there exists a positive constant $\tilde{c}_p$ such that

$$\begin{aligned} Q_2 &\leq p\tilde{c}_pE[\int_{t_0}^{t}\int_U|x(s)-D(x_s)|^{2p-2}|h(x_s,u)|^2\pi(du)ds]^{\frac{1}{2}} \\ &\leq p\tilde{c}_pE[\sup_{t_0\leq s\leq t}|x(s)-D(x_s)|^p(\int_{t_0}^{t}\int_U|x(s)-D(x_s)|^{p-2} \\ &\quad \times|h(x_s,u)|^2\pi(du)ds)]^{\frac{1}{2}}. \end{aligned} \quad (69)$$

Further, for any $\varepsilon>0$, the Young inequality implies that

$$\begin{aligned} Q_2 &\leq p\tilde{c}_p[\varepsilon E\sup_{t_0\leq s\leq t}|x(s)-D(x_s)|^p]^{\frac{1}{2}}[\frac{1}{\varepsilon}E(\int_{t_0}^{t}\int_U|x(s)-D(x_s)|^{p-2} \\ &\quad \times|h(x_s,u)|^2\pi(du)ds)]^{\frac{1}{2}} \\ &\leq \frac{p\tilde{c}_p\varepsilon}{2}E\sup_{t_0\leq s\leq t}|x(s)-D(x_s)|^p+\frac{p\tilde{c}_p}{2\varepsilon}E\int_{t_0}^{t}\int_U|x(s)-D(x_s)|^{p-2} \\ &\quad \times|h(x_s,u)|^2\pi(du)ds. \end{aligned} \quad (70)$$

Letting $\varepsilon = \frac{1}{p\tilde{c}_p}$, we obtain

$$Q_2 \leq \frac{1}{2}E\sup_{t_0\leq s\leq t}|x(s)-D(x_s)|^p + \frac{1}{2}p^2\tilde{c}_p^2 E\int_{t_0}^t\int_U |x(s)-D(x_s)|^{p-2} \times|h(x_s,u)|^2\pi(du)ds. \quad (71)$$

By the following inequality (see Mao[2]),

$$a^{p-2}b^2 \leq \frac{\varepsilon_2(p-2)}{p}a^p + \frac{1}{p\varepsilon_2^{\frac{p-2}{2}}}b^p, \quad a,b,\varepsilon_2>0,$$

and condition (60), we have

$$\begin{aligned} & E\int_{t_0}^t\int_U |x(s)-D(x_s)|^{p-2}|h(x_s,u)|^2\pi(du)ds \\ \leq\ & \frac{(p-2)\varepsilon_2}{p}E\int_{t_0}^t\int_U |x(s)-D(x_s)|^p\pi(du)ds \\ & +\frac{2}{p\varepsilon_2^{\frac{p-2}{2}}}E\int_0^t\int_U |h(x_s,u)|^p\pi(du)ds] \\ \leq\ & [\frac{(p-2)\varepsilon_2(1+k_0)^p}{p}]E\int_{t_0}^t\int_U ||x_s||^p\pi(du)ds \\ & +\frac{2}{p\varepsilon_2^{\frac{p-2}{2}}}KE\int_0^t ||x_s||^p ds] \end{aligned} \quad (72)$$

Letting $\varepsilon_2 = \frac{1}{(1+k_0)^2}$,

$$\begin{aligned} & E\int_{t_0}^t\int_U |x(s)-D(x_s)|^{p-2}|h(x_s,u)|^2\pi(du)ds \\ \leq\ & [\frac{(p-2)}{p}\pi(U)+\frac{2K}{p}](1+k_0)^{p-2}E\int_{t_0}^t ||x_s||^p ds. \end{aligned} \quad (73)$$

