(a, b)-rectangle patterns in permutations and words

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Abstract

In this paper, we introduce the notion of an (a, b)-rectangle pattern on permutations which is closely related to the notion of successive elements (bonds) in permutations and to mesh patterns introduced recently by Brändén and Claesson. We call the (k, k)-rectangle pattern the k-box pattern. We show that we can derive an explicit formula for the number of permutations of S_n which have the maximum possible occurrences of the 1-box pattern by using a new enumerative result on pattern-avoidance in signed permutations.

We also study the notion of (a, b)-rectangle patterns in words. In particular, we give a general method for computing the generating function for the distribution of $(1, b)$-rectangle patterns on words over an alphabet of size k for $b \in \{1, 2\}$. Our method requires to invert a certain matrix depending on b and k, and can be used to give explicit formuals for such generating functions for $k = 2, \ldots, 7$. We also provide similar results for the distribution of bonds over words. As a corollary to our studies, we prove a conjecture of Mathar on the number of "stable LEGO walls" of width 7, as well as prove three conjectures due to Hardin and a conjecture due to Barker. We provide generating functions for two sequences published by Hardin in the On-Line Encyclopedia of Integer Sequences.

Keywords: (a, b)-rectangle patterns, k-box patterns, bond, k-bond, mesh patterns, permutations, words, distribution, successions in permutations, Fibonacci numbers, LEGO

1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [2] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns (see [4] for a comprehensive introduction to the theory of patterns in permutations and words and also appropriate portions of [3]). This notion was studied further in a series of papers, e.g. in [1, 5, 6, 7, 12].

In this paper, we introduce the notion of an (a, b)-*rectangle* pattern in permutations and words. Formally, given a sequence $\alpha = \alpha_1 \dots \alpha_n$ of postive integers, we say that α_i matches the (a, b)-rectangle pattern in α if and only if there exists a j such that $0 < |i - j| \leq a$ and $|\alpha_i - \alpha_j| \leq b$. The (a, b)-rectangle pattern has a nice pictorial description in permutations. That is, let $\sigma = \sigma_1 \dots \sigma_n \in S_n$ be a permutation written in one-line notation where S_n denotes the set of all permutations of length n. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \dots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1.

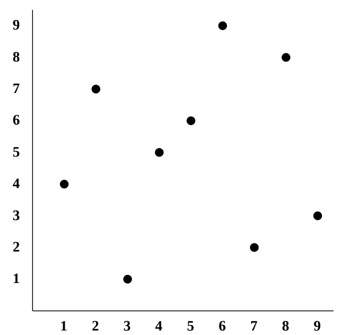

Figure 1: The graph of $\sigma = 471569283$.

We will be interested in the points that lie in the $(2a) \times (2b)$ rectangle centred at a point (i, σ_i), that is, in the set of points $(i \pm r, \sigma_i \pm s)$ such that $r \in \{0, 1, \dots, a\}$ and $s \in \{0, 1, \dots, b\}$. Then σ_i *matches the* (a, b)-*rectangle pattern* in σ, if there is at least one point in the $(2a) \times (2b)$ rectangle centered at the point (i, σ_i) in $G(\sigma)$ other than (i, σ_i). For example, when we look for matches of the (2,3)-rectangle patterns, we would look at 4×6 rectangles centered at points (i, σ_i) for $i = 1, \dots, n$, as pictured in Figure 2 for the point $(4, 5)$ where the 4×6 rectangle is represented by the circled points.

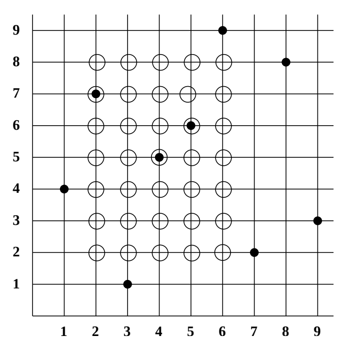

Figure 2: The 4×6 rectangle centered at the point $(4, 5)$ in the graph of $\sigma = 471569283$.

We shall refer to the (k, k)-rectangle pattern as the k-box pattern. For example, if $\sigma = 471569283$, then the 2-box centered at the point $(4, 5)$ in $G(\sigma)$ is the set of circled points pictured in Figure 3. Hence, σ_i matches the k-box pattern in σ, if there is at least one point in the k-box centered at the point (i, σ_i) in $G(\sigma)$ other than (i, σ_i). For example, σ_4 matches the pattern k-box for all $k \geq 1$ in $\sigma = 471569283$ since the point $(5, 6)$ is present in the k-box centered at the point $(4, 5)$ in $G(\sigma)$ for all $k \geq 1$. However, σ_3 only matches the k-box pattern in $\sigma = 471569283$ for $k \geq 3$ since there are no points in 1-box or 2-box centered at $(3, 1)$ in $G(\sigma)$, but the point $(1, 4)$ is in the 3-box centered at $(3, 1)$ in $G(\sigma)$. For $k \geq 1$, we let $k\text{box}(\sigma)$ denote the set of all i such that σ_i matches the k-box pattern in $\sigma = \sigma_1 \ldots \sigma_n$.

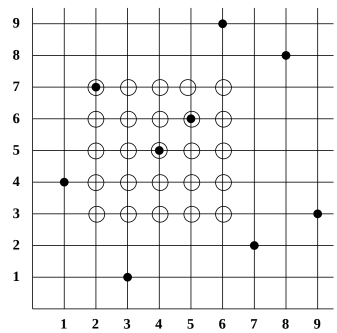

Figure 3: The 2-box centered at the point $(4, 5)$ in the graph of $\sigma = 471569283$.

In this paper, we shall mainly be interested in the 1-box patterns in permutations and words. Note that σ_i matches the 1-box pattern in a permutation $\sigma = \sigma_1 \ldots \sigma_n$ if either $|\sigma_i - \sigma_{i+1}| = 1$ or $|\sigma_{i-1} - \sigma_i| = 1$, while if $w = w_1 \ldots w_n$ is a word of postive integers, then w_i matches the 1-box pattern in w if either $|w_i - w_{i+1}| \leq 1$ or $|w_{i-1} - w_i| \leq 1$. For any permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, let $1\text{box}(\sigma)$ denote the number of i such that σ_i matches the 1-box pattern in σ. We let $(a, b)\text{rec}(\sigma)$ denote the number of i such that σ_i matches the (a, b)-rectangle pattern in σ.

Avoidance of the 1-box pattern is given by permutations without rising or falling successions which are also called bonds. That is, a *bond* in a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ is a pair $\sigma_i \sigma_{i+1}$ of the form $s(s+1)$ or $(s+1)s$ for some s. We let $\text{bond}(\sigma)$ denote the number of bonds in σ. We note that in general $1\text{box}(\sigma) \neq \text{bond}(\sigma)$. For example, if $\sigma = 214365$, then $1\text{box}(\sigma) = 6$ while $\text{bond}(\sigma) = 3$. However, for any permutation $\sigma \in S_n$, $1\text{box}(\sigma) = 0$ if and only if $\text{bond}(\sigma) = 0$.

The distributions of $1\text{box}(\sigma)$ and $\text{bond}(\sigma)$ for S_2, S_3, and S_4 are given below.

σ	$1\text{box}(\sigma)$	$\text{bond}(\sigma)$
12	2	1
21	2	1

σ	$1\text{box}(\sigma)$	$\text{bond}(\sigma)$
123	3	2
132	2	1
213	2	1
231	2	1
312	2	1
321	3	2

σ	1box(σ)	bond(σ)		σ	1box(σ)	bond(σ)
1234	4	3		2134	4	2
1243	4	2		2143	4	2
1324	2	1		2314	2	1
1342	2	1		2341	3	2
1423	2	1		2413	0	0
1432	3	2		2431	2	1
3124	2	1		4123	3	2
3142	0	0		4132	2	1
3214	3	2		4213	2	1
3241	2	1		4231	2	1
3412	4	2		4312	4	2
3421	4	2		4321	4	3

Finding the number of permutations σ of length n with bond(σ) = 0 (equivalently, 1box(σ) = 0) is equivalent to solving the *problem of Hertzsprung*, which is finding the number of ways to arrange n non-attacking kings on an $n \times n$ board, with one in each row and column. Riordan [10] first derived a recurrence relation for the number a_n of such permutations in 1965: $a_0 = a_1 = 1$, $a_2 = a_3 = 0$, and for $n \geq 4$,

$$a_n = (n+1)a_{n-1} - (n-2)a_{n-2} - (n-5)a_{n-3} + (n-3)a_{n-4}.$$

The initial values for a_n are

$$1, 1, 0, 0, 2, 14, 90, 646, 5242, 47622, 479306, 5296790, 63779034, \ldots.$$

We refer to the sequence A002464 in the *On-Line Encyclopedia of Integer Sequences* (*OEIS*) for many references and for other interpretations/properties of this sequence of numbers. In particular, the generating function for these numbers was derived by Flajolet:

$$\sum_{n\geq 0} \frac{n!x^n(1-x)^n}{(1+x)^n}.$$

Riordan [10] obtained a more general result. That is, let $S_{n,m}$ be the number of permutations in S_n with exactly m bonds, and let $S[n] := S[n](t) = \sum_{m\geq 0} S_{n,m}t^m$. Then $S[0] = 1$, $S[1] = 1$, $S[2] = 2t$, $S[3] = 4t + 2t^2$, and for $n \geq 4$,

$$S[n] = (n+1-t)S[n-1]-(1-t)(n-2+3t)S[n-2]-(1-t)^2(n-5+t)S[n-3]+(1-t)^3(n-3)S[n-4].$$

In particular, the coefficient of t in $S[n](t)$ gives the number of permutations of length n with exactly one bond, which, in our terminology, is the number of permutations in S_n with exactly two occurrences of the 1-box pattern. This is the sequence A086852 in the OEIS. Clearly, there are no permutations with exactly one occurrence of the 1-box pattern.

It is straightforward to see that the number of permutations of length $n + 1$ with exactly three occurrences of the 1-box pattern is equal to the number of permutations of length n with exactly two occurrences of the 1-box pattern. Indeed, to have exactly three occurrences of the pattern in a permutation π means to have in π a factor either of the form $a(a+1)(a+2)$ or of the form $(a+2)(a+1)a$, and no other consecutive successive elements. Removing $(a+1)$ from π and decreasing by 1 all elements that are larger than $(a+1)$, we get a permutation containing exactly two occurrences of the 1-box pattern. This procedure is obviously reversible. Thus, the coefficient of t in $S[n](t)$ also gives the number of permutations of length $n+1$ with exactly three occurrences of the 1-box pattern.

Thus, our study of 1-box/k-box patterns cannot only be seen as an extension of the study of mesh patterns, but also as an extension of the study of consecutive successive elements (bonds) conducted in the literature. We do not define the notation of mesh patterns in this paper; however, the relevance of these patterns to our patterns is that in both cases we look for presence of points in specified regions in graphical representation of permutations.

