

SPECTRAL ASYMPTOTICS FOR RESOLVENT DIFFERENCES OF ELLIPTIC OPERATORS WITH δ AND δ' -INTERACTIONS ON HYPERSURFACES

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ABSTRACT. We consider self-adjoint realizations of a second-order elliptic differential expression on \mathbb{R}^n with singular interactions of δ and δ' -type supported on a compact closed smooth hypersurface in \mathbb{R}^n . In our main results we prove spectral asymptotics formulae with refined remainder estimates for the singular values of the resolvent difference between the standard self-adjoint realizations and the operators with a δ and δ' -interaction, respectively. Our technique makes use of general pseudodifferential methods, classical results on spectral asymptotics of ψ do's on closed manifolds and Krein-type resolvent formulae.

1. INTRODUCTION

In this paper we study self-adjoint operator realizations of the formally symmetric, uniformly strongly elliptic differential expression

$$(\mathcal{A}u)(x) := - \sum_{j,k=1}^n \partial_j (a_{jk}(x) \partial_k u)(x) + a(x)u(x), \quad x \in \mathbb{R}^n,$$

with singular interactions of δ and δ' -type supported on a C^∞ -smooth compact hypersurface $\Sigma \subset \mathbb{R}^n$, which splits \mathbb{R}^n into a bounded open set Ω_- and an unbounded open set Ω_+ . More precisely, denote by \mathcal{A}_\pm the restrictions of \mathcal{A} to Ω_\pm , let γ^\pm and ν^\pm be the trace and conormal trace, respectively, on the boundary Σ of Ω_\pm , let $\alpha, \beta \in C^\infty(\Sigma)$ be real functions with $\beta(x') \neq 0$ for all $x' \in \Sigma$, and consider the elliptic realizations

$$\begin{aligned} A_{\delta,\alpha} u &= \mathcal{A}_+ u_+ \oplus \mathcal{A}_- u_-, \\ \text{dom } A_{\delta,\alpha} &= \{u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-): \\ &\quad \gamma^+ u_+ = \gamma^- u_-, \quad \nu^+ u_+ + \nu^- u_- = \alpha \gamma^+ u_+\}, \end{aligned}$$

and

$$\begin{aligned} A_{\delta',\beta} u &= \mathcal{A}_+ u_+ \oplus \mathcal{A}_- u_-, \\ \text{dom } A_{\delta',\beta} &= \{u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-): \\ &\quad \nu^+ u_+ + \nu^- u_- = 0, \quad \beta \nu^+ u_+ = \gamma^+ u_+ - \gamma^- u_-\}, \end{aligned}$$

which are self-adjoint and bounded from below in $L_2(\mathbb{R}^n)$; cf. Theorem 2.1. Our main goal is to compare the resolvents of $A_{\delta,\alpha}$ and $A_{\delta',\beta}$ with the resolvent of the 'free' or 'unperturbed' self-adjoint realization

$$A_0 u = \mathcal{A}u, \quad \text{dom } A_0 = H^2(\mathbb{R}^n),$$

and to prove spectral asymptotics formulae with refined remainder estimates for the singular values of the corresponding resolvent differences. Without loss of generality we may assume that a sufficiently large positive constant is added to \mathcal{A} such that all operators under consideration have a positive lower bound; hence we consider

$$G_{\delta,\alpha} = A_{\delta,\alpha}^{-1} - A_0^{-1} \quad \text{and} \quad G_{\delta',\beta} = A_{\delta',\beta}^{-1} - A_0^{-1}. \quad (1.1)$$

It is known that both operators $G_{\delta,\alpha}$ and $G_{\delta',\beta}$ are compact in $L_2(\mathbb{R}^n)$, and estimates for the decay of the singular values $s_k(G_{\delta,\alpha})$ and $s_k(G_{\delta',\beta})$ were recently obtained in Behrndt et al. [6] and [7] (for the special case $\mathcal{A} = -\Delta + a$). In our main results Theorem 5.1 and Theorem 5.2 we shall prove the more precise asymptotic results with estimates of the remainder of the form

$$\begin{aligned} s_k(G_{\delta,\alpha}) &= C_{\delta,\alpha} k^{-\frac{3}{n-1}} + O(k^{-\frac{4}{n-1}}), & k \rightarrow \infty, \\ s_k(G_{\delta',\beta}) &= C_{\delta'} k^{-\frac{2}{n-1}} + O(k^{-\frac{3}{n-1}}), & k \rightarrow \infty, \end{aligned} \quad (1.2)$$

with positive constants $C_{\delta,\alpha}$ and $C_{\delta'}$ which are given explicitly in terms of the coefficients of \mathcal{A} and α ; the constant $C_{\delta'}$ is independent of β . Note that the singular values of $G_{\delta,\alpha}$ converge faster than the singular values of $G_{\delta',\beta}$. We mention that for the first result in (1.2) it is assumed that the function α does not vanish on Σ ; if this assumption is dropped, the estimate holds with remainder $o(k^{-\frac{3}{n-1}})$; cf. Theorem 5.1. In the course of our work we also make use of the direct sum A_ν of the self-adjoint Neumann operators in $L_2(\Omega_+)$ and $L_2(\Omega_-)$ and we show that the singular values of $G_{\delta',\beta,\nu} = A_{\delta',\beta}^{-1} - A_\nu^{-1}$ satisfy

$$s_k(G_{\delta',\beta,\nu}) = C_{\delta',\beta,\nu} k^{-\frac{3}{n-1}} + O(k^{-\frac{4}{n-1}}), \quad k \rightarrow \infty, \quad (1.3)$$

with the constant $C_{\delta',\beta,\nu} > 0$ explicitly given; cf. Theorem 5.3. The proofs of (1.2) and (1.3) are mainly based on pseudodifferential techniques and classical results on spectral asymptotics of ψ do's on closed C^∞ -smooth manifolds due to Seeley [47], Hörmander [37] and Grubb [28]. We also refer to Boutet de Monvel [11], Hörmander [38], Taylor [50], Rempel and Schulze [46] and Grubb [30, 31] for general pseudodifferential methods. A further ingredient in our analysis is a Krein-type resolvent formula, which provides a factorization of the operators in (1.1) and is discussed in detail in Section 4; cf. Brasche et al. [13], Alpay and Behrndt [1], Behrndt et al. [6, 7].

Our results in this paper contribute to a prominent field in the analysis of partial differential operators: asymptotic estimates for the resolvent difference of elliptic operators subject to different boundary conditions were first obtained by Povzner [45] and Birman [8]. These estimates were sharpened to spectral asymptotics formulae by Grubb [26] for bounded domains and by Birman and Solomjak [9, 10] for exterior domains, further generalized by Grubb [28, 29], and more recently in [32, 34, 36]; see also the review paper [35]. For the case of two Robin Laplacians a faster convergence of the singular values was observed in Behrndt et al. [5], and further refined to spectral asymptotics in Grubb [33]. We also list the closely related works Deift and Simon [16], Bardos et al. [3], Gorbachuk and Kutovoi [25], Brasche [12], Carron [15], Malamud [43] and Lotoreichik and Rohleder [42] with spectral estimates for resolvent differences and resolvent power differences.

We wish to emphasize that the operators $A_{\delta,\alpha}$ and $A_{\delta',\beta}$ have attracted considerable interest in the last two decades from more applied branches of mathematics and mathematical physics. In the special case $\mathcal{A} = -\Delta + a$ we refer to the review paper [18] by Exner for an overview on Schrödinger operators with δ -interactions supported on curves and hypersurfaces. Such Hamiltonians are physically relevant in quantum mechanics, where they are employed in many-body problems and in the description of various nanostructures, as well as in the theory of photonic crystals; see, e.g. Figotin and Kuchment [24], Popov [44] and Brummelhuis and Duclos [14]. At the same time there is a mathematical motivation to study Schrödinger operators with δ -interactions on hypersurfaces because these operators exhibit non-trivial and interesting spectral properties; for more details we refer to Brasche et al. [13], Exner et al. [19, 20, 22, 23], Suslina and Shterenberg [49], Kondej and Veselić [40], Kondej and Krejčířík [39], Duchêne and Raymond [17] and the references therein. Schrödinger operators with δ' -interactions supported on hypersurfaces are much less studied than their δ -counterparts. They have been rigorously defined (in a general setting) only recently in [7]; the works Behrndt et al. [4] and Exner and Jex [21] on their spectral properties appeared subsequently. We also mention that for very special geometries such operators were considered earlier in Antoine et al. [2] and Shabani [48].