Inserting (73) into (71), we obtain

$$Q_2 \leq c_{10}E\int_{t_0}^t ||x_s||^p ds + \frac{1}{2}E\sup_{t_0\leq s\leq t}|x(s)-D(x_s)|^p, \quad (74)$$

where $c_{10} = \frac{1}{2}p\tilde{c}_p^2[(p-2)\pi(U) + 2K](1+k_0)^{p-2}$. Finally, let us estimate Q_3. Since $N_{\bar{p}}(dt,du) = \tilde{N}_{\bar{p}}(dt,du) + \pi(du)dt$ and $\tilde{N}_{\bar{p}}(dt,du)$ is a martingale measure, we get

$$\begin{aligned} Q_3 &= E\int_{t_0}^t \int_U \{|x(s) - D(x_s) + h(x_s,u)|^p - |x(s) - D(x_s)|^p \\ &\quad -p|x(s) - D(x_s)|^{p-2}[x(s) - D(x_s]^\top h(x_s,u)\}\pi(du)ds. \end{aligned} \tag{75}$$

We note that it has the form

$$\begin{aligned} E\int_{t_0}^t \int_U &\{f(x(s) - D(x_s) + h(x_s,u)) - f(x(s) - D(x_s)) \\ &\qquad -f'(x(s) - D(x_s))h(x_s,u)\}\pi(du)ds, \end{aligned} \tag{76}$$

where $f(x) = |x|^p$. Using the Taylor formula, there exists a positive constant M_p, such that for $p \geq 2$

$$\begin{aligned} & f(x(s) - D(x_s) + h(x_s,u)) - f(x(s) - D(x_s)) \\ & -f'(x(s) - D(x_s))h(x_s,u) \\ = \; & |x(s) - D(x_s) + h(x_s,u)|^p - |x(s) - D(x_s)|^p \\ & -p|x(s) - D(x_s)|^{p-2}[x(s) - D(x_s)]^\top h(x_s,u) \\ \leq \; & M_p[|x(s) - D(x_s) + h(x_s,u)|^{p-2}|h(x_s,u)|^2]. \end{aligned} \tag{77}$$

Again the basic inequality $|a+b|^{p-2} \leq 2^{p-3}(|a|^{p-2} + |b|^{p-2})$ and the Young inequality implies that

$$\begin{aligned} Q_3 &\leq M_p E\int_{t_0}^t \int_U [|x(s) - D(x_s) + h(x_s,u)|^{p-2}|h(x_s,u)|^2]\pi(du)ds \\ &\leq M_p 2^{p-3} E\int_{t_0}^t \int_U [(|x(s) - D(x_s)|^{p-2} + |h(x_s,u)|^{p-2})|h(x_s,u)|^2]\pi(du)ds \\ &\leq M_p 2^{p-3}\frac{p-2}{p}(1+k_0)^{p-2}\pi(U)E\int_{t_0}^t ||x_s||^p ds \\ &\quad + M_p 2^{p-3}(1 + \frac{2(1+k_0)^{p-2}}{p})E\int_{t_0}^t \int_U |h(x_s,u)|^p \pi(du)ds. \end{aligned} \tag{78}$$

By (60), we have

$$E\int_{t_0}^t \int_U |h(x_s,u)|^p \pi(du)ds \;\leq\; KE\int_{t_0}^t ||x_s||^p ds. \tag{79}$$

Substituting (79) into (78),

$$Q_3 \leq c_{11} \int_{t_0}^{t} E||x_s||^p ds, \tag{80}$$

where $c_{11} = M_p 2^{p-3}[\frac{p-2}{p}(1+k_0)^{p-2}\pi(U) + (1 + \frac{2(1+k_0)^{p-2}}{p})K]$. Combing (64), (68), (74) and (80) together, we obtain that

$$\begin{aligned} E \sup_{t_0 \leq s \leq t}(1 + |x(s) - D(x_s)|^p) &\leq 2 + 2(1+k_0)^p E||\xi||^p \\ &\quad + 2(c_9 + c_{10} + c_{11}) \int_{t_0}^{t} E(1 + ||x_s||^p) ds. \end{aligned} \tag{81}$$