In Theorem 3, we will enumerate permutations having the maximum number of occurrences of the 1-box pattern. To achieve this result, we obtain a result on pattern-avoiding signed permutations (see Theorem 1) thus contributing to the theory of permutation patterns (see [4]). Theorem 1 is strengthened by Theorem 2 to actually derive the generating function for the number of occurrences, rather than for avoidance, of what we call *bad pairs* in hyperoctahedral group B_n.

In Section 3 we not only provide a general solution (in matrix form) for finding the distribution of bonds and 1-box patterns over words (see Theorems 4 and 5) but also apply our studies to settle a conjecture of Mathar on the number of "stable LEGO walls" of width 7 (see Subsection 3.4), as well as to settle three conjectures of Hardin (see Subsection 3.3) and a conjecture of Barker (see Subsection 3.5). Also, in Subsection 3.5, we find generating functions for two sequences published by Hardin in the OEIS.

Given a word $w_1 \dots w_n \in [\ell]^n$, where $[\ell] = \{1, \dots, \ell\}$, we say that the pair $w_i w_{i+1}$ is a *k-bond* if $|w_i - w_{i+1}| \leq k$. In Subsection 3.5, we study the distribution of 2-bonds and (1,2)-rectangle patterns in words.

2 Permutations with the maximum number of occurrences of the 1-box pattern

Note that if $\sigma = 1 \dots n$, then $1\text{box}(\sigma) = n$ so that the maximum possible number of occurrences of the 1-box pattern in a permutation of length n is n.

In order to enumerate permutations with the maximum number of occurrences of the 1-box pattern, we need to prove a pattern avoidance result on the *hyperoctahedral group* B_n which is the group of signed permutations of length n. We will indicate the negative elements of an element of B_n by placing a bar over that element and we will define $|\bar{i}| = i$. For example, if $\alpha = 3\ \bar{4}\ 2\ \bar{1}$, then 2 and 3 are the positive elements of α and $\bar{1}$ and $\bar{4}$ are the negative elements of α. If $\alpha = \alpha_1 \dots \alpha_n \in B_n$, then we let $|\alpha| = |\alpha_1| \dots |\alpha_n|$ be the permutation of S_n that results by removing all the bars. Every element of B_n can be constructed by starting with an element $\sigma = \sigma_1 \dots \sigma_n$ in S_n and placing bars on top of some of the σ_is so that $|B_n| = 2^n n!$. Given $\alpha = \alpha_1 \dots \alpha_n \in B_n$, we shall say that $\alpha_j\ \alpha_{j+1}$ is a *bad pair* if either $\alpha_j\ \alpha_{j+1} = i\ (i+1)$ or $\alpha_j\ \alpha_{j+1} = \overline{(i+1)}\ \bar{i}$ for some i. We let $a(n,k)$ be the number of elements of B_n with exactly k bad pairs.

Theorem 1. *The exponential generating function for $a_n := a(n,0)$, the number of elements in B_n with no bad pairs, is given by*

$$A(t) = \sum_{n \geq 0} \frac{a_n t^n}{n!} = 1 + 2\int_0^t \frac{e^{-z}}{(1-2z)^2}\ dz. \tag{1}$$

The initial values for a_n are

$$1, 2, 6, 34, 262, 2562, 30278, 419234, 6651846, 118950658, 2366492038, \dots.$$

Proof. Clearly, $a(0,0) = 1$ and $a(1,0) = 2$ since the empty signed permutation, as well as 1 and $\bar{1}$, avoid the prohibited factors. Our goal is to show that for $n \geq 2$,

$$a(n,0) = (2n-1)a(n-1,0) + 2(n-2)a(n-2,0). \tag{2}$$

Given a permutation $\alpha = \alpha_1 \dots \alpha_n \in B_n$, we construct a permutation $D_i(\alpha)$ which we call the *doubling of σ at i* as follows. First, replace each j which appears in α by j if $j < i$ and by $j+1$ if $j > i$. Similarly, replace each $\bar{j}$ which appears in α by $\bar{j}$ if $j < i$ and by $\overline{j+1}$ if $j > i$. Then, if i appears in α, replace i by the pair $i\ (i+1)$, and if $\bar{i}$ appears in α, replace $\bar{i}$ by the pair $\overline{(i+1)}\ \bar{i}$. For example, if $\alpha = 3\ \bar{4}\ 2\ \bar{5}\ 6\ \bar{1}$, then $D_3(\alpha) = 3\ 4\ \bar{5}\ 2\ \bar{6}\ 7\ \bar{1}$ and $D_4(\alpha) = 3\ \bar{5}\ \bar{4}\ 2\ \bar{6}\ 7\ \bar{1}$. We claim that $D_i(\alpha)$ has exactly one more bad pair than α. That is, suppose that $\alpha = \alpha_1 \dots \alpha_n \in B_n$ and $|\alpha_j| = i$. Now, if $\alpha_j = i$, then

$$D_i(\alpha) = \alpha'_1 \dots \alpha'_{j-1}\ i\ (i+1)\ \alpha'_{j+2} \dots \alpha'_{n+1}.$$

It is then easy to check that if $s < j-1$, then our definitions ensure that $\alpha_s\ \alpha_{s+1}$ is a bad pair in α if and only if $\alpha'_s\ \alpha'_{s+1}$ is a bad pair in $D_i(\alpha)$. Similarly, it is easy to check that if $s \geq j+1$, then our definitions ensure that $\alpha_s\ \alpha_{s+1}$ is a bad pair in α if and only if $\alpha'_{s+1}\ \alpha'_{s+2}$ is a bad pair in $D_i(\alpha)$. Note that $\alpha_{j-1}\ \alpha_j$ is a bad pair in α if and only if $\alpha_{j-1} = i-1 = \alpha'_{j-1}$ if and only if $\alpha'_{j-1}\ \alpha'_j$ is a bad pair in $D_i(\alpha)$. Finally, $\alpha_j\ \alpha_{j+1}$ is a bad pair in α if and only if $\alpha_{j+1} = i+1$ if and only if $\alpha'_{j+2} = i+2$ if and only if $\alpha'_{j+1}\ \alpha'_{j+2}$ is a bad pair in $D_i(\alpha)$. This shows that there is a one-to-one correspondence between the bad pairs in α and the bad pairs in $D_i(\alpha)$ which are not equal to $i\ (i+1)$. Thus, $D_i(\alpha)$ has one more bad pair than α. A similar argument will show that if $\alpha_j = \bar{i}$, then $D_i(\alpha)$ has one more bad pair than α.

It follows that if $\alpha \in B_{n-1}$ has k bad pairs, then $D_1(\alpha), D_2(\alpha), \dots, D_{n-1}(\alpha)$ are $(n-1)$ distinct signed permutations in B_n which have $k+1$ bad pairs. Clearly, we can recover α from $D_i(\alpha)$ and i. Thus, it follows that if $\beta = \beta_1 \dots \beta_n \in B_n$ has $k+1$ bad pairs, then for each bad pair $\beta_s\ \beta_{s+1}$ where $\{|\beta_s|, |\beta_{s+1}|\} = \{i, i+1\}$, we can reconstruct a $\gamma \in B_{n-1}$ with k bad pairs such that $D_i(\gamma) = \beta$. Thus, it follows that for all $n \geq 0$,

$$a(n,k+1) = \frac{n-1}{k+1} a(n-1,k). \tag{3}$$

In particular, if $b_n = a(n,1)$ is the number of elements in B_n with exactly one bad pair, then

$$b_n = (n-1)a_{n-1}. \tag{4}$$

Next, we claim that

$$a_n = 2b_{n-1} + 2na_{n-1} - a_{n-1}. \tag{5}$$

That is, suppose that $\alpha = \alpha_1 \ldots \alpha_n$ is an element of B_n which has no bad pairs and $|\alpha_j| = n$. Then, it is easy to see that $\alpha' = \alpha_1 \ldots \alpha_{j-1}\alpha_{j+1} \ldots \alpha_n$ is an element of B_{n-1} which has at most one bad pair. Now, if $\beta = \beta_1 \ldots \beta_{n-1} \in B_{n-1}$ has exactly one bad pair, say $\beta_s\ \beta_{s+1}$, then we can insert either n or $\overline{n}$ between β_s and β_{s+1} to create a permutation in B_n with no bad pairs. That is, if $\beta_s\ \beta_{s+1} = \underline{i\ (i+1)}$, then $\underline{i\ n\ (i+1)}$ or $\underline{i\ \overline{n}\ i+1}$ will kill the bad pair and, similarly, if $\beta_s\ \beta_{s+1} = \overline{(i+1)}\ \overline{i}$, then $\overline{(i+1)}\ n\ \overline{i}$ or $\overline{(i+1)}\ \overline{n}\ \overline{i}$ will kill the bad pair. Thus, there are $2b_{n-1}$ αs in B_n such that α' has one bad pair. On the other hand, if $\gamma = \gamma_1 \ldots \gamma_{n-1}$ is an element of B_{n-1} with no bad pairs and $|\gamma_j| = n-1$, then if $\gamma_j = n-1$, we can insert n in any place in γ except immediately after γ_j to create an element of B_n with no bad pairs, and we can insert $\overline{n}$ in any place in γ to create an element of B_n with no bad pairs. Similarly, if $\gamma_j = \overline{n-1}$, then we can insert $\overline{n}$ in any place in γ except immediately before γ_j to create an element of B_n with no bad pairs, and we can insert n in any place in γ to create an element of B_n with no bad pairs. Thus, in each case, there are $2n-1$ elements $\alpha \in B_n$ with no bad pairs such that $\alpha' = \gamma$. Hence,

$$a_n = 2b_{n-1} + (2n-1)a_{n-1} = 2b_{n-1} + 2na_{n-1} - a_{n-1}.$$

Using (4) and (5), we obtain (2).