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2. THE DIFFERENTIAL OPERATORS

Throughout this paper let \mathcal{A} be the following second-order formally symmetric differential expression on \mathbb{R}^n :

$$(\mathcal{A}u)(x) := - \sum_{j,k=1}^n \partial_j (a_{jk}(x) \partial_k u)(x) + a(x)u(x), \quad x \in \mathbb{R}^n, \quad (2.1)$$

with real-valued $a_{jk} \in C^\infty(\mathbb{R}^n)$ satisfying $a_{jk}(x) = a_{kj}(x)$ for all $x \in \mathbb{R}^n$, $j, k = 1, \dots, n$, and a bounded real-valued coefficient $a \in C^\infty(\mathbb{R}^n)$. We assume that a_{jk} and all their derivatives are bounded and that \mathcal{A} is uniformly strongly elliptic, i.e.

$$\sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k \geq C |\xi|^2, \quad x, \xi \in \mathbb{R}^n,$$

for some constant $C > 0$.

Further, let $\Sigma \subset \mathbb{R}^n$ be a C^∞ -smooth $(n-1)$ -dimensional manifold that separates the Euclidean space \mathbb{R}^n into a bounded open set Ω_- and an unbounded open set Ω_+ . In the following we denote by u_+ and u_- the restrictions of $u \in L_2(\mathbb{R}^n)$ to Ω_+ and Ω_- , respectively; the restrictions of the differential expression \mathcal{A} to Ω_\pm are denoted by \mathcal{A}_\pm . For functions $u_\pm \in H^2(\Omega_\pm)$

denote by $\gamma^\pm u_\pm$ the traces (boundary values on Σ) and by $\nu^\pm u_\pm$ the outward conormal derivatives of u_\pm :

$$\nu^\pm u_\pm = \sum_{j,k=1}^n a_{jk} \nu_{\pm,j} \gamma^\pm \partial_k u_\pm, \quad (2.2)$$

where $(\nu_{\pm,1}(x), \dots, \nu_{\pm,n}(x))$ is the exterior unit normal to Ω_\pm at $x \in \Sigma$. If $u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-)$ and $\gamma^+ u_+ = \gamma^- u_-$, then $u \in H^1(\mathbb{R}^n)$ and we write γu for $\gamma^+ u_+ = \gamma^- u_-$.

Let us introduce the following operators: the free realization of \mathcal{A} in $L_2(\mathbb{R}^n)$,

$$A_0 u := \mathcal{A} u, \quad \text{dom } A_0 := H^2(\mathbb{R}^n), \quad (2.3)$$

the Dirichlet realizations on Ω_+ and Ω_- ,

$$A_{\pm,\gamma} u_\pm := \mathcal{A}_\pm u_\pm, \quad \text{dom } A_{\pm,\gamma} := \{u_\pm \in H^2(\Omega_\pm) : \gamma^\pm u_\pm = 0\},$$

and the Neumann realizations,

$$A_{\pm,\nu} u_\pm := \mathcal{A}_\pm u_\pm, \quad \text{dom } A_{\pm,\nu} := \{u_\pm \in H^2(\Omega_\pm) : \nu^\pm u_\pm = 0\}.$$

It is well known that the operators A_0 , $A_{+,\gamma}$, $A_{-,\gamma}$, $A_{+,\nu}$ and $A_{-,\nu}$ are self-adjoint and bounded below.

Let us also introduce direct sums of operators on Ω_+ and Ω_- :

$$A_\gamma := A_{+,\gamma} \oplus A_{-,\gamma}, \quad A_\nu := A_{+,\nu} \oplus A_{-,\nu}, \quad (2.4)$$

which are self-adjoint operators in $L_2(\mathbb{R}^n) = L_2(\Omega_+) \oplus L_2(\Omega_-)$. Note also that the domain of A_0 can be written with interface conditions:

$$\begin{aligned} \text{dom } A_0 &= \{u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-) : \\ &\quad \gamma^+ u_+ = \gamma^- u_-, \quad \nu^+ u_+ = -\nu^- u_-\}. \end{aligned}$$

Moreover, let us fix a real-valued function $\alpha \in C^\infty(\Sigma)$, and define the δ -operator with strength α by

$$\begin{aligned} A_{\delta,\alpha} u &:= \mathcal{A}_+ u_+ \oplus \mathcal{A}_- u_-, \\ \text{dom } A_{\delta,\alpha} &:= \{u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-) : \\ &\quad \gamma^+ u_+ = \gamma^- u_-, \quad \nu^+ u_+ + \nu^- u_- = \alpha \gamma u\}. \end{aligned} \quad (2.5)$$

Let us also fix a real-valued function $\beta \in C^\infty(\Sigma)$ such that β is non-zero on Σ , and define the δ' -operator with strength β by

$$\begin{aligned} A_{\delta',\beta} u &:= \mathcal{A}_+ u_+ \oplus \mathcal{A}_- u_-, \\ \text{dom } A_{\delta',\beta} &:= \{u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-) : \\ &\quad \nu^+ u_+ + \nu^- u_- = 0, \quad \beta \nu^+ u_+ = \gamma^+ u_+ - \gamma^- u_-\}. \end{aligned} \quad (2.6)$$

The statements in the next theorem were shown in [6, Theorem 4.17] and [7, Theorem 3.11, 3.14, and 3.16] for the special case $\mathcal{A} = -\Delta + a$; the general case can be shown in a similar way. For the self-adjointness of $A_{\delta,\alpha}$ and $A_{\delta',\beta}$ one can also use the symmetry together with elliptic regularity theory as done in a related situation in [26, Theorem 7.3].

Theorem 2.1. *The operators $A_{\delta,\alpha}$ and $A_{\delta',\beta}$ are self-adjoint and bounded below in $L_2(\mathbb{R}^n)$.*

Since all these operators are bounded below, we can assume without loss of generality (by adding a sufficiently large real constant to a) that A_0 , $A_{\pm,\gamma}$, $A_{\pm,\nu}$, $A_{\delta,\alpha}$ and $A_{\delta',\beta}$ are positive with 0 in the resolvent set.

We shall often tacitly identify $H^s(\Omega_+) \oplus H^s(\Omega_-)$ with $H^s(\Omega_+) \times H^s(\Omega_-)$ and write the operators in matrix form.

3. PSEUDODIFFERENTIAL METHODS

In order to show the spectral asymptotics formulae we are aiming for, we have to go deeper into the definitions of the entering operators by pseudodifferential techniques. Pseudodifferential operators (ψ do's) P are defined on \mathbb{R}^n by formulae

$$(Pu)(x) = \text{Op}(p(x, \xi))u = p(x, D)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) dx,$$

where $\hat{u}(\xi) = \mathcal{F}u = \int e^{-ix \cdot \xi} u(x) dx$ is the Fourier transform; $p(x, \xi)$ is called the *symbol* of P . There are various conditions on p , interpretations to distributions u , and rules of calculus, in particular behaviour under coordinate changes that allow the definition on manifolds, for which we refer to the vast literature, e.g. [38, 50, 31]. The symbols we consider are ‘‘classical’’ or ‘‘polyhomogeneous’’, meaning that $p(x, \xi)$ is an asymptotic series of functions $p_{d-j}(x, \xi)$, $j = 0, 1, 2, \dots$, homogeneous of degree $d - j$ in ξ ; p (and P) is then said to be of order d , and the principal symbol is the first term $p^0 = p_d$. It has an invariant meaning in the manifold situation.