On the other hand, by lemma 3.1, we have

$$\begin{aligned} E \sup_{t_0 \leq s \leq t} |x(s)|^p &\leq \frac{k_0}{1-k_0} E||\xi||^p \\ &\quad + \frac{1}{(1-k_0)^p} E \sup_{t_0 \leq s \leq t}(1 + |x(s) - D(x_s)|^p) \\ &\leq c_{12} E||\xi||^p + \frac{2(c_9 + c_{10} + c_{11})}{(1-k_0)^p} \int_{t_0}^{t} E(1 + ||x_s||^p) ds, \end{aligned} \tag{82}$$

where $c_{12} = \frac{k_0}{1-k_0} + \frac{2+2(1+k_0)^p}{(1-k_0)^p}$. Consequently,

$$\begin{aligned} E(1 + \sup_{t_0 - \tau \leq s \leq t} |x(s)|^p) &\leq 1 + (1 + c_{12}) E||\xi||^p \\ &\quad + \frac{2(c_9 + c_{10} + c_{11})}{(1-k_0)^p} \int_{t_0}^{t} E(1 + \sup_{t_0 - \tau \leq \sigma \leq s} |x(\sigma)|^p) ds. \end{aligned} \tag{83}$$

Therefore, we apply the Gronwall inequality to get

$$E(1 + \sup_{t_0 - \tau \leq s \leq t} |x(s)|^p) \leq [1 + (1 + c_{12}) E||\xi||^p] e^{M(t-t_0)},$$

where $M = \frac{2(c_9+c_{10}+c_{11})}{(1-k_0)^p}$. This completes the proof.

The next result shows that exponential estimations implies almost surely asymptotic estimations, and we give an upper bound for the sample Lyapunov exponent.

Theorem 4.2 Under the conditions (H1)-(H2), we have

$$\limsup_{t\to\infty}\frac{1}{t}log|x(t)| \le \frac{L[(1+k_0^2)+3+2\pi(U)+2\tilde{c}_2^2]}{(1-k_0)^2}, \quad a.s. \tag{84}$$

That is, the sample Lyapunov exponent of the solution should not be greater than $\frac{L[(1+k_0^2)+3+2\pi(U)+2\tilde{c}_2^2]}{(1-k_0)^2}$.

Proof: For each $n = 1, 2, \ldots$, it follows from Theorem 4.1 (taking $p = 2$) that

$$E(\sup_{t_0+n-1\le t\le t_0+n}|x(t)|^2) \;\le\; \beta e^{\gamma n},$$

where $\beta = \frac{k_0}{1-k_0}E||\xi||^2 + \frac{2[3+2\pi(U)+2\tilde{c}_2^2]L(T-t_0)}{(1-k_0)^2}$ and $\gamma = \frac{2L[(1+k_0^2)+3+2\pi(U)+2\tilde{c}_2^2]}{(1-k_0)^2}$. Hence, for any $\varepsilon > 0$, by the chebysher inequality, it follows that

$$P\{\omega : \sup_{t_0+n-1\le t\le t_0+n}|x(t)|^2 > e^{(\gamma+\epsilon)n}\} \;\le\; \beta e^{-\epsilon n}.$$

Since $\Sigma_{n=0}^{\infty}\beta e^{-\varepsilon n} < \infty$, by the Borel-Cantelli lemma, we deduce that, there exists a integer n_0 such that

$$\sup_{t_0+n-1\le t\le t_0+n}|x(t)|^2 \le e^{(\gamma+\varepsilon)n} \quad a.s \quad n \ge n_0.$$

Thus, for almost all $\omega \in \Omega$, if $t_0 + n - 1 \le t \le t_0 + n$ and $n \ge n_0$, then

$$\frac{1}{t}log|x(t)| = \frac{1}{2t}log(|x(t)|^2) \le \frac{(\gamma+\varepsilon)n}{2(t_0+n-1)}. \tag{85}$$

Taking limsup in (85) leads to almost surely exponential estimate, that is,

$$\limsup_{t\to\infty}\frac{1}{t}log|x(t)| \le \frac{\gamma+\varepsilon}{2} = \frac{L[(1+k_0^2)+3+2\pi(U)+2\tilde{c}_2^2]}{(1-k_0)^2}, \qquad a.s.$$

Required assertion (84) follows because $\varepsilon > 0$ is arbitrary.

5. Conclusion

In this paper, we prove the existence and uniqueness of the solution to NSFDEs with pure jumps under the Local Lipschitz condition. Meanwhile, by using the Itô formula, Taylor formula and the Burkholder-Davis inequality, we establish the p-th exponential estimations and almost surely asymptotic estimations of the solution to NSFDEs with pure jumps.

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