Note that the second derivative of $A(t)$ is given by

$$\begin{aligned}
A''(t) &= \sum_{n\geq 0} a_{n+2}\frac{t^n}{n!} = \sum_{n\geq 0}((2n+3)a_{n+1} + 2na_n)\frac{t^n}{n!} \\
&= 2t\sum_{n\geq 1} a_{n+1}\frac{t^{n-1}}{(n-1)!} + 3\sum_{n\geq 0} a_{n+1}\frac{t^n}{n!} + 2t\sum_{n\geq 1} a_n\frac{t^{n-1}}{(n-1)!} \\
&= 2tA''(t) + 3A'(t) + 2tA'(t).
\end{aligned}$$

Solving for $A''(t)$, we see that

$$A''(t) = \frac{2t+3}{1-2t}A'(t)$$

or, equivalently,

$$\frac{A''(t)}{A'(t)} = -1 + \frac{4}{1-2t}. \tag{6}$$

Integrating both sides of (6) and using the fact that $A'(0) = 2$, we see that

$$\ln(A'(t)) = -t - 2\ln(1-2t) + \ln(2).$$

Thus

$$A'(t) = 2\frac{e^{-t}}{(1-2t)^2}. \tag{7}$$

Integrating both sides of (7) and using the fact that $A(0) = 1$, we see that

$$A(t) = 1 + 2\int_0^t \frac{e^{-t}}{(1-2t)^2}\mathrm{d}t.$$

□

Remark 1. *Theorem 1 is a result on pattern avoidance in signed permutations (see [4, Chapter 9.6] for relevant results). In fact, avoidance of factors of the form $i(i+1)$ and $\overline{(i+1)}\,\overline{i}$ can be expressed in terms of avoidance of* bivincular patterns *(see [4, Chapter 1.4] for definition; bars can be incorporated in the definition in an obvious way extending it from S_n to B_n), and thus, Theorem 1 seems to be the first instance of enumerative results on signed permutations avoiding bivincular patterns.*

In fact, we can use Theorem 1 to find the distribution of bad pairs in elements of B_n. That is, given $\alpha \in B_n$, let $\mathrm{bp}(\alpha)$ denote the number of bad pairs in α and let

$$BP_n(x) = \sum_{\alpha \in B_n} x^{\mathrm{bp}(\alpha)} = \sum_{k=0}^{n} a(n,k)x^k.$$

Theorem 2.

$$BP(x,t) := 1 + \sum_{n\geq 1} BP_n(x)\frac{t^n}{n!} = 1 + 2\int_0^t \frac{e^{z(x-1)}}{(1-2z)^2}dz. \tag{8}$$

Proof. Note that if $\alpha \in B_n$ has k bad pairs where $k \geq 1$, then $n \geq k+1$. Thus

$$BP(x,t) = \sum_{k\geq 0} x^k \sum_{n\geq k+1} a(n,k)\frac{t^n}{n!}.$$

We claim that for $k \geq 1$,

$$\sum_{n\geq k+1} a(n,k)\frac{t^n}{n!} = 2\int_0^t \frac{z^k}{k!}\frac{e^{-z}}{(1-2z)^2}dz. \tag{9}$$

By Theorem 1,

$$\overline{A}(t) := \sum_{n\geq 1} a(n,0)\frac{t^n}{n!} = 2\int_0^t \frac{e^{-z}}{(1-2z)^2}dz. \tag{10}$$

Iterating (3), we have that for $n \geq k+1$,

$$a(n,k) = \frac{(n-1)\downarrow_k}{k!}a(n-k,0), \tag{11}$$

where for any m, $m\downarrow_k = m(m-1)\cdots(m-k+1)$. Thus

$$\begin{aligned}
\sum_{n\geq k+1} a(n,k)\frac{t^n}{n!} &= \sum_{n\geq k+1} \frac{(n-1)\downarrow_k}{k!} a(n-k,0)\frac{t^n}{n!} \\
&= \int_0^t \sum_{n\geq k+1} \frac{(n-1)\downarrow_k}{k!} a(n-k,0)\frac{z^{n-1}}{(n-1)!}dz \\
&= \int_0^t \frac{z^k}{k!} \sum_{n\geq k+1} a(n-k,0)\frac{z^{n-k-1}}{(n-k-1)!}dz \\
&= \int_0^t \frac{z^k}{k!} \left(\frac{d}{dz}\overline{A}(z)\right) dz \\
&= \int_0^t \frac{z^k}{k!} 2\frac{e^{-z}}{(1-2z)^2}dz \\
&= 2\int_0^t \frac{z^k}{k!} \frac{e^{-z}}{(1-2z)^2}dz.
\end{aligned}$$

Hence

$$\begin{aligned}
BP(x,t) &= A(t) + \sum_{k\geq 1} x^k \sum_{n\geq k+1} a(n,k)\frac{t^n}{n!} \\
&= 1 + 2\int_0^t \frac{e^{-z}}{(1-2z)^2}dz + \sum_{k\geq 1} x^k 2\int_0^t \frac{z^k}{k!}\frac{e^{-z}}{(1-2z)^2}dz \\
&= 1 + 2\int_0^t \left(\sum_{k\geq 0} \frac{x^k z^k}{k!}\right) \frac{e^{-z}}{(1-2z)^2}dz \\
&= 1 + 2\int_0^t e^{xz}\frac{e^{-z}}{(1-2z)^2}dz \\
&= 1 + 2\int_0^t \frac{e^{(x-1)z}}{(1-2z)^2}dz.
\end{aligned}$$

□

Using Mathematica, we have computed the following initial values of the polynomials $BP_n(x)$:

$$\begin{aligned}
BP_0(x) &= 1, \\
BP_1(x) &= 2, \\
BP_2(x) &= 2(3+x), \\
BP_3(x) &= 2(17+6x+x^2), \\
BP_4(x) &= 2(131+51x+9x^2+x^3), \\
BP_5(x) &= 2(1281+524x+102x^2+12x^3+x^4), \\
BP_6(x) &= 2(15139+6405x+1310x^2+170x^3+15x^4+x^3), \text{ and} \\
BP_7(x) &= 2(209617+90834x+19215x^2+2620x^3+255x^4+18x^5+x^6).
\end{aligned}$$

Theorem 3. *The number of permutations in S_n with the maximum number of occurrences of the* 1*-box pattern (which is n) is given by*

$$\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j-1}{j-1} a_j, \tag{12}$$

where a_j's are given by the recurrence (2) or by the exponential generating function (1). The initial values for the number of such permutations starting with the case $n = 0$ are

$$1, 1, 2, 2, 8, 14, 54, 128, 498, 1426, 5736, 18814, 78886, 287296, 1258018, \ldots.$$

Proof. Each permutation $\pi \in S_n$ having the maximum number of occurrences of the 1-box pattern can be uniquely decomposed into maximal factors of consecutive elements of size at least 2, since each element of π must be staying next to a consecutive element. For example, the permutation $\pi = 543126798$ is decomposed into maximal factors 543, 12, 67 and 98. Let a permutation π' be obtained from π by substituting the ith largest factor with i if it is increasing, and with $\bar{i}$ if it is decreasing. We refer to π' as the *basis permutation* for π and, clearly, $\pi' \in B_m$ for some m. For π as above, $\pi' = \bar{2}13\bar{4}$. Since the decomposition factors are of maximal possible length, basis permutations must avoid factors of the form $i(i+1)$ and $(\overline{i+1})\bar{i}$, and these permutations were counted by us in Theorem 1.

Finally, to create permutations of length n with the maximum number of occurrences of the 1-box pattern, we choose basis permutations of length j, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, and decide on the lengths of the j decomposition factors to be made decreasing or increasing depending on the respective elements to have or not to have bars, respectively. These lengths must be of size at least 2, and it is a standard combinatorial problem to see that the number of ways to make such a decision is $\binom{n-j-1}{j-1}$ (indeed, we reserve $2j$ elements to make sure each decomposition factor will contain at least two elements; the remaining $n - 2j$ elements can be distributed among j factors in the desired number of ways). Note that all permutations of interest will be generated in a bijective manner, which completes our proof of (12). □

3 Distribution of bonds and 1-box patterns over words

Given a word $w = w_1 \ldots w_n$, let $|w| = n$ be the length of the w and $\text{1box}(w)$ denote the number of occurences of the 1-box pattern in w. A *bond* in w is a pair $w_i w_{i+1}$ of the form $s(s+1)$, $(s+1)s$, or ss for some s. We let $\text{bond}(w)$ denote the number of bonds in w.

In Subsection 3.1, we study distribution of bonds over words, while in Subsection 3.2, we study distribution of 1-box patterns over words. Three relevant conjectures of Hardin are settled in Subsection 3.3, and a conjecture of Mathar on stable LEGO walls is settled in Subsection 3.4. In Subsection 3.5, we consider $(1, k)$-rectangle patterns for $k \geq 2$, which led us to solving a conjecture of Barker and enumerating two sequences of Hardin published in the OEIS.

3.1 Distribution of bonds over words.

As in the case of permutations, it is relatively straightforward to find the generating functions for the number of bonds in words over $[\ell]$ for any $\ell \geq 1$. That is, let

$$A_{\ell,1}(x,t) = \sum_{w \in [\ell]^*} x^{\mathrm{bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_{\ell,1}(m,n) x^m t^n,$$

where $[\ell]^*$ is the set of all words over the alphabet $[\ell]$. Thus $a_{\ell,1}(m,n)$ is the number of words w of length n over the alphabet $[\ell]$ such that $\mathrm{bond}(w) = m$. Note that $A_{\ell,1}$ is a particular case of the function $A_{\ell,k}$ defined in Subsection 3.5.

The following theorem gives the distribution of bonds over words in matrix form.

Theorem 4. *The generating function $A_{\ell,1}(x,t)$ is equal to*

$$1 + (\underbrace{1,\ldots,1}_{\ell})\mathbb{A}_{\ell,1}^{-1}(\underbrace{-t,\ldots,-t}_{\ell})^T,$$

where $\mathbb{A}_{\ell,1}$ is the following $\ell \times \ell$ matrix:

$$\mathbb{A}_{\ell,1} = \begin{pmatrix} xt-1 & xt & t & t & t & \cdots & t & t \\ xt & xt-1 & xt & t & t & \cdots & t & t \\ t & xt & xt-1 & xt & t & \cdots & t & t \\ t & t & xt & xt-1 & xt & \cdots & t & t \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t & t & t & t & t & \cdots & xt & xt-1 \end{pmatrix}.$$

Proof. Let $i[\ell]^*$ denote the set of words over $[\ell]$ that begin with a letter i. For $1 \leq i \leq \ell$, let

$$A_{\ell,1}^{(i)}(x,t) = \sum_{w \in i[\ell]^*} x^{\mathrm{bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_{\ell,1}^{(i)}(m,n) x^m t^n.$$

Thus $a_{\ell,1}^{(i)}(m,n)$ is the number of words of length n over $[\ell]$ such that w begins with the letter i and $\mathrm{bond}(w) = m$. Clearly,

$$A_{\ell,1}(x,t) = 1 + \sum_{1 \leq i \leq \ell} A_{\ell,1}^{(i)}(x,t). \tag{13}$$

(The term 1 in (13) comes from the empty word.) Also, we have the following system of equations, where to obtain $A_{\ell,1}^{(i)}(x,t)$, we can think of taking words counted by $A_{\ell,1}^{(j)}(x,t)$, $1 \leq j \leq \ell$, and adjoining the letter i to the left of them; these functions are then to be multiplied by xt if $|i-j| \leq 1$ (indicating that the length of such words is increased by 1 and one more bond is created), and by t otherwise (to indicate change of the length keeping the number of occurrences of bonds the same); we also need to add t corresponding to the one-letter word i:

ℓ	generating function for distribution of the number of bonds
$A_{3,1}(x,t)$	$\frac{1-2(x-1)t-(x-1)^2t^2}{1-t-2xt-x(x-1)t^2}$
$A_{4,1}(x.t)$	$\frac{1-3(x-1)t+(x-1)^2t^2}{1-(3x+1)t+(x^2-1)t^2}$
$A_{5,1}(x,t)$	$\frac{1-3(x-1)t+2(x-1)^3t^3}{1-(3x+2)t+2(x-1)t^2+2(x+1)(x-1)^2t^3}$
$A_{6,1}(x,t)$	$\frac{1-4(x-1)t+3(x-1)^2t^2+(x-1)^3t^3}{1-2(2x+1)t+(3x^2+2x-5)t^2+(x+1)(x-1)^2t^3}$
$A_{7,1}(x,t)$	$\frac{1-4(x-1)t+2(x-1)^2t^2+4(x-1)^3t^3-(x-1)^4t^4}{1-(4x+3)t-(7-5x-2x^2)t^2+(4x+5)(x-1)^2t^3-(x+2)(x-1)^3t^4}$

Table 1: Distribution of the number of bonds on ℓ-ary words, $\ell = 3, ..., 7$.

$$\begin{aligned}
A^{(1)}_{\ell,1}(x,t) &= t + xtA^{(1)}_{\ell,1}(x,t) + xtA^{(2)}_{\ell,1}(x,t) + tA^{(3)}_{\ell,1}(x,t) + tA^{(4)}_{\ell,1}(x,t) + \cdots + tA^{(\ell)}_{\ell,1}(x,t);\\
A^{(2)}_{\ell,1}(x,t) &= t + xtA^{(1)}_{\ell,1}(x,t) + xtA^{(2)}_{\ell,1}(x,t) + xtA^{(3)}_{\ell,1}(x,t) + tA^{(4)}_{\ell,1}(x,t) + \cdots + tA^{(\ell)}_{\ell,1}(x,t);\\
A^{(3)}_{\ell,1}(x,t) &= t + tA^{(1)}_{\ell,1}(x,t) + xtA^{(2)}_{\ell,1}(x,t) + xtA^{(3)}_{\ell,1}(x,t) + xtA^{(4)}_{\ell,1}(x,t) + \cdots + tA^{(\ell)}_{\ell,1}(x,t);\\
&\vdots\\
A^{(\ell)}_{\ell,1}(x,t) &= t + tA^{(1)}_{\ell,1}(x,t) + tA^{(2)}_{\ell,1}(x,t) + \cdots + tA^{(\ell-2)}_{\ell,1}(x,t) + xtA^{(\ell-1)}_{\ell,1}(x,t) + xtA^{(\ell)}_{\ell,1}(x,t).
\end{aligned}$$

Solving the system for the functions $A^{(i)}_{\ell,1}(x,t)$ and applying (13) we get the desired result. □

As corollaries to Theorem 4, we can obtain, e.g. using Mathematica, explicit generating functions for ℓ letter alphabets, where $3 \le \ell \le 7$. These are presented in Table 1. Note that $A_{1,1}(x,t)$ and $A_{2,1}(x,t)$ are trivial since any word w of length n over the alphabet $\{1\}$ or the alphabet $\{1,2\}$ has $n-1$ bonds. We also give expansions of the functions $A_{\ell,1}(x,t)$ for $\ell = 3, ..., 7$:

$$\begin{aligned}
A_{3,1}(x,t) &= 1 + 3t + (2+7x)t^2 + \left(2 + 8x + 17x^2\right)t^3 + \left(2 + 10x + 28x^2 + 41x^3\right)t^4\\
&+ \left(2 + 12x + 42x^2 + 88x^3 + 99x^4\right)t^5 + \left(2 + 14x + 58x^2 + 154x^3 + 262x^4 + 239x^5\right)t^6\\
&+ \left(2 + 16x + 76x^2 + 240x^3 + 524x^4 + 752x^5 + 577x^6\right)t^7\\
&+ \left(2 + 18x + 96x^2 + 348x^3 + 908x^4 + 1692x^5 + 2104x^6 + 1393x^7\right)t^8 + \cdots;
\end{aligned}$$

$$\begin{aligned}
A_{4,1}(x,t) &= 1 + 4t + 2(3+5x)t^2 + 2\left(5 + 14x + 13x^2\right)t^3 + 4\left(4 + 17x + 26x^2 + 17x^3\right)t^4\\
&+ 2\left(13 + 72x + 162x^2 + 176x^3 + 89x^4\right)t^5\\
&+ 2\left(21 + 145x + 422x^2 + 662x^3 + 565x^4 + 233x^5\right)t^6\\
&+ 4\left(17 + 140x + 503x^2 + 1016x^3 + 1239x^4 + 876x^5 + 305x^6\right)t^7\\
&+ 2\left(55 + 527x + 2247x^2 + 5567x^3 + 8717x^4 + 8757x^5 + 5301x^6 + 1597x^7\right)t^8 + \cdots;
\end{aligned}$$

ℓ	generating function for ℓ-ary words avoiding the 1-box pattern
$A_{3,1}(0,t)$	$\frac{1+2t-t^2}{1-t}$
$A_{4,1}(0.t)$	$\frac{1+3t+t^2}{1-t-t^2}$
$A_{5,1}(0,t)$	$\frac{1+3t-2t^3}{1-2t-2t^2+2t^3}$
$A_{6,1}(0,t)$	$\frac{1+4t+3t^2-t^3}{1-2t-5t^2+t^3}$
$A_{7,1}(0,t)$	$\frac{1+4t+2t^2-4t^3-t^4}{1-3t-7t^2+5t^3+2t^4}$

Table 2: Distribution of ℓ-ary words which avoid the 1-box pattern for $\ell = 3, ..., 7$.

$$\begin{aligned}
A_{5,1}(x,t) &= 1 + 5t + (12+13x)t^2 + 5\left(6+12x+7x^2\right)t^3 + \left(74+222x+234x^2+95x^3\right)t^4 \\
&+ \left(184+724x+1134x^2+824x^3+259x^4\right)t^5 \\
&+ \left(456+2236x+4574x^2+4902x^3+2750x^4+707x^5\right)t^6 \\
&+ \left(1132+6624x+16800x^2+23480x^3+19290x^4+8868x^5+1931x^6\right)t^7 \\
&+ \left(2808+19124x+57696x^2+99716x^3+106666x^4+71418x^5+27922x^6+5275x^7\right)t^8 + \cdots;
\end{aligned}$$

$$\begin{aligned}
A_{6,1}(x,t) &= 1 + 6t + 4(5+4x)t^2 + 4\left(17+26x+11x^2\right)t^3 + 2\left(115+263x+209x^2+61x^3\right)t^4 \\
&+ 4\left(195+590x+696x^2+378x^3+85x^4\right)t^5 \\
&+ 2\left(1321+4987x+7742x^2+6218x^3+2585x^4+475x^5\right)t^6 \\
&+ 2\left(4477+20230x+39031x^2+41156x^3+25211x^4+8534x^5+1329x^6\right)t^7 \\
&+ 2\left(15169+79871x+183933x^2+240507x^3+193107x^4+95997x^5+27503x^6+3721x^7\right)t^8 + \cdots;
\end{aligned}$$

$$\begin{aligned}
A_{7,1}(x,t) &= 1 + 7t + (30+19x)t^2 + \left(130+160x+53x^2\right)t^3 + \left(562+1034x+656x^2+149x^3\right)t^4 \\
&+ \left(2432+5940x+5598x^2+2416x^3+421x^4\right)t^5 \\
&+ \left(10520+32068x+39942x^2+25526x^3+8400x^4+1193x^5\right)t^6 \\
&+ \left(45514+166236x+257634x^2+217088x^3+105512x^4+28172x^5+3387x^6\right)t^7 \\
&+ \left(196898+838274x+1553178x^2+1625554x^3+1039904x^4+409176x^5+92190x^6+9627x^7\right)t^8 + \cdots.
\end{aligned}$$

As noted in the introduction, the number of permutations $\sigma \in S_n$ such that $1\text{box}(\sigma) = 0$ equals the number of permutations $\sigma \in S_n$ such that $\text{bond}(\sigma) = 0$. The same applies to words. Thus, plugging in $x = 0$ in the functions in Table 1, one gets generating functions for avoidance of the 1-box pattern (alternatively, we can plug in $x = 0$ in the matrix $\mathbb{A}_{\ell,1}$ in Theorem 4 to get the most general case and to work out particular small values of ℓ); in Table 3, we list initial values of the respective sequences indicating connections to the OEIS [11]. In particular, the connection to the sequence A118649 led us to solving a conjecture of Mathar (published in [11, A118649]) to be discussed in Subsection 3.4.

In [9], Knopfmacher et al. studied generating functions for smooth ℓ words where a word $w = w_1 \dots w_n \in [\ell]^n$ is smooth if $|w_i - w_{i+1}| \leq 1$ for $1 \leq i < n$. Thus in our notation,

ℓ	number of ℓ-ary words avoiding the 1-box pattern	sequence in [11]
3	1, 3, 2, 2, 2, 2, 2, 2, 2, 2, ...	
4	1, 4, 6, 10, 16, 26, 42, 68, 110, 178, ...	A006355, $n \geq 1$
5	1, 5, 12, 30, 74, 184, 456, 1132, 2808, 6968, ...	A118649, $n \geq 1$
6	1, 6, 20, 68, 230, 780, 2642, 8954, 30338, 102804, ...	
7	1, 7, 30, 130, 562, 2432, 10520, 45514, 196898, 851828, ...	

Table 3: Avoidance of the 1-box patterns in ℓ-ary words for lengths n up to 9.