The following theorem is an important ingredient in the proofs in Section 5. The first part is essentially due to Seeley [47]. This paper treats the elliptic case; how the estimate can be extended to the general case is discussed, e.g. in [28, Lemma 4.5 and following paragraph] with more references given as well. The second part is due to Hörmander [37]. The transition between his formulation in terms of the counting function for P^{-1} and the eigenvalue asymptotics for P is accounted for, e.g. in [27, Lemma 6.2] and [30, Lemma A.5].

Theorem 3.1. *Let P be a classical pseudodifferential operator of negative order $-t$ on Σ , with principal symbol $p^0(x, \xi)$. Then the following statements hold.*

- (i) P is a compact operator in $L_2(\Sigma)$, and its singular values satisfy

$$s_k(P) = (c(P))^{\frac{t}{n-1}} k^{-\frac{t}{n-1}} + o(k^{-\frac{t}{n-1}}), \quad k \rightarrow \infty,$$

where

$$c(P) = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma} \int_{|\xi|=1} |p^0(x, \xi)|^{\frac{n-1}{t}} d\omega(\xi) d\sigma(x);$$

here σ and ω are the surface measures on the hypersurfaces Σ and $\{\xi \in \mathbb{R}^{n-1} : |\xi| = 1\}$, respectively.

- (ii) If, moreover, P is elliptic and invertible, then the asymptotic estimate can be sharpened to the form

$$s_k(P) = (c(P))^{\frac{t}{n-1}} k^{-\frac{t}{n-1}} + O(k^{-\frac{t+1}{n-1}}), \quad k \rightarrow \infty.$$

A general systematic theory covering the boundary value problems we are considering, as well as much more general situations, was introduced by Boutet de Monvel [11]: the theory of pseudodifferential boundary operators (ψ dbo's). Besides working with pseudodifferential operators P on \mathbb{R}^n and their versions P_+ truncated to smooth subsets Ω (in particular to \mathbb{R}_+^n), the theory includes Poisson operators K (going from $\partial\Omega$ to Ω), trace operators T (going from Ω to $\partial\Omega$), ψ do's S on $\partial\Omega$, and the so-called singular Green operators G , essentially in the form of finite sums or infinite series: $\sum K_j T_j$. We shall not use the ψ dbo calculus in full generality, but rather its notation and elementary composition rules. Details on the ψ dbo calculus are found, e.g. in [11, 46, 28, 30, 31].

In this theory, the operators are described by use of local coordinate systems, carrying the study of the operators over to the situation of $\Omega = \mathbb{R}_+^n := \mathbb{R}^{n-1} \times \mathbb{R}_+$. Here a differential operator $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ with the symbol $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$ has the principal symbol $p^0(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$, and the model operator at a point $(x', 0) \in \mathbb{R}^{n-1} \times \{0\}$ is $p^0(x', 0, \xi', D_n) = \sum_{|\alpha|=m} a_\alpha(x', 0) \xi'^{\alpha'} D_n^{\alpha_n}$. The solution operator for the Dirichlet problem for our \mathcal{A} on Ω_+ with non-zero boundary data, zero interior data, is a Poisson operator. Such operators, carried over to \mathbb{R}_+^n , are generally of the form

$$(K\varphi)(x) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \tilde{k}(x', x_n, \xi') \hat{\varphi}(\xi') d\xi', \quad x = (x', x_n),$$

where $\tilde{k}(x', x_n, \xi')$ is called the *symbol-kernel* of K ; it is a C^∞ -function on $\mathbb{R}_+^n \times \mathbb{R}^{n-1}$ that is rapidly decreasing for $x_n \rightarrow \infty$, with

$$\sup_{x_n \in (0, \infty)} |x_n^l \partial_n^{l'} \partial_{x'}^\beta \partial_{\xi'}^\alpha \tilde{k}(x, \xi')| \leq C_{l, l', \alpha, \beta} (1 + |\xi'|)^{d-l+l'-|\alpha|}, \quad (3.1)$$

$$l, l' \in \mathbb{N}_0, \alpha, \beta \in \mathbb{N}_0^{n-1},$$

for some d , and is then said to be of *order* d . More information on the structure of symbol-kernels is found, e.g. in [31, Section 10.1]. The symbol-kernel \tilde{k} is a series of terms with certain quasi-homogeneities in (ξ', x_n) , corresponding to falling homogeneities in ξ of the terms in the Fourier transform w.r.t. x_n , the *symbol* $k = \mathcal{F}_{x_n \rightarrow \xi_n} e^+ \tilde{k}$. There is a principal part, the top order term.

It is also known from the general calculus that the adjoint of a Poisson operator is a trace operator (of the general form defined in the ψ dbo calculus), and that a Poisson operator composed to the left with a trace operator gives a pseudodifferential operator on $\partial\Omega$ (on \mathbb{R}^{n-1} in local coordinates).

In Lemma 3.2 below we describe the principal symbol-kernel of the Poisson solution operators and the principal symbols of the Dirichlet-to-Neumann and Neumann-to-Dirichlet maps corresponding to \mathcal{A} from (2.1) both on Ω_+ and on Ω_- . For this purpose, let us write the principal symbol $a^0(x, \xi)$ of \mathcal{A} in local coordinates $(x', x_n) = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}_+^n$ at the boundary of Ω_+ :

$$\underline{a}^0(x', 0, \xi) = \sum_{j, k=1}^n \underline{a}_{jk}(x') \xi_j \xi_k = \underline{a}_{nn}(x') \xi_n^2 + 2b(x', \xi') \xi_n + c(x', \xi'). \quad (3.2)$$

Here $\xi' = (\xi_1, \dots, \xi_{n-1})$, $\underline{a}_{nn}(x') > 0$,

$$b(x', \xi') := \sum_{j=1}^{n-1} \underline{a}_{jn}(x') \xi_j, \quad c(x', \xi') := \sum_{j,k=1}^{n-1} \underline{a}_{jk}(x') \xi_j \xi_k \quad (3.3)$$

and $\underline{a}_{nn}c > b^2$ when $\xi' \neq 0$ since $\underline{a}^0 > 0$ for $\xi \neq 0$. The roots of the second-order polynomial in ξ_n on the right-hand side of (3.2) are

$$\lambda_{\pm}(x', \xi') = \frac{-b(x', \xi') \pm i \sqrt{\underline{a}_{nn}(x')c(x', \xi') - (b(x', \xi'))^2}}{\underline{a}_{nn}(x')},$$

lying in the upper, respectively lower, complex half-plane and being homogeneous of degree 1 in ξ' . Define

$$\begin{aligned} \kappa_0 &:= \sqrt{\underline{a}_{nn}c - b^2} \quad (> 0 \text{ when } \xi' \neq 0), \\ \kappa_{\pm} &:= \mp i \lambda_{\pm} = \frac{\kappa_0 \pm ib}{\underline{a}_{nn}}. \end{aligned} \quad (3.4)$$

Clearly, κ_{\pm} are complex conjugates and have positive real part: $\operatorname{Re} \kappa_{\pm} = \kappa_0/\underline{a}_{nn}$, and satisfy $\kappa_+ \bar{\kappa}_+ = c/\underline{a}_{nn}$. With these expressions we can factorize \underline{a}^0 :

$$\underline{a}^0(x', 0, \xi', \xi_n) = \underline{a}_{nn}(x')(\kappa_+ + i\xi_n)(\kappa_- - i\xi_n), \quad (3.5)$$

$$\underline{a}^0(x', 0, \xi', D_n) = \underline{a}_{nn}(x')(\kappa_+ + \partial_n)(\kappa_- - \partial_n). \quad (3.6)$$

Let $K_{\gamma}^{\pm} : H^{3/2}(\Sigma) \rightarrow H^2(\Omega_{\pm})$ be the Poisson solution operators that map a $\varphi \in H^{3/2}(\Omega_{\pm})$ onto the solutions $u_{\pm} \in H^2(\Omega_{\pm})$ of the boundary value problems

$$\mathcal{A}_{\pm} u_{\pm} = 0, \quad \gamma^{\pm} u_{\pm} = \varphi. \quad (3.7)$$

Similarly, let $K_{\nu}^{\pm} : H^{1/2}(\Sigma) \rightarrow H^2(\Omega_{\pm})$ be the Poisson solution operators corresponding to the Neumann problems

$$\mathcal{A}_{\pm} u_{\pm} = 0, \quad \nu^{\pm} u_{\pm} = \psi. \quad (3.8)$$

Moreover, we define the Dirichlet-to-Neumann and Neumann-to-Dirichlet operators by

$$P_{\gamma, \nu}^{\pm} := \nu^{\pm} K_{\gamma}^{\pm}, \quad P_{\nu, \gamma}^{\pm} := \gamma^{\pm} K_{\nu}^{\pm}. \quad (3.9)$$

In the next lemma we collect properties of these operators, which are needed in the proofs of our main results.