$w \in [\ell]^n$ is smooth if $\text{bond}(w) = n-1$. Let $M_{n,1,\ell}$ denote the number of $w \in [\ell]^n$ such that $\text{bond}(w) = n-1$ and $sm_\ell(t) = 1 + \sum_{n\geq 1} M_{n,1,\ell}t^n$ (by definition, a bond is the k-bond for $k = 1$, and the subindex 1 in $M_{n,1,\ell}$ indicates this). Then Knopfmacher et al. [9, Theorem 2.2] proved that

$$sm_\ell(t) = 1 + \frac{t(\ell-(3\ell+2)t)}{(1-3t)^2} + \frac{2t^2}{(1-3t)^2}\frac{1+U_{\ell-1}\left(\frac{1-t}{2t}\right)}{U_\ell\left(\frac{1-t}{2t}\right)}, \tag{14}$$

where $U_r(t)$ is the Chebyshev polynomial of the second kind defined by

$$U_r(\cos(\theta)) = \frac{\sin((r+1)\theta)}{\sin(\theta)}.$$

Alternatively, one can define the polynomials by recursion by setting $U_0(t) = 1$, $U_1(t) = 2t$, and

$$U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for } r \geq 2.$$

We can also obtain a formula for $sm_\ell(t)$ from our generating function $A_{\ell,1}(x,t)$. That is, clearly

$$A_{\ell,1}(1/x, xt) = 1 + \sum_{n\geq 1}\sum_{w\in[\ell]^n} x^{n-\text{bond}(w)}t^n$$

so that

$$C_{\ell,1}(x,t) := \frac{1}{x}(A_{\ell,1}(1/x,xt)-1) = \sum_{n\geq 1}\sum_{w\in[\ell]^n} x^{n-1-\text{bond}(w)}t^n.$$

Hence

$$sm_\ell(t) = 1 + C_{\ell,1}(0,t).$$

3.2 Distribution of 1-box patterns over words.

One can use similar methods to find the distribution of $1\text{box}(w)$ for $w \in [\ell]^*$. In this case we have to keep track of more information. This is due to the fact that the extra contribution to $1\text{box}(w)$ caused by adding an extra letter at the front of a word w depends on the first two letters of w. For example, $1\text{box}(12) = x^2t^2$ and $1\text{box}(112) = x^3t^3$ so that adding 1 to the front of $w = 12$ increased $x^{1\text{box}(w)}t^{|w|}$ by a factor of xt. However, $1\text{box}(13) = t^2$ and

$1\mathrm{box}(113) = x^2t^3$ so that adding 1 to the front of $w = 13$ increased $x^{1\mathrm{box}(w)}t^{|w|}$ by a factor of x^2t.

For $1 \leq i,j \leq \ell$, let

$$B_{\ell,1}^{(ij)} = \sum_{w \in ij[\ell]^*} WT(w),$$

where $WT(w) = x^{1\mathrm{box}(w)}t^{|w|}$ and $ij[\ell]^*$ denotes the set of words over $[\ell]$ that begin with letters ij. For any statement S, let $\chi(S) = 1$ if S is true and $\chi(S) = 0$ if S is false. Then we claim that for all $1 \leq i,j \leq \ell$,

$$\begin{aligned} B_{\ell,1}^{(ij)}(x,t) &= x^{2\chi(|i-j|\leq 1)}t^2 + \\ \sum_{k=1}^{\ell}&(t\chi(|i-j|>1) + xt\chi(|i-j|\leq 1)\chi(|j-k|\leq 1) + \\ &\qquad x^2t\chi(|i-j|\leq 1)\chi(|j-k|>1))B_{\ell,1}^{(jk)}(x,t). \end{aligned} \tag{15}$$

That is, the words in $ij[\ell]^*$ are of the form ij plus words $ijkv$ where $k \in [\ell]$ and $v \in [\ell]^*$. Now

$$WT[ij] = \begin{cases} t^2 & \text{if } |i-j| > 1 \text{ and} \\ x^2t^2 & \text{if } |i-j| \leq 1. \end{cases}$$

Similarly,

$$WT[ijkv] = \begin{cases} tWT[jkv] & \text{if } |i-j| > 1, \\ xtWT[jkv] & \text{if } |i-j| \leq 1 \text{ and } |j-k| \leq 1, \text{ and} \\ x^2tWT[jkv] & \text{if } |i-j| \leq 1 \text{ and } |j-k| > 1. \end{cases}$$

The set of equations of the form (15) can be written out in matrix form. That is, let $\vec{B}_{\ell,1}$ be the row vector of length ℓ^2 of the $B_{\ell,1}^{(ij)}(t,x)$ where the elements are listed in the lexicographic order of the pairs (ij). For example, $\vec{B}_{3,1}$ equals

$$(B_{3,1}^{(11)}(x,t), B_{3,1}^{(12)}(x,t), B_{3,1}^{(13)}(x,t), B_{3,1}^{(21)}(x,t), B_{3,1}^{(22)}(x,t), B_{3,1}^{(23)}(x,t), B_{3,1}^{(31)}(x,t), B_{3,1}^{(32)}(x,t), B_{3,1}^{(33)}(x,t)).$$

Similarly, let $\vec{I}_{\ell,1}$ be the row vector of length ℓ^2 of the terms $t^2x^{2\chi(|i-j|\leq 1)}$ again listed in the lexicographic order on the pairs ij. For example,

$$\vec{I}_{3,1} = (x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2).$$

Then one can write a set of equations of the form (15) in the form

$$(\vec{I}_{\ell,1})^T = \mathbb{B}_{\ell,1}(\vec{B}_{\ell,1})^T,$$

where $\mathbb{B}_{\ell,1}$ is an $\ell^2 \times \ell^2$ matrix. For example, $\mathbb{B}_{3,1}$ is the matrix

$$\begin{pmatrix} xt-1 & xt & x^2t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & xt & xt & xt & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & t & t & t \\ xt & xt & x^2t & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & xt & xt-1 & xt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & x^2t & xt & xt \\ t & t & t & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & xt & xt & xt & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2t & xt & xt-1 \end{pmatrix}.$$

Note that since setting $x = t = 0$ in $\mathbb{B}_{\ell,1}$ will gives an $\ell \times \ell$ diagonal matrix with -1s on the diagonal, $\mathbb{B}_{\ell,1}$ is invertible. Thus

$$(\vec{B}_{\ell,1})^T = \mathbb{B}_{\ell,1}^{-1}(\vec{I}_{\ell,1})^T.$$

Let $\vec{1}_{\ell,1}$ denote the vector of length ℓ^2 consisting of all 1s. Then

$$\sum_{1 \le i,j \le \ell} B_{\ell,1}^{(ij)}(x,t) = \vec{1}_{\ell,1}\mathbb{B}_{\ell,1}^{-1}(\vec{I}_{\ell,1})^T.$$

Taking into account the empty word and all the words of length 1 will yeild the following theorem.

Theorem 5. *For all $\ell \ge 2$,*

$$B_{\ell,1}(x,t) := \sum_{w \in [\ell]^*} x^{\text{1box}(w)} t^{|w|} = 1 + \ell t + \vec{1}_{\ell,1}\mathbb{B}_{\ell,1}^{-1}(\vec{I}_{\ell,1})^T.$$

We have used Theorem 5 to compute $B_{\ell,1}(x,t)$ for $\ell = 3, 4,$ and 5.

$$B_{3,1}(x,t) = \frac{1 + 2(1-x)t - (1+4x-5x^2)t^2 + 2x(1-x)^2t^3 + x^2(1-x)^2t^4}{1 - (1+2x)t + 2x(1-x)t^2 + x^2(1-x)t^3};$$

$$B_{4,1}(x,t) = \frac{1 + 3(1-x)t + (1-9x+8x^2)\,t^2 - 3x(1-x)^2t^3 + x^2(1-x)^2t^4}{1 - (1+3x)t - (1-3x+2x^2)\,t^2 - x\,(3-4x+x^2)\,t^3 - x^2(1-x)^2t^4};$$

$$B_{5,1}(x,t) = \frac{f_{5,1}(x,t)}{g_{5,1}(x,t)},$$

where

$$\begin{aligned} f_{5,1}(x,t) &= 1 + 3(1-x)t + 9x(1-x)t^2 - 2(1-x)^2(1+2x)t^3 + \\ &\quad 6x(1-x)^2(1+x)t^4 - 4(1-x)^3x^3t^6 \end{aligned}$$

and

$$
\begin{aligned}
g_{5,1}(x,t) \;=\;& 1-(2+3x)t-\left(2-6x+4x^2\right)t^2-\left(-2-6x+8x^2\right)t^3- \\
& 6(1-x)^2x(1+x)t^4-4(1-x)^2x^3t^5+4(1-x)^3x^3t^6.
\end{aligned}
$$

Using the generating functions above, we have computed some of the initial terms in their Taylor series expansions:

$$
\begin{aligned}
& B_{3,1}(x,t) = 1+3t+(2+7x^2)t^2+(2+8x^2+17x^3)t^3+ \\
& (2+10x^2+20x^3+49x^4)t^4+(2+12x^2+26x^3+64x^4+139x^5)t^5+ \\
& (2+14x^2+32x^3+88x^4+200x^5+393x^6)t^6+ \\
& (2+16x^2+38x^3+114x^4+290x^5+614x^6+1113x^7)t^7+ \\
& (2+18x^2+44x^3+142x^4+392x^5+932x^6+1880x^7+3151x^8)t^8+\cdots;
\end{aligned}
$$

$$
\begin{aligned}
& B_{4,1}(x,t) = 1+4t+(6+10x^2)t^2+(10+28x^2+26x^3)t^3+ \\
& (16+68x^2+72x^3+100x^4)t^4+(26+144x^2+174x^3+338x^4+342x^5)t^5+ \\
& (42+290x^2+368x^3+930x^4+1256x^5+1210x^6)t^6+ \\
& (68+560x^2+740x^3+2232x^4+3612x^5+4932x^6+4240x^7)t^7+ \\
& (110+1054x^2+1428x^3+4996x^4+8984x^5+15246x^6+18820x^7+14898x^8)t^8+\cdots;
\end{aligned}
$$

$$
\begin{aligned}
& B_{5,1}(x,t) = 1+5t+\left(12+13x^2\right)t^2+5\left(6+12x^2+7x^3\right)t^3+ \\
& \left(74+222x^2+160x^3+169x^4\right)t^4+\left(184+724x^2+592x^3+974x^4+651x^5\right)t^5+ \\
& \left(456+2236x^2+1932x^3+4238x^4+4048x^5+2715x^6\right)t^6+ \\
& \left(1132+6624x^2+5968x^3+16036x^4+18372x^5+18982x^6+11011x^7\right)t^7+ \\
& \left(2808+19124x^2+17688x^3+56072x^4+71724x^5+94282x^6+83828x^7+45099x^8\right)t^8+\cdots.
\end{aligned}
$$

3.3 Solving three conjectures of Hardin.

Define

$$\overline{B}_{\ell,1}(x,t) := B_{\ell,1}(1/x, xt) = \sum_{w\in[\ell]^*} x^{|w|-(1\mathrm{box}(w))}t^{|w|},$$

so that $\overline{B}_{\ell,1}(0,t)$ is the generating function of all words $w = w_1\dots w_n \in [\ell]^*$ such that $1\mathrm{box}(w) = n$, i.e. each letter of w differs from at least one neighbor by 1 or less. We have computed $\overline{B}_{\ell,1}(0,t)$ for $\ell = 3,4,5$.