Lemma 3.2. *Let the operators K_{γ}^{\pm} , K_{ν}^{\pm} , $P_{\gamma, \nu}^{\pm}$ and $P_{\nu, \gamma}^{\pm}$ be as above. Then the following statements hold.*

- (i) *The operators K_{γ}^+ and K_{ν}^+ are Poisson operators of orders 0 and -1 , respectively. Their principal symbol-kernels are, in local coordinates,*

$$\tilde{k}_{\gamma}^{+0}(x', x_n, \xi') = e^{-\kappa_+(x', \xi')x_n}, \quad (3.10)$$

$$\tilde{k}_{\nu}^{+0}(x', x_n, \xi') = \frac{1}{\kappa_0(x', \xi')} e^{-\kappa_+(x', \xi')x_n}. \quad (3.11)$$

- (ii) *The operators $P_{\gamma, \nu}^+$ and $P_{\nu, \gamma}^+$ are pseudodifferential operators of orders 1 and -1 , respectively. Their principal symbols are*

$$p_{\gamma, \nu}^{+0}(x', \xi') = \kappa_0(x', \xi') \quad \text{and} \quad p_{\nu, \gamma}^{+0}(x', \xi') = \frac{1}{\kappa_0(x', \xi')}, \quad (3.12)$$

which are positive.

- (iii) The compositions $(K_\gamma^+)^* K_\gamma^+$ and $(K_\nu^+)^* K_\nu^+$ are pseudodifferential operators of orders -1 and -3 , respectively. Their principal symbols are

$$\frac{\underline{a}_{nn}(x')}{2\kappa_0(x', \xi')} \quad \text{and} \quad \frac{\underline{a}_{nn}(x')}{2(\kappa_0(x', \xi'))^3}.$$

For \mathcal{A} on Ω_- the formulae hold with κ_- instead of κ_+ .

Proof. (i) To find the principal symbol-kernel of K_γ^+ , we have to solve the following model problem for each (x', ξ') with $\xi' \neq 0$ on the one-dimensional level:

$$\underline{a}^0(x', 0, \xi', D_n)u(x_n) = 0 \quad \text{on } \mathbb{R}_+, \quad u(0) = \varphi \in \mathbb{C}, \quad (3.13)$$

where \underline{a}^0 is the principal symbol of \mathcal{A} in local coordinates. It follows from (3.6) and the inequality $\text{Re } \kappa_+ > 0$ that the $L_2(\mathbb{R}_+)$ -solution of (3.13) is

$$u(x_n) = \varphi e^{-\kappa_+ x_n}.$$

Hence $\tilde{k}_\gamma^{+0} = e^{-\kappa_+ x_n}$ is the principal symbol-kernel of the Poisson operator K_γ^+ . This shows (3.10). In view of (3.1), it has order 0.

Let us now consider the Neumann problem. In local coordinates the conormal derivative takes the form

$$\underline{\nu}^+ u = -\underline{a}_{nn} \partial_{x_n} u|_{x_n=0} - \sum_{k=1}^{n-1} \underline{a}_{nk} i \xi_k u(0);$$

cf. (2.2) and observe that the outward normal is $(0, \dots, 0, -1)$. Since

$$\underline{\nu}^+ e^{-\kappa_+ x_n} = -\underline{a}_{nn}(-\kappa_+) - \sum_{k=1}^{n-1} \underline{a}_{nk} i \xi_k = \underline{a}_{nn} \frac{\kappa_0 + ib}{\underline{a}_{nn}} - ib = \kappa_0, \quad (3.14)$$

the solution of $\underline{a}^0(x', 0, \xi', D_n)u(x_n) = 0$, $\underline{\nu}^+ u = \psi$ is

$$u(x_n) = \psi \frac{1}{\kappa_0} e^{-\kappa_+ x_n},$$

which yields (3.11). Moreover, \tilde{k}_ν^{+0} has order -1 .

(ii) It follows from (3.14) that the Dirichlet-to-Neumann operator $P_{\gamma, \nu}^+$ has the principal symbol $p_{\gamma, \nu}^{+0} = \underline{\nu}^+ \tilde{k}_\gamma^{+0} = \kappa_0$, which is of order 1 since κ_0 is homogeneous of degree 1 in ξ' . The principal symbol of $P_{\nu, \gamma}^+$ is $p_{\nu, \gamma}^{+0} = \tilde{k}_\nu^{+0}|_{x_n=0} = 1/\kappa_0$.

(iii) The adjoint of K_γ^+ has the principal part in local coordinates acting like

$$u(x) \mapsto \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_0^\infty e^{-\bar{\kappa}_+ x_n} \mathcal{F}_{x' \rightarrow \xi'} u(x', x_n) dx_n d\xi',$$

with symbol-kernel \bar{k}_γ^{+0} and order -1 ; for the latter see, e.g. [31, Theorem 10.29 and Remark 10.6]. Then the composition of the principal part of $(K_\gamma^+)^*$ with the principal part of K_γ^+ is the ψ do on \mathbb{R}^{n-1} with symbol

$$\int_0^\infty e^{-\bar{\kappa}_+ x_n} e^{-\kappa_+ x_n} dx_n = \frac{1}{\bar{\kappa}_+ + \kappa_+} = \frac{\underline{a}_{nn}}{2\kappa_0}$$

and order -1 by [31, Proposition 10.10 (v)].

For K_ν^+ , the principal part of the adjoint acts like

$$u(x) \mapsto \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_0^\infty \frac{1}{\kappa_0} e^{-\bar{\kappa}_+ x_n} \mathcal{F}_{x' \rightarrow \xi'} u(x', x_n) dx_n d\xi',$$

so we find that $(K_\nu^+)^* K_\nu^+$ has the principal symbol

$$\int_0^\infty \frac{1}{\kappa_0} e^{-\bar{\kappa}_+ x_n} \frac{1}{\kappa_0} e^{-\kappa_+ x_n} dx_n = \frac{1}{\kappa_0^2} \cdot \frac{1}{\bar{\kappa}_+ + \kappa_+} = \frac{a_{nn}}{2\kappa_0^3}.$$

For invariance with respect to coordinate changes see, e.g. [31, Theorem 8.1] for ψ do's and [30, Theorem 2.4.11] for ψ dbo's.

For the same operator \mathcal{A} considered on Ω_- , the symbol in local coordinates at a boundary point is as above with the direction of x_n , and hence also of ξ_n , reverted. Then κ_+ and κ_- exchange roles. \square

The following lemma collects some properties of the operators K_γ^\pm and K_ν^\pm considered as operators between L_2 -spaces.

Lemma 3.3. *Let K_γ^\pm and K_ν^\pm be as above. Then the L_2 -adjoints of K_γ^\pm and K_ν^\pm satisfy*

$$(K_\gamma^\pm)^* = -\nu^\pm A_{\pm, \gamma}^{-1}, \quad (K_\nu^\pm)^* = \gamma^\pm A_{\pm, \nu}^{-1}. \quad (3.15)$$

Proof. Let $\varphi \in H^{3/2}(\Sigma)$ and set $u_+ := K_\gamma^+ \varphi$, which satisfies (3.7). Moreover, let $f \in L_2(\Omega_+)$ and set $v_+ := A_{+, \gamma}^{-1} f$. Then Green's identity implies

$$\begin{aligned} (f, K_\gamma^+ \varphi) &= (\mathcal{A}_+ v_+, u_+) = (\mathcal{A}_+ v_+, u_+) - (v_+, \mathcal{A}_+ u_+) \\ &= (\gamma^+ v_+, \nu^+ u_+) - (\nu^+ v_+, \gamma^+ u_+) = -(\nu^+ A_{+, \gamma}^{-1} f, \varphi) \end{aligned}$$

which yields the relation for K_γ^+ in (3.15) since $\nu^+ A_{+, \gamma}^{-1}$ is bounded. The other relations in (3.15) are shown in a similar way. \square

Example 3.4. In the special, important case $\mathcal{A} = -\Delta$ we derive from (3.2) and (3.3) that

$$a_{nn}(x') \equiv 1, \quad b(x', \xi') \equiv 0, \quad c(x', \xi') = |\xi'|^2;$$

note that the principal symbol at the boundary point is unchanged under the transformation to local coordinates. Hence, following (3.4) we get

$$\kappa_\pm(x', \xi') = \kappa_0(x', \xi') = |\xi'|.$$

Thus, according to Lemma 3.2 the principal symbol-kernels of K_γ^\pm , K_ν^\pm are

$$\tilde{k}_\gamma^{\pm 0} = e^{-|\xi'| x_n} \quad \text{and} \quad \tilde{k}_\nu^{\pm 0} = \frac{e^{-|\xi'| x_n}}{|\xi'|};$$

the principal symbols of $P_{\gamma, \nu}^\pm$, $P_{\nu, \gamma}^\pm$ are

$$p_{\gamma, \nu}^{\pm 0} = |\xi'| \quad \text{and} \quad p_{\nu, \gamma}^{\pm 0} = \frac{1}{|\xi'|};$$

and the principal symbols of $(K_\gamma^\pm)^* K_\gamma^\pm$, $(K_\nu^\pm)^* K_\nu^\pm$ are

$$\frac{1}{2|\xi'|} \quad \text{and} \quad \frac{1}{2|\xi'|^3}.$$

4. KREIN-TYPE FORMULAE

In this section we provide Krein-type formulae for differences between inverses of self-adjoint realizations of \mathcal{A} defined in Section 2. First we derive Krein-type formulae in a general setting, and then we simplify these formulae for particular δ and δ' -couplings. Similar formulae for systems acting on a single bounded domain can be found in the paper [26]. For coupled problems the reader may also consult [6, Section 4] and [7, Section 3].

Let us define the trace mapping

$$\varrho: H^2(\Omega_+) \oplus H^2(\Omega_-) \rightarrow H^{3/2}(\Sigma) \times H^{3/2}(\Sigma) \times H^{1/2}(\Sigma) \times H^{1/2}(\Sigma) \quad (4.1)$$

by

$$\varrho(u_+ \oplus u_-) := \begin{pmatrix} \gamma^+ u_+ \\ \gamma^- u_- \\ \nu^+ u_+ \\ \nu^- u_- \end{pmatrix}, \quad (4.2)$$

where γ^\pm are the traces on Σ from the two sides of it, and ν^\pm are the outward conormal derivatives; cf. Section 2. Note that the map ϱ is surjective by the classical trace theorem; see, e.g. [41].

Let in the following $\alpha, \beta \in C^\infty(\Sigma)$ be real-valued with β non-vanishing on Σ . We consider realizations of \mathcal{A} in $L_2(\mathbb{R}^n)$ defined by

$$\begin{aligned} A_* u &:= \mathcal{A}_+ u_+ \oplus \mathcal{A}_- u_-, \\ \text{dom } A_* &:= \{u \in H^2(\Omega_+) \oplus H^2(\Omega_-): B_* \varrho u = 0\}, \end{aligned} \quad (4.3)$$

where B_* is one of the following matrices

$$\begin{aligned} B_0 &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad B_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_\nu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ B_{\delta, \alpha} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha & 0 & 1 & 1 \end{pmatrix}, \quad B_{\delta', \beta} = \begin{pmatrix} 1 & -1 & -\beta & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.4)$$

The self-adjoint operators $A_0, A_\gamma, A_\nu, A_{\delta, \alpha}$ and $A_{\delta', \beta}$ defined in (2.3), (2.4), (2.5) and (2.6) correspond, respectively, to $B_0, B_\gamma, B_\nu, B_{\delta, \alpha}$ and $B_{\delta', \beta}$, in the sense of (4.3). The boundary conditions that are induced by the matrices $B_0, B_\gamma, B_\nu, B_{\delta, \alpha}$ and $B_{\delta', \beta}$ are analogues of what are called *normal boundary conditions* in [26]. A typical property of such boundary conditions is given in the lemma below, which follows from the fact that ϱ defined in (4.1), (4.2) is surjective.

Lemma 4.1. *With ϱ defined by (4.1) and (4.2), let B_* be one of the matrices in (4.4) and let us define*

$$R_* := H^s(\Sigma) \times H^t(\Sigma) \quad (4.5)$$

with

$$\begin{aligned} s = 3/2, t = 1/2 & \quad \text{if } B_* = B_0, B_{\delta, \alpha}, \text{ or } B_{\delta', \beta}, \\ s = t = 3/2 & \quad \text{if } B_* = B_\gamma, \\ s = t = 1/2 & \quad \text{if } B_* = B_\nu. \end{aligned}$$

Then $B_* \varrho$ is surjective from $H^2(\Omega_+) \oplus H^2(\Omega_-)$ onto R_* .

As discussed at the end of Section 2 we can assume without loss of generality that the operators $A_0, A_\gamma, A_\nu, A_{\delta,\alpha}$ and $A_{\delta',\beta}$ are positive with 0 in the resolvent set. Then the semi-homogeneous Dirichlet and Neumann boundary value problems (3.7) and (3.8) for the differential expression \mathcal{A}_\pm on Ω_\pm are uniquely solvable in $H^2(\Omega_\pm)$, and K_γ^\pm and K_ν^\pm are the corresponding solution operators. It follows that for all $\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \in H^{3/2}(\Sigma) \times H^{3/2}(\Sigma)$ the problem

$$\mathcal{A}_+u_+ \oplus \mathcal{A}_-u_- = 0, \quad B_\gamma \varrho u = \varphi,$$

with B_γ as in (4.4), has a unique solution $u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-)$. The corresponding solution operator is

$$K_\gamma = \begin{pmatrix} K_\gamma^+ & 0 \\ 0 & K_\gamma^- \end{pmatrix}, \quad (4.6)$$

i.e. $u = K_\gamma \varphi$. Note that for $u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-)$ one has

$$\mathcal{A}_+u_+ \oplus \mathcal{A}_-u_- = 0 \quad \Rightarrow \quad u = K_\gamma B_\gamma \varrho u. \quad (4.7)$$

Similarly, for all $\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \in H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ the problem

$$\mathcal{A}_+u_+ \oplus \mathcal{A}_-u_- = 0, \quad B_\nu \varrho u = \psi,$$

with B_ν as in (4.4), has a unique solution $u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-)$; the solution operator is given by

$$K_\nu = \begin{pmatrix} K_\nu^+ & 0 \\ 0 & K_\nu^- \end{pmatrix}. \quad (4.8)$$

In the next proposition we investigate the solvability of the boundary value problems associated with the matrices B_* in (4.4) and derive related properties of the 2×2 matrix ψ do's

$$\Phi_* := B_* \varrho K_\gamma \quad \text{and} \quad \Psi_* := B_* \varrho K_\nu. \quad (4.9)$$

Proposition 4.2. *Let B_* be one of the matrices in (4.4) with the associated space R_* as in (4.5), and let the matrix ψ do's Φ_* and Ψ_* be defined by (4.9). Then the following statements hold.*

(i) *For all $\psi \in R_*$ the boundary value problem*

$$\mathcal{A}_+u_+ \oplus \mathcal{A}_-u_- = 0, \quad B_* \varrho u = \psi, \quad (4.10)$$

has a unique solution $u = u_+ \oplus u_- \in H^2(\Omega_+) \oplus H^2(\Omega_-)$.