$$\overline{B}_{3,1}(0,t) = \frac{1-2t+5t^2+2t^3+t^4}{1-2t-2t^2-t^3}.$$

The initial terms of this series are $1, 0, 7, 17, 49, 139, 393, 1113, 3151, 8921, \dots$. This is the sequence A221591 which was apparently computed directly from its combinatorial definition by R. H. Hardin. If $\overline{B}_{3,1}(0,t) = \sum_{n\geq 0} b_{3,1,n}t^n$, then Hardin observed empirically that $b_{3,1,n} = 2b_{3,1,n-1} + 2b_{3,1,n-2} + b_{3,1,n-3}$ for $n > 4$. This recursion follows immediately from the generating function for $\overline{B}_{3,1}(0,t)$ so that we have proved Hardin's conjecture:

$$\overline{B}_{4,1}(0,t) = \frac{1-3t+8t^2-3t^3+t^4}{1-3t-2t^2+t^3-t^4}.$$

The initial terms of this series are $1, 0, 10, 26, 100, 342, 1210, 4240, 14898, 52306, \ldots$. This is the sequence A221569 which was also computed directly from its combinatorial definition by R. H. Hardin. If $\overline{B}_{4,1}(0,t) = \sum_{n\geq 0} b_{4,1,n}t^n$, then Hardin observed empirically that $b_{4,1,n} = 3b_{4,1,n-1} + 2b_{4,1,n-2} - b_{4,1,n-3} + b_{4,1,n-4}$ for $n > 5$. Again, this recursion follows immediately from the generating function for $\overline{B}_{4,1}(0,t)$ so that we have also proved this conjecture of Hardin:

$$\overline{B}_{5,1}(0,t) = \frac{1 - 3t + 9t^2 - 4t^3 + 6t^4 + 4t^6}{1 - 3t - 4t^2 - 6t^4 - 4t^5 - 4t^6}.$$

The initial terms of this series are $1, 0, 13, 35, 169, 651, 2715, 11011, 45099, 184063, \ldots$. This is the sequence A221592 which was also computed directly form its combinatorial definition by R. H. Hardin. If $\overline{B}_{5,1}(0,t) = \sum_{n\geq 0} b_{5,1,n}t^n$, then Hardin observed empirically that $b_{5,1,n} = 3b_{5,1,n-1} + 4b_{5,1,n-2} + 6b_{5,1,n-4} + 4b_{5,1,n-5} + 4b_{5,1,n-6}$ for $n > 6$. As was the case for $\overline{B}_{3,1}(0,t)$ and $\overline{B}_{4,1}(0,t)$, this recursion follows immediately from the the generating function for $\overline{B}_{5,1}(0,t)$ so that we have also proved this conjecture of Hardin.

3.4 Solving an enumerative conjecture on LEGO.

A "stable LEGO wall" is a wall in which seams do not match up from one level to the next. Stable LEGO walls of width 7 and heights 1 and 2 when using bricks of length 2, 3, and 4 can be found in Figure 4 (the numbers should be ignored there for the moment).

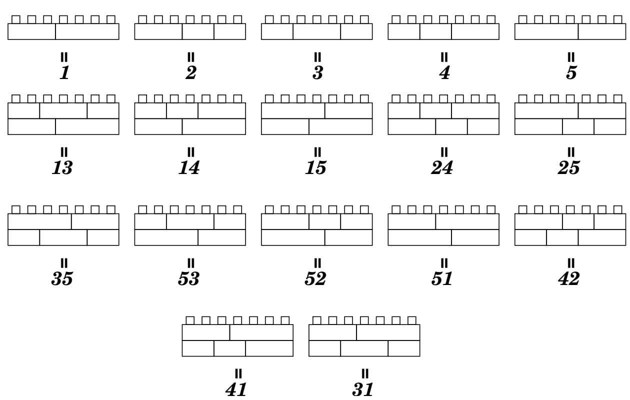

Figure 4: Stable LEGO walls of width 7 and heights 1 and 2.

Lemma 6. *There is a bijection between words over the alphabet $A = \{1, 2, 3, 4, 5\}$ of length n that avoid the 1-box pattern and stable LEGO walls of width 7 and height n when using bricks of length 2, 3, and 4.*

Proof. Encode the eligible LEGO configurations of height 1 by the elements of A as shown in Figure 4, which gives a bijection between the objects in the case of $n = 1$.

More generally, given a word $w = w_1w_2 \ldots w_n$ avoiding the 1-box pattern, we let the i-th level from the bottom of the wall corresponding to w be given by the configuration corresponding to the letter w_i defined in Figure 4. For example, the correspondence for the case $n = 2$ is shown in Figure 4.

It is straightforward to check that the prohibited factors of words, namely 12, 23, 34, 45, 54, 43, 32, and 21, correspond to the prohibited configurations in LEGO, and vice versa. □

Using Lemma 6, the function corresponding to $\ell = 5$ and $x = 0$ in Table 2, and taking care of the offset (removing the number 2 in the sequence [11, A118649] and shifting down the indices of the larger numbers), we can confirm a conjecture of R. J. Mathar that stable LEGO walls satisfying the assumptions of Lemma 6 are counted by the following generating function:

$$\frac{1+3t-2t^3}{1-2t-2t^2+2t^3}.$$

3.5 $(1,k)$-rectangle patterns for $k \geq 2$; solving a conjecture of Barker and enumerating two sequences of Hardin.

Given a word $w = w_1 \dots w_n \in [\ell]^n$ and an integer $k \geq 2$, we let $k\text{bond}(w) = |\{i : |w_i - w_{i+1}| \leq k\}|$. It is straightforward to generalize Theorems 4 and 5 to find the distribution of $k\text{bond}(w)$ and $(1,k)\text{rec}(w)$, the number of $(1,k)$-rectangle patterns in w, over words w in $[\ell]^*$. That is, we claim that the same method of proof can also be used to find the generating function

$$A_{\ell,k}(x,t) = \sum_{w \in [\ell]^*} x^{k\text{bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_{\ell,k}(m,n) x^m t^n$$

for $k \geq 2$. Thus $a_{\ell,k}(m,n)$ is the number of words $w \in [\ell]^n$ such that $k\text{bond}(w) = m$.

Let $\mathbb{A}_{\ell,k}$ be the $\ell \times \ell$ matrix whose entries on the main diagonal consists of all $xt-1$'s, whose entries on the first k superdiagonals and the first k subdiagonals are xt, and whose remaining entries are t. Then we have the following theorem.

Theorem 7. *For all $\ell, k \geq 1$,*

$$A_{\ell,k}(x,t) = 1 + (\underbrace{1,\dots,1}_{\ell})\mathbb{A}_{\ell,k}^{-1}(\underbrace{-t,\dots,-t}_{\ell})^T.$$

Proof. For $i = 1, 2, \dots, \ell$, let

$$A_{\ell,k}^{(i)}(x,t) = \sum_{w \in i[\ell]^*} x^{k\text{bond}(w)} t^{|w|} = \sum_{m,n \geq 0} a_{\ell,k}^{(i)}(m,n) x^m t^n.$$

When $k \geq 2$, we can follow the proof of Theorem 4 and find simple recurrences for the functions $A_{\ell,k}^{(i)}(x,t)$. Indeed, in this case we may have more possibilities to create an occurrence of the k-box pattern while adjoining letter i from the left side, so that in the terminology of the proof of Theorem 4,

$$\begin{aligned}
A^{(i)}_{\ell,k}(x,t) &= t + tA^{(1)}_{\ell,k}(x,t) + \cdots + tA^{(i-k-1)}_{\ell,k}(x,t) + \\
&\quad xtA^{(i-k)}_{\ell}(x,t) + \cdots + xtA^{(i+k)}_{\ell,k}(x,t) + \\
&\quad tA^{(i+k+1)}_{\ell,k}(x,t) + \cdots + tA^{(\ell,k)}_{\ell}(x,t).
\end{aligned}$$

Thus, for an arbitrary k, the first row in the matrix $\mathbb{A}$ in Theorem 4 is the vector

$$(xt-1, \underbrace{xt,\dots,xt}_{k}, t,\dots,t),$$

the second row is the vector

$$(xt, xt-1, \underbrace{xt,\dots,xt}_{k}, t,\dots,t),$$

and, more generally, any middle row in A in this case is of the form

$$(t,\dots,t,\underbrace{xt,\dots,xt}_{k}, xt-1, \underbrace{xt,\dots,xt}_{k}, t,\dots,t).$$

□

For example, the generating function $A_{\ell,2}(x,t)$ is equal to

$$1 + (\underbrace{1,\dots,1}_{\ell})A^{-1}_{\ell,2}(\underbrace{-t,\dots,-t}_{\ell})^T,$$

where $A_{\ell,2}$ is the following $\ell \times \ell$ matrix:

$$\mathbb{A}_{\ell,2} = \begin{pmatrix}
xt-1 & xt & xt & t & t & t & t & \cdots & t & t & t \\
xt & xt-1 & xt & xt & t & t & t & \cdots & t & t & t \\
xt & xt & xt-1 & xt & xt & t & t & \cdots & t & t & t \\
t & xt & xt & xt-1 & xt & xt & t & \cdots & t & t & t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
t & t & t & t & t & t & t & \cdots & xt & xt & xt-1
\end{pmatrix}.$$

We have used Theorem 7 to compute the generating functions $A_{\ell,2}(x,t)$ for $\ell = 4,5,6,7$:

$$\begin{aligned}
A_{4,2}(x,t) &= 1 + 4t + 2(1+7x)t^2 + 2\left(1+6x+25x^2\right)t^3 + 2\left(1+7x+31x^2+89x^3\right)t^4 + \\
&\quad 2\left(1+8x+42x^2+144x^3+317x^4\right)t^5 + 2\left(1+9x+54x^2+222x^3+633x^4+1129x^5\right)t^6 + \\
&\quad 2\left(1+10x+67x^2+316x^3+1095x^4+2682x^5+4021x^6\right)t^7 + \\
&\quad 2\left(1+11x+81x^2+427x^3+1707x^4+5145x^5+11075x^6+14321x^7\right)t^8 + \cdots;
\end{aligned}$$

ℓ	generating function $A_{\ell,2}(x,t)$ for $\ell = 4, 5, 6, 7$.
4	$\frac{1-3t(-1+x)-2t^2(-1+x)^2}{1-t-3tx-2t^2(-1+x)x}$
5	$\frac{1-4t(-1+x)+t^3(-1+x)^3}{1+t^2(-1+x)+t^3(-1+x)^2x-t(1+4x)}$
6	$\frac{1-4t(-1+x)-t^2(-1+x)^2+t^3(-1+x)^3}{1-t^2(-1+x)^2+t^3(-1+x)^2(1+x)-2t(1+2x)}$
7	$\frac{1-5t(-1+x)+2t^2(-1+x)^2+4t^3(-1+x)^3-2t^4(-1+x)^4}{1-2t^4(-1+x)^3(1+x)+2t^3(-1+x)^2(1+2x)-t(2+5x)+2t^2(-2+x+x^2)}$

Table 4: Distribution of the 2-bond on ℓ-ary words, $\ell = 4, 5, 6, 7$.