(ii) *The matrix ψ do Φ_* is bijective from $H^{3/2}(\Sigma) \times H^{3/2}(\Sigma)$ onto R_* .*

(iii) *The matrix ψ do Ψ_* is bijective from $H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ onto R_* .*

Proof. In the following, A_* is the self-adjoint operator in $L_2(\mathbb{R}^n)$ corresponding to the matrix B_* in the sense of (4.3). By our assumptions A_* is strictly positive. This implies that the semi-homogeneous boundary value problem

$$\mathcal{A}_+w_+ \oplus \mathcal{A}_-w_- = f, \quad B_* \varrho w = 0, \quad (4.11)$$

is uniquely solvable for all $f \in L_2(\mathbb{R}^n)$, and the unique solution is given by $w := A_*^{-1}f \in H^2(\Omega_+) \oplus H^2(\Omega_-)$.

(i) Let $\psi \in R_*$ and choose $v \in H^2(\Omega_+) \oplus H^2(\Omega_-)$ such that $B_* \varrho v = \psi$, which is possible by Lemma 4.1. The boundary value problem

$$\mathcal{A}_+z_+ \oplus \mathcal{A}_-z_- = -\mathcal{A}_+v_+ \oplus \mathcal{A}_-v_-, \quad B_* \varrho z = 0,$$

has a unique solution $z \in H^2(\Omega_+) \oplus H^2(\Omega_-)$; cf. (4.11). It follows that

$$u := z + v \in H^2(\Omega_+) \oplus H^2(\Omega_-)$$

is a solution of (4.10). Moreover, this solution is unique. In fact, suppose that $\tilde{u} \in H^2(\Omega_+) \oplus H^2(\Omega_-)$ is also a solution of (4.10). Then $u - \tilde{u} \in \text{dom } A_*$ and $A_*(u - \tilde{u}) = 0$. As A_* is strictly positive we conclude that $u = \tilde{u}$.

(ii) First we verify that Φ_* is injective. Suppose that $\Phi_*\varphi = B_*\varrho K_\gamma\varphi = 0$ holds for some $\varphi \in H^{3/2}(\Sigma) \times H^{3/2}(\Sigma)$, $\varphi \neq 0$. Then the function $K_\gamma\varphi \neq 0$ belongs to $\text{dom } A_*$ and satisfies $A_*K_\gamma\varphi = 0$, a contradiction to the strict positivity of A_* . In order to show that Φ_* is surjective, let $\psi \in R_*$. By item (i) the boundary value problem

$$\mathcal{A}_+u_+ \oplus \mathcal{A}_-u_- = 0, \quad B_*\varrho u = \psi,$$

has a unique solution $u \in H^2(\Omega_+) \oplus H^2(\Omega_-)$. Define

$$\varphi := B_\gamma\varrho u \in H^{3/2}(\Sigma) \times H^{3/2}(\Sigma).$$

Note that by item (i) the boundary value problem

$$\mathcal{A}_+v_+ \oplus \mathcal{A}_-v_- = 0, \quad B_\gamma\varrho v = \varphi,$$

has a unique solution in $H^2(\Omega_+) \oplus H^2(\Omega_-)$. Observe that $K_\gamma\varphi$ and u both are solutions of the above problem. Hence $u = K_\gamma\varphi$ by the uniqueness. From

$$\Phi_*\varphi = B_*\varrho K_\gamma\varphi = B_*\varrho u = \psi$$

we conclude that $\psi \in \text{ran } \Phi_*$. It follows that Φ_* is surjective onto R_* .

(iii) The proof of this item is analogous to the proof of (ii). \square

We mention that the matrix pseudo's Φ_* and Ψ_* in (4.9) are elliptic. This is essentially a consequence of the bijectivity shown in Proposition 4.2 above.

In the next theorem we give Krein-type factorizations for differences of inverses between either A_γ or A_ν and one of the operators A_0 , A_ν , $A_{\delta,\alpha}$ and $A_{\delta',\beta}$.

Theorem 4.3. *Let B_* be one of the matrices in (4.4), let A_* , A_γ and A_ν be the self-adjoint operators in $L_2(\mathbb{R}^n)$ corresponding to the matrices B_* , B_γ and B_ν as in (4.3), and let ϱ be the trace map in (4.1), (4.2). Then the following statements are true.*

(i) *The formula*

$$A_*^{-1} - A_\gamma^{-1} = -K_\gamma\Phi_*^{-1}B_*\varrho A_\gamma^{-1}$$

holds, where K_γ and Φ_ are as in (4.6) and (4.9), respectively.*

(ii) *The formula*

$$A_*^{-1} - A_\nu^{-1} = -K_\nu\Psi_*^{-1}B_*\varrho A_\nu^{-1}$$

holds, where K_ν and Ψ_ are as in (4.8) and (4.9), respectively.*

Proof. (i) Recall that, by our assumptions, A_* and A_γ are strictly positive. Let $f \in L_2(\mathbb{R}^n)$ and note that $u := A_*^{-1}f$ is the unique solution of the semi-homogeneous boundary value problem

$$\mathcal{A}_+u_+ \oplus \mathcal{A}_-u_- = f, \quad B_*\varrho u = 0,$$

in $H^2(\Omega_+) \oplus H^2(\Omega_-)$. Define the functions

$$v := A_\gamma^{-1}f \quad \text{and} \quad z := u - v. \quad (4.12)$$

The latter satisfies

$$\mathcal{A}_+ z_+ \oplus \mathcal{A}_- z_- = 0, \quad B_* \varrho z = -B_* \varrho v, \quad (4.13)$$

which, by (4.7), implies that

$$z = K_\gamma B_\gamma \varrho z. \quad (4.14)$$

Now it follows from (4.14), (4.9), (4.13) and (4.12) that

$$\begin{aligned} z &= K_\gamma B_\gamma \varrho z = K_\gamma \Phi_*^{-1} B_* \varrho K_\gamma B_\gamma \varrho z = K_\gamma \Phi_*^{-1} B_* \varrho z \\ &= -K_\gamma \Phi_*^{-1} B_* \varrho v = -K_\gamma \Phi_*^{-1} B_* \varrho A_\gamma^{-1} f. \end{aligned}$$

This, together with $A_*^{-1} f = u = v + z$ yields the formula in (i).