ℓ	generating function for ℓ-ary words avoiding the $(1,2)$-rectangle pattern
$A_{4,2}(0,t)$	$\frac{1+3t-2t^2}{1-t}$
$A_{5,2}(0,t)$	$\frac{1+4t-t^3}{1-t-t^2}$
$A_{6,2}(0,t)$	$\frac{1+4t-t^2-t^3}{1-2t-t^2+t^3}$
$A_{7,2}(0,t)$	$\frac{1+5t+2t^2-4t^3-2t^4}{1-2t-4t^2+2t^3+2t^4}$

Table 5: Enumeration of ℓ-ary words which avoid the $(1,2)$-rectangle pattern for $\ell = 4, 5, 6, 7$.

$$\begin{aligned} A_{5,2}(x,t) \quad = \quad & 1 + 5t + (6+19x)t^2 + 5\left(2+8x+15x^2\right)t^3 + \left(16+88x+226x^2+295x^3\right)t^4 + \\ & \left(26+176x+606x^2+1156x^3+1161x^4\right)t^5 + \left(42+342x+1428x^2+3644x^3+5600x^4+4569x^5\right)t^6 + \\ & \left(68+644x+3170x^2+9840x^3+20250x^4+26172x^5+17981x^6\right)t^7 + \\ & \left(110+1190x+6708x^2+24456x^3+61446x^4+106686x^5+119266x^6+70763x^7\right)t^8 + \cdots; \end{aligned}$$

$$\begin{aligned} A_{6,2}(x,t) \quad = \quad & 1 + 6t + 12(1+2x)t^2 + 4\left(7+22x+25x^2\right)t^3 + \left(62+294x+522x^2+418x^3\right)t^4 + \\ & 4\left(35+214x+552x^2+706x^3+437x^4\right)t^5 + 2\left(157+1191x+3926x^2+7154x^3+7245x^4+3655x^5\right)t^6 + \\ & \left(706+6364x+25702x^2+59624x^3+85166x^4+71804x^5+30570x^6\right)t^7 + \\ & 2\left(793+8295x+39525x^2+111571x^3+202491x^4+239637x^5+173575x^6+63921x^7\right)t^8 + \cdots; \end{aligned}$$

$$\begin{aligned} A_{7,2}(x,t) \quad = \quad & 1 + 7t + (20+29x)t^2 + \left(62+156x+125x^2\right)t^3 + \left(186+710x+962x^2+543x^3\right)t^4 + \\ & \left(566+2820x+5658x^2+5400x^3+2363x^4\right)t^5 + \left(1712+10648x+27710x^2+38526x^3+28766x^4+10287x^5\right)t^6 + \\ & \left(5192+38520x+124086x^2+222928x^3+239930x^4+148100x^5+44787x^6\right)t^7 + \\ & \left(15728+135852x+519888x^2+1149548x^3+1594738x^4+1409754x^5+744298x^6+194995x^7\right)t^8 + \cdots. \end{aligned}$$

Clearly the number of words $w \in [\ell]^n$ such that $k\text{bond}(w) = 0$ equals the number of words $w \in [\ell]^n$ such that $(1,k)\text{rec}(w) = 0$. Plugging in $x = 0$ in the functions in Table 4

ℓ	number of ℓ-ary words avoiding the (1,2)-rectangle pattern	sequence in [11]
4	1, 4, 2, 2, 2, 2, 2, 2, 2, 2, ...	
5	1, 5, 6, 10, 16, 26, 42, 68, 110, 178, ...	A006355, $n \geq 2$
6	1, 6, 12, 28, 62, 140, 314, 706, 1586, 3564, ...	A052994, $n \geq 2$
7	1, 7, 20, 62, 186, 566, 1712, 5192, 15728, 47688, ...	

Table 6: Avoidance of the (1,2)-rectangle patterns in ℓ-ary words for lengths n up to 9.

one gets generating functions for avoidance of the $(1,2)$-rectangle pattern. In Table 6, we list initial values of the respective sequences indicating connections to the OEIS [11].

We note that the sequence A052994 has no combinatorial interpretation in the OEIS so now we have given a combinatorial interpretation to this sequence. Also, comparing Tables 3 and 6, and using an interpretation of [11, A006355], one has the truth of the following proposition that we explain combinatorially.

Proposition 1. *For $n \geq 2$, the following objects are equinumerous:*

(i) words of length n over the alphabet $[5]$ that avoid the $(1,2)$-rectangle pattern;

(ii) words of length n over the alphabet $[4]$ that avoid the 1-box pattern;

(iii) binary words of length $n+3$ that contain no singletons, that is, any 0 has a 0 next to it, and any 1 has a 1 next to it.

Thus, according to [11, A006355], any of these objects is counted by $F_{n-1} + F_{n+2}$ where F_n is the nth Fibonacci number defined as $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Proof. Equinumeration of (i) and (ii) follows directly from the observation that the letter 3 never appears in words described by (i), so that we can take any such word, make the substitution of letters $4 \to 3$ and $5 \to 4$ to get a proper word described by (ii); this operation is clearly reversible.

Equinumeration of (ii) and (iii) is established by the following bijective map from (ii) to (iii). Let a word $w = w_1 w_2 \dots w_n$ described by (ii) be given and we want to obtain its binary image $u = u_1 u_2 \dots u_{n+3}$. If $w_1 \in \{1,2\}$ then $u_1 u_2 = 00$; if $w_1 \in \{3,4\}$ then $u_1 u_2 = 11$. Also, no matter what u_{n+2} is, we set $u_{n+3} = u_{n+2}$. To obtain the letters $u_3, u_4, \dots, u_{n+2}$, we read w from left to right letter by letter: if $w_i \in \{1,4\}$, then $u_{i+2} = u_{i+1}$; if $w_i \in \{2,3\}$, then $u_{i+2} \neq u_{i+1}$. For example, the word 3142413 avoiding the 1-box pattern is mapped to 1100011100. In Table 7, we provide our map for all words of length $n = 2, 3$.

We do not provide a proper proof of the fact that the map described by us from (ii) to (iii) is a bijection just giving a couple of remarks why this is the case. Indeed, if $w_i \in \{2,3\}$ and $i < n$ then $w_{i+1} \in \{1,4\}$ and thus $u_{i+2} = u_{i+3}$. This, together with the fact that $u_{n+3} = u_{n+2}$ makes sure that u has no singletons. □

We say that a word $w = w_1 \dots w_n \in [\ell]^n$ is *k-smooth* if $|w_i - w_{i+1}| \leq k$ for $1 \leq i < n$. Thus in our notation, $w \in [\ell]^n$ is k-smooth if $k\text{bond}(w) = n - 1$. Let $M_{n,k,\ell}$ denote the

13	00011		131	000111
14	00000		141	000000
24	00111		142	000011
31	11000		241	001111
41	11111		242	001100
42	11100		313	110011
			314	110000
			413	111100
			414	111111
			424	111000

Table 7: Mapping 1-box avoiding permutations over [4] to binary strings without singletons.

ℓ	generating function for words $w \in [\ell]^n$ such that $2\text{bond}(w) = n-1$.
$sm_{4,2}(t)$	$\frac{1+t}{1-3t-2t^2}$
$sm_{5,2}(t)$	$\frac{1+t-t^2}{1-4t+t^3}$
$sm_{6,2}(t)$	$\frac{1+2t-t^2-t^3}{1-4t-t^2+t^3}$
$sm_{7,2}(t)$	$\frac{1+2t-4t^2-2t^3+2t^4}{1-5t+2t^2+4t^3-2t^4}$

Table 8: Distribution of words $w \in [\ell]^n$ such that $2\text{bond}(w) = n-1$, $\ell = 4, 5, 6, 7$.

number of $w \in [\ell]^n$ such that $k\text{bond}(w) = n-1$ and $sm_{\ell,k}(t) = 1 + \sum_{n\geq 1} M_{n,k,\ell} t^n$. Clearly,

$$A_{\ell,k}(1/x, xt) = 1 + \sum_{n\geq 1} \sum_{w\in[\ell]^n} x^{n-k\text{bond}(w)} t^n$$

so that

$$C_{\ell,k}(x,t) := \frac{1}{x}(A_{\ell,k}(1/x, xt) - 1) = \sum_{n\geq 1} \sum_{w\in[\ell]^n} x^{n-1-k\text{bond}(w)} t^n.$$

Hence

$$sm_{\ell,k}(t) = 1 + C_{\ell,k}(0,t).$$

We have used our generating functions for $A_{\ell,2}(x,t)$ to compute $sm_{\ell,2}(t)$ for $\ell = 4, 5, 6, 7$, which we record in Table 9. In the case $\ell = 4$, our objects match a combinatorial interpretation for the sequence A055099. For the sequence A126392, the generating function $sm_{5,2}(t) = \frac{1+t-t^2}{1-4t+t^3}$ was conjectured by Colin Barker, so we have proved his conjecture. The sequences A126393 and A126394 were apparently computed from their combinatorial definitions by R. H. Hardin, so that we now have found explicit formulas for their generating functions.

ℓ	number of words $w \in [\ell]^n$ such that 2bond$(w) = n-1$	sequence in [11]
4	1, 4, 14, 50, 178, 634, 2258, 8042, 28642,102010, ...	A055099, $n \geq 0$
5	1, 5, 19, 75, 295, 1161, 4569, 17981, 70763, 278483, ...	A126392, $n \geq 0$
6	1, 6, 24, 100, 418, 1748, 7310, 30570, 127842, 534628, ...	A126393, $n \geq 0$
7	1, 7, 29, 125, 543, 2363, 10287, 44787, 194995, 848979, ...	A126394, $n \geq 0$

Table 9: Number of words $w \in [\ell]^n$ such that $k$bond$(w) = n-1$ for n up to 9.

One can also modify the proof of Theorem 5 to find the generating function for the distribution of $(1,k)\text{rec}(w)$ for $w \in [\ell]^*$. That is, suppose $k \geq 2$, and for $1 \leq i,j \leq \ell$,

$$B^{(ij)}_{\ell,k} = \sum_{w \in ij[\ell]^*} WT_k(w),$$

where $WT_k(w) = x^{(1,k)\text{rec}(w)} t^{|w|}$. Then we claim that for all $1 \leq i,j \leq \ell$,

$$\begin{aligned} B^{(ij)}_{\ell,k}(x,t) &= x^{2\chi(|i-j|\leq k)}t^2 + \\ &\sum_{k=1}^{\ell}(t\chi(|i-j|>k) + xt\chi(|i-j|\leq k)\chi(|j-k|\leq k) + \\ &\qquad x^2t\chi(|i-j|\leq k)\chi(|j-k|>k))B^{(jk)}_{\ell,k}(x,t). \end{aligned} \tag{16}$$

That is, the words in $ij[\ell]^*$ are of the form ij plus words $ijmv$ where $m \in [\ell]$ and $v \in [\ell]^*$. Now

$$WT_k[ij] = \begin{cases} t^2 & \text{if } |i-j| > k \text{ and} \\ x^2t^2 & \text{if } |i-j| \leq k. \end{cases}$$

Similarly,

$$WT_k[ijmv] = \begin{cases} tWT_k[jmv] & \text{if } |i-j| > k, \\ xtWT_k[jmv] & \text{if } |i-j| \leq k \text{ and } |j-m| \leq k, \text{ and} \\ x^2tWT_k[jmv] & \text{if } |i-j| \leq k \text{ and } |j-m| > k. \end{cases}$$

The set of equations of the form (16) can be written out in matrix form. That is, let $\vec{B}_{\ell,k}$ be the row vector of length ℓ^2 of the $B^{(ij)}_{\ell,k}(t,x)$ where the elements are listed in the lexicographic order of the pairs (ij). Let $\vec{I}_{\ell,k}$ be the row vector of length ℓ^2 of the terms $t^2x^{2\chi(|i-j|\leq k)}$ again listed in the lexicographic order on the pairs ij. For example,

$$\vec{I}_{4,2} = (x^2t^2, x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, x^2t^2, t^2, x^2t^2, x^2t^2, x^2t^2).$$

Then one can write a set of equations of the form (16) in the form

$$(\vec{I}_{\ell,x})^T = \mathbb{B}_{\ell,k}(\vec{B}_{\ell,k})^T,$$

where $\mathbb{B}_{\ell,k}$ is an $\ell^2 \times \ell^2$ matrix. For example, $\mathbb{B}_{4,2}$ is the matrix

$$\begin{pmatrix}
xt-1 & xt & xt & x^2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \\
xt & xt & xt & x^2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & xt & xt-1 & xt & xt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & x^2t & xt & xt & xt \\
xt & xt & xt & x^2t & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xt & xt & xt-1 & xt & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & x^2t & xt & xt & xt \\
t & t & t & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & xt & xt & xt & xt & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2t & xt & xt & xt-1
\end{pmatrix}$$

Note that $\mathbb{B}_{\ell,k}$ is invertible since setting $x = t = 0$ in $\mathbb{B}_{\ell,k}$ will give the $\ell \times \ell$ diagonal matrix with -1s on the diagonal. Thus

$$(\vec{B}_{\ell,k})^T = \mathbb{B}_{\ell,k}^{-1}(\vec{I}_{\ell,k})^T.$$

Let $\vec{1}_{\ell,1}$ denote the vector of length ℓ^2 consisting of all 1s. Then

$$\sum_{1\le i,j\le \ell} B_{\ell,1}^{(ij)}(x,t) = \vec{1}_{\ell,1}\mathbb{B}_{\ell,k}^{-1}(\vec{I}_{\ell,k})^T.$$

Taking into account the empty word and all the words of length 1 will yeild the following theorem.

Theorem 8. *For all $\ell \ge 2$,*

$$B_{\ell,k}(x,t) := \sum_{w\in[\ell]^*} x^{(1,k)\text{rec}(w)} t^{|w|} = 1 + \ell t + \vec{1}_{\ell,k}\mathbb{B}_{\ell,k}^{-1}(\vec{I}_{\ell,k})^T.$$

Note that $B_{\ell,2}(x,t) = \frac{1}{1-\ell xt}$ for $\ell = 1, 2, 3$ since in such words every letter matches the $(1, 2)$-rectangle pattern. We have used Theorem 8 to compute $B_{\ell,2}(x,t)$ for $\ell = 4, 5$:

$$B_{4,2}(x,t) = \frac{1 - 3t(-1+x) + 6t^3(-1+x)^2x + 4t^4(-1+x)^2x^2 + t^2\left(2 + 9x - 11x^2\right)}{1 - t - 3tx - 3t^2(-1+x)x + 2t^3(-1+x)x^2};$$

$$B_{5,2}(x,t) = \\ -\frac{1 + t(-1+x)\left(-4 + t\left(16x + t(-1+x)\left(-1 + x\left(-2 + t^3(-1+x)x^2 + 4t(1+x)\right)\right)\right)\right)}{-1 + t\left(1 + 4x + t(-1+x)\left(-1 + x\left(3 + t^3(-1+x)x^2 + t(4+x)\right)\right)\right)}.$$

Using the generating functions above, we have computed some of the initial terms in their Taylor series expansions:

$$B_{4,2}(x,t) = 1 + 4t + 2\left(1 + 7x^2\right)t^2 + 2\left(1 + 6x^2 + 25x^3\right)t^3 + 2\left(1 + 7x^2 + 22x^3 + 98x^4\right)t^4 + \\ 2\left(1 + 8x^2 + 27x^3 + 93x^4 + 383x^5\right)t^5 + 2\left(1 + 9x^2 + 32x^3 + 117x^4 + 396x^5 + 1493x^6\right)t^6 + \\ 2\left(1 + 10x^2 + 37x^3 + 142x^4 + 519x^5 + 1659x^6 + 5824x^7\right)t^7 + \\ 2\left(1 + 11x^2 + 42x^3 + 168x^4 + 652x^5 + 2247x^6 + 6930x^7 + 22717x^8\right)t^8 + \cdots;$$

$$
\begin{aligned}
&B_{5,2}(x,t) = 1 + 5t + \left(6 + 19x^2\right) t^2 + 5\left(2 + 8x^2 + 15x^3\right) t^3 + \left(16 + 88x^2 + 160x^3 + 361x^4\right) t^4 + \\
&\left(26 + 176x^2 + 358x^3 + 876x^4 + 1689x^5\right) t^5 + \\
&\left(42 + 342x^2 + 724x^3 + 2106x^4 + 4496x^5 + 7915x^6\right) t^6 + \\
&\left(68 + 644x^2 + 1416x^3 + 4586x^4 + 11328x^5 + 22976x^6 + 37107x^7\right) t^7 + \\
&\left(110 + 1190x^2 + 2680x^3 + 9562x^4 + 25712x^5 + 60762x^6 + 116672x^7 + 173937x^8\right) t^8 + \cdots .
\end{aligned}
$$

We also can compute the generating functions of the number of words that avoid the $(1,2)$-rectangle patterns for words $w \in [5]^*$. That is, we have that

$$
\begin{aligned}
B_{5,2}(0,t) &= \frac{1+4t-t^3}{1-t-t^2} \\
&= 1 + 5t + 6t^2 + 10t^3 + 16t^4 + 26t^5 + 42t^6 + 68t^7 + 110t^8 + \cdots .
\end{aligned}
$$

Note that

$$
\overline{B}_{\ell,k}(x,t) := B_{\ell,k}(1/x, xt) = \sum_{w\in[\ell]^*} x^{n-((1,k)\mathrm{rec}(w))} t^n,
$$

so that $\overline{B}_{\ell,k}(0,t)$ is the generating function of all words in $w = w_1 \dots w_n \in [k]^*$ such that $(1,k)\mathrm{rec}(w) = n$, i.e., each letter of w differs from at least one neighbor by k or less. We have computed $\overline{B}_{\ell,2}(0,t)$ for $k = 4, 5$.

$$
\overline{B}_{4,2}(0,t) = \frac{1 - 3t + 11t^2 + 6t^3 + 4t^4}{1 - 3t - 3t^2 - 2t^3}.
$$

The initial terms of this series are $1, 0, 14, 50, 196, 766, 2986, 11648, 44343, 177218, 691252, \dots$. This sequence does not appear in the OEIS.

$$
\overline{B}_{5,1}(0,t) = \frac{1 - 3t + 9t^2 - 4t^3 + 6t^4 + 4t^6}{1 - 3t - 4t^2 - 6t^4 - 4t^5 - 4t^6}.
$$

The initial terms of this series are $1, 0, 19, 75, 361, 1689, 7915, 37107, 173937, 815345, \dots$. This sequence also does not appear in the OEIS.

Our methods obviously extend to allow us to write a matrix equation for the generating function $B_{\ell,a,b}(x,t) = \sum_{w\in[\ell]^*} x^{(a,b)\mathrm{rec}(w)} t^{|w|}$. However, it becomes computationally unfeasiable even in the case of $2\mathrm{box}(w)$. That is, one has to keep track of the first four letters to be able to compute the necessary recursions. For example, let

$$
B^{rstu}_{\ell,2\mathrm{box}}(x,t) = \sum_{w\in rstu[\ell]^*} x^{2\mathrm{box}(w)} t^{|w|},
$$

where $rstu[\ell]^*$ is the set of all words over $[\ell]$ that begin with letters $rstu$. Then it is easy to see that

$$
B^{rstu}_{\ell,2\mathrm{box}}(x,t) = x^{2\mathrm{box}(rstu)} t^4 + \sum_{v=1}^{\ell} \theta(rstuv) B^{stuv}_{\ell,2\mathrm{box}}(x,t),
$$

where $\theta(rstuv)$ is computed according the following four cases.

Case 1. $|r-s|>2$ and $|r-t|>2$. In this case, $\theta(rstuv)=t$.

Case 2. $|r-s|>2$ and $|r-t|\leq 2$. In this case, $\theta(rstuv)=xt$ if t matches the 2-box pattern in $stuv$ and $\theta(rstuv)=x^2t$ if t does not match the 2-box pattern in $stuv$. That is, for any word $w\in[\ell]^*$, the presence of r does not effect whether s will match the 2-box pattern in $rstuv$, but it does effect the question of whether t matches the 2-box pattern in $rstuv$.

Case 3. $|r-s|\leq 2$ and $|r-t|>2$. In this case, $\theta(rstuv)=xt$ if s matches the 2-box pattern in stu and $\theta(rstuv)=x^2t$ if s does not match the 2-box pattern in stu. That is, for any word $w\in[\ell]^*$, the presence of r does not effect whether t will match the 2-box pattern in $rstuv$, but it does effect the question of whether s matches the 2-box pattern in $rstuv$.

Case 4. $|r-s|\leq 2$ and $|r-t|\leq 2$. In this case $\theta(rstuv)=xt$ if both s and t match the 2-box pattern in $stuv$, $\theta(rstuv)=x^2t$ if exactly one of s and t match the 2-box pattern in $stuv$, and $\theta(rstuv)=x^3t$ if neither s nor t match the 2-box pattern in $stuv$.

This recursion allows us to write a simple matrix type equation for the generating function $B_{\ell,2\text{box}}(x,t)$; however, it requires that we have to invert an $\ell^4\times\ell^4$ matrix which is not really feasible even for small ℓ. Indeed, the generating function $B_{\ell,2\text{box}}(x,t)$ is trivial for $\ell\leq 3$, so the smallest non-trivial ℓ is $\ell=4$ which requires we would have to invert a $4^4\times 4^4$-matrix.

4 Conclusion

The goal of this paper was to introduce k-box patterns and to study them, mainly in the case of $k=1$, on permutations and words. In particular, we proved a conjecture of Mathar on the number of “stable LEGO walls” of width 7, as well as proved three conjectures due to Hardin and a conjecture due to Barker. In [8], we study 1-box patterns on pattern-avoiding permutations (more precisely, on 132-avoiding permutations and on *separable permutations*).

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