Item (ii) can be proved in the same way as item (i) when A_γ , B_γ , K_γ and Φ_* are replaced by A_ν , B_ν , K_ν and Ψ_* , respectively. \square

In the next proposition we simplify the formula from Theorem 4.3 (i) for the difference between the inverses of the self-adjoint operators $A_{\delta, \alpha}$ and A_γ . In the formulation and the proof of this proposition we employ the column operator

$$\tilde{K}_\gamma := \begin{pmatrix} K_\gamma^+ \\ K_\gamma^- \end{pmatrix} : H^{3/2}(\Sigma) \rightarrow H^2(\Omega_+) \times H^2(\Omega_-). \quad (4.15)$$

Proposition 4.4. *Let $A_{\delta, \alpha}$ and A_γ be as above, and let $P_{\gamma, \nu}^\pm$ be the Dirichlet-to-Neumann maps in (3.9) with principal symbols $p_{\gamma, \nu}^{\pm 0}$ in (3.12). Then the following statements hold.*

- (i) *The ψ do $P_{\gamma, \nu}^+ + P_{\gamma, \nu}^- - \alpha$ is elliptic of order 1 with principal symbol $p_{\gamma, \nu}^{+0} + p_{\gamma, \nu}^{-0} = 2\kappa_0$, and it maps $H^{3/2}(\Sigma)$ bijectively onto $H^{1/2}(\Sigma)$.*
- (ii) *The formula*

$$A_{\delta, \alpha}^{-1} - A_\gamma^{-1} = \tilde{K}_\gamma (P_{\gamma, \nu}^+ + P_{\gamma, \nu}^- - \alpha)^{-1} \tilde{K}_\gamma^*$$

holds, where \tilde{K}_γ is as in (4.15) and \tilde{K}_γ^ is the L_2 -adjoint of \tilde{K}_γ .*

Proof. (i) Both operators $P_{\gamma, \nu}^+$ and $P_{\gamma, \nu}^-$ are symmetric first-order elliptic ψ do's on Σ (see Lemmas 3.2 (ii) and 3.3) with principal symbols $p_{\gamma, \nu}^{\pm 0}$ as in (3.12). Hence the ψ do

$$P_{\gamma, \nu}^+ + P_{\gamma, \nu}^- - \alpha \quad (4.16)$$

is also symmetric of order 1 with principal symbol $p_{\gamma, \nu}^{+0} + p_{\gamma, \nu}^{-0} = 2\kappa_0$. According to Lemma 3.2 (ii) we have $p_{\gamma, \nu}^{\pm 0} > 0$ for $\xi' \neq 0$, and hence (4.16) is elliptic. By [31, Theorem 8.11] the index of the ψ do (4.16) as a mapping from $H^{3/2}(\Sigma)$ to $H^{1/2}(\Sigma)$ is 0. Hence, in order to prove bijectivity of (4.16) from $H^{3/2}(\Sigma)$ onto $H^{1/2}(\Sigma)$, it remains to show that $\ker(P_{\gamma, \nu}^+ + P_{\gamma, \nu}^- - \alpha)$ is trivial. Suppose for a moment that this were not the case. Then, by [31, Theorem 8.11], there exists a non-trivial $\varphi \in C^\infty(\Sigma)$ such that

$$(P_{\gamma, \nu}^+ + P_{\gamma, \nu}^- - \alpha)\varphi = 0.$$

Let us consider the non-trivial function $u := \tilde{K}_\gamma \varphi$. Then we have $u \in H^2(\Omega_+) \oplus H^2(\Omega_-)$, $\mathcal{A}_+ u_+ \oplus \mathcal{A}_- u_- = 0$, and

$$\begin{aligned} \gamma^+ u_+ - \gamma^- u_- &= 0, \\ \nu^+ u_+ + \nu^- u_- - \alpha \gamma u &= \nu^+ K_\gamma^+ \varphi + \nu^- K_\gamma^- \varphi - \alpha \varphi \\ &= P_{\gamma,\nu}^+ \varphi + P_{\gamma,\nu}^- \varphi - \alpha \varphi = 0. \end{aligned}$$

From (2.5) we conclude that $u \in \text{dom } A_{\delta,\alpha}$ and, moreover, $A_{\delta,\alpha} u = 0$, which contradicts the invertibility of $A_{\delta,\alpha}$.

(ii) Let us consider the matrix ψ do $\Phi_{\delta,\alpha} = B_{\delta,\alpha} \varrho K_\gamma$ (cf. (4.9)). For $\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \in H^{3/2}(\Sigma) \times H^{3/2}(\Sigma)$ we have

$$\begin{aligned} \Phi_{\delta,\alpha} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha & 0 & 1 & 1 \end{pmatrix} \varrho K_\gamma \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \\ P_{\gamma,\nu}^+ \varphi_+ \\ P_{\gamma,\nu}^- \varphi_- \end{pmatrix} \\ &= \begin{pmatrix} \varphi_+ - \varphi_- \\ -\alpha \varphi_+ + P_{\gamma,\nu}^+ \varphi_+ + P_{\gamma,\nu}^- \varphi_- \end{pmatrix}, \end{aligned}$$

and hence $\Phi_{\delta,\alpha}$ can be written in matrix form as

$$\Phi_{\delta,\alpha} = \begin{pmatrix} 1 & -1 \\ -\alpha + P_{\gamma,\nu}^+ & P_{\gamma,\nu}^- \end{pmatrix}.$$

By item (i) the operator $P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha$ is bijective from $H^{3/2}(\Sigma)$ onto $H^{1/2}(\Sigma)$. It follows that the matrix operator

$$(P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \begin{pmatrix} P_{\gamma,\nu}^- & 1 \\ \alpha - P_{\gamma,\nu}^+ & 1 \end{pmatrix} \quad (4.17)$$

is well defined as a mapping from $H^{3/2}(\Sigma) \times H^{1/2}(\Sigma)$ into $H^{3/2}(\Sigma) \times H^{3/2}(\Sigma)$ and that it is the inverse of $\Phi_{\delta,\alpha}$. Indeed, we have

$$\begin{aligned} &(P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \begin{pmatrix} P_{\gamma,\nu}^- & 1 \\ \alpha - P_{\gamma,\nu}^+ & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -\alpha + P_{\gamma,\nu}^+ & P_{\gamma,\nu}^- \end{pmatrix} \\ &= (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \begin{pmatrix} P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha & 0 \\ 0 & P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

on $H^{3/2}(\Sigma) \times H^{3/2}(\Sigma)$.

For $f = f_+ \oplus f_- \in L_2(\mathbb{R}^n)$ we obtain from Theorem 4.3, (3.15), (4.17) and (4.4) that

$$\begin{aligned} A_{\delta,\alpha}^{-1} f - A_\gamma^{-1} f &= - \begin{pmatrix} K_\gamma^+ & 0 \\ 0 & K_\gamma^- \end{pmatrix} \Phi_{\delta,\alpha}^{-1} B_{\delta,\alpha} \varrho A_\gamma^{-1} f \\ &= - \begin{pmatrix} K_\gamma^+ & 0 \\ 0 & K_\gamma^- \end{pmatrix} \Phi_{\delta,\alpha}^{-1} B_{\delta,\alpha} \begin{pmatrix} 0 \\ 0 \\ \nu^+ A_{+,\gamma}^{-1} f_+ \\ \nu^- A_{-,\gamma}^{-1} f_- \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} K_\gamma^+ & 0 \\ 0 & K_\gamma^- \end{pmatrix} \Phi_{\delta,\alpha}^{-1} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ (K_\gamma^+)^* f_+ \\ (K_\gamma^-)^* f_- \end{pmatrix} \\
&= \begin{pmatrix} K_\gamma^+ & 0 \\ 0 & K_\gamma^- \end{pmatrix} \Phi_{\delta,\alpha}^{-1} \begin{pmatrix} 0 \\ \tilde{K}_\gamma^* f \end{pmatrix} \\
&= \begin{pmatrix} K_\gamma^+ (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \tilde{K}_\gamma^* f \\ K_\gamma^- (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \tilde{K}_\gamma^* f \end{pmatrix} \\
&= \tilde{K}_\gamma (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \tilde{K}_\gamma^* f,
\end{aligned}$$

which proves item (ii). \square

Note that the operator $A_{\delta,\alpha}$ with $\alpha \equiv 0$ coincides with the operator A_0 ; cf. Section 2. This observation, together with Proposition 4.4 and the relation

$$\begin{aligned}
&(P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} - (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^-)^{-1} \\
&= (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \alpha (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^-)^{-1},
\end{aligned} \tag{4.18}$$

yields the following corollary.

Corollary 4.5. *Under the assumptions of Proposition 4.4 and with A_0 as in (2.3),*

$$A_{\delta,\alpha}^{-1} - A_0^{-1} = \tilde{K}_\gamma (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \alpha (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^-)^{-1} \tilde{K}_\gamma^*$$

holds.

In the next proposition we simplify the formula from Theorem 4.3 (ii) for the difference between the inverses of the self-adjoint operators $A_{\delta',\beta}$ and A_ν . The proof follows the same strategy as the proof of Proposition 4.4. For the convenience of the reader we provide the essential arguments. We also mention that the formula in item (ii) below is similar to the one in [7, Theorem 3.11 (ii)]. Here we employ the column operator

$$\tilde{K}_\nu := \begin{pmatrix} K_\nu^+ \\ -K_\nu^- \end{pmatrix} : H^{1/2}(\Sigma) \rightarrow H^2(\Omega_+) \times H^2(\Omega_-). \tag{4.19}$$

Proposition 4.6. *Let A_ν and $A_{\delta',\beta}$ be as defined above, and let $P_{\nu,\gamma}^\pm$ be the Neumann-to-Dirichlet maps in (3.9). Then the following statements hold.*

- (i) *The ψ do $\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)$ is elliptic of order 0 with principal symbol β , and it maps $H^{1/2}(\Sigma)$ bijectively onto $H^{1/2}(\Sigma)$.*
- (ii) *The formula*

$$A_{\delta',\beta}^{-1} - A_\nu^{-1} = \tilde{K}_\nu (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1} \tilde{K}_\nu^*$$

holds, where \tilde{K}_ν is as in (4.19) and \tilde{K}_ν^* is the L_2 -adjoint of \tilde{K}_ν .

Proof. (i) The operators $P_{\nu,\gamma}^\pm$ are symmetric elliptic ψ do's on Σ of order -1 (see Lemmas 3.2 (ii) and 3.3). Since β is real-valued and non-zero on Σ , $\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)$ is a symmetric and elliptic ψ do of order 0 with principal symbol β . Hence, by [31, Theorem 8.11], its index as a mapping from $H^{1/2}(\Sigma)$ into $H^{1/2}(\Sigma)$ is 0. Therefore it suffices to verify the injectivity of

$\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)$. Suppose that this were not the case. As in the proof of Proposition 4.4 (i) it follows that there exists a non-trivial $\psi \in C^\infty(\Sigma)$ such that $(\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))\psi = 0$ and the function $\tilde{K}_\nu\psi \neq 0$ belongs to $\ker A_{\delta',\beta}$; this is a contradiction to the invertibility of $A_{\delta',\beta}$.

(ii) A simple calculation shows that the matrix ψ do $\Psi_{\delta',\beta} = B_{\delta',\beta}\varrho K_\nu$ acts as

$$\Psi_{\delta',\beta} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 1 & -1 & -\beta & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \varrho K_\nu \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} P_{\nu,\gamma}^+ - \beta & -P_{\nu,\gamma}^- \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

and a similar consideration as in the proof of Proposition 4.4 (ii) yields

$$\Psi_{\delta',\beta}^{-1} = (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^- - \beta)^{-1} \begin{pmatrix} 1 & P_{\nu,\gamma}^- \\ -1 & P_{\nu,\gamma}^+ - \beta \end{pmatrix}. \quad (4.20)$$

It is seen from the form of $B_{\delta',\beta}$ in (4.4), relations (3.15), (4.20) and Theorem 4.3 (ii) that

$$\begin{aligned} A_{\delta',\beta}^{-1}f - A_\nu^{-1}f &= - \begin{pmatrix} K_\nu^+ & 0 \\ 0 & K_\nu^- \end{pmatrix} \Psi_{\delta',\beta}^{-1} B_{\delta',\beta} \begin{pmatrix} (K_\nu^+)^* f_+ \\ (K_\nu^-)^* f_- \\ 0 \\ 0 \end{pmatrix} \\ &= - \begin{pmatrix} K_\nu^+ & 0 \\ 0 & K_\nu^- \end{pmatrix} (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^- - \beta)^{-1} \begin{pmatrix} (K_\nu^+)^* f_+ - (K_\nu^-)^* f_- \\ (K_\nu^-)^* f_- - (K_\nu^+)^* f_+ \end{pmatrix} \\ &= \tilde{K}_\nu (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1} \tilde{K}_\nu^* f \end{aligned}$$

holds for all $f \in L_2(\mathbb{R}^n)$, which proves (ii). \square

Finally we provide a more explicit formula for the difference between the inverses of the self-adjoint operators A_0 and A_ν . Again the proof follows the same strategy as the proofs of Propositions 4.4 and 4.6.

Proposition 4.7. *Let A_0 and A_ν be as above, and let $P_{\nu,\gamma}^\pm$ be the Neumann-to-Dirichlet maps $P_{\nu,\gamma}^\pm$ in (3.9) with principal symbols $p_{\nu,\gamma}^{\pm 0}$ in (3.12). Then the following statements hold.*

(i) *The ψ do $P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-$ is elliptic of order -1 with principal symbol $p_{\nu,\gamma}^{+0} + p_{\nu,\gamma}^{-0}$, and it maps $H^{1/2}(\Sigma)$ bijectively onto $H^{3/2}(\Sigma)$.*

(ii) *The formula*

$$A_\nu^{-1} - A_0^{-1} = \tilde{K}_\nu (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \tilde{K}_\nu^*$$

holds, where \tilde{K}_ν is as in (4.19).

Proof. (i) Following the arguments in the proofs of Proposition 4.4 and Proposition 4.6 we conclude that the ψ do $P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-$ is elliptic of order -1 with principal symbol $p_{\nu,\gamma}^{+0} + p_{\nu,\gamma}^{-0}$, and its index as a mapping from $H^{1/2}(\Sigma)$ into $H^{3/2}(\Sigma)$ is 0. Again it is sufficient for the bijectivity to verify that $\ker(P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)$ is trivial. Suppose that this were not the case. Then it follows that there exists a non-trivial $\psi \in C^\infty(\Sigma)$ such that $(P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)\psi = 0$ and the function $\tilde{K}_\nu\psi \neq 0$ belongs to $\ker A_0$, a contradiction to the invertibility of A_0 .

(ii) The matrix ψ do $\Psi_0 = B_0 \varrho K_\nu$ and its inverse have the form

$$\Psi_0 = \begin{pmatrix} P_{\nu,\gamma}^+ & -P_{\nu,\gamma}^- \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Psi_0^{-1} = (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \begin{pmatrix} 1 & P_{\nu,\gamma}^- \\ -1 & P_{\nu,\gamma}^+ \end{pmatrix}.$$

Hence it follows from (3.15), the form of B_0 in (4.4) and Theorem 4.3 (ii) that

$$\begin{aligned} A_\nu^{-1} f - A_0^{-1} f &= \begin{pmatrix} K_\nu^+ & 0 \\ 0 & K_\nu^- \end{pmatrix} \Psi_0^{-1} B_0 \begin{pmatrix} (K_\nu^+)^* f_+ \\ (K_\nu^-)^* f_- \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} K_\nu^+ & 0 \\ 0 & K_\nu^- \end{pmatrix} (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \begin{pmatrix} (K_\nu^+)^* f_+ - (K_\nu^-)^* f_- \\ (K_\nu^-)^* f_- - (K_\nu^+)^* f_+ \end{pmatrix} \\ &= \tilde{K}_\nu (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \tilde{K}_\nu^* f \end{aligned}$$

holds for all $f \in L_2(\mathbb{R}^n)$, which proves (ii). \square

Corollary 4.8. *Under the assumptions of Proposition 4.7 and with $A_{\delta',\beta}$ as in (2.6),*

$$A_{\delta',\beta}^{-1} - A_0^{-1} = \tilde{K}_\nu (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \beta (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1} \tilde{K}_\nu^*$$

holds.

5. SPECTRAL ASYMPTOTICS FOR RESOLVENT DIFFERENCES

In this section we present and prove the main results of this note, namely, we obtain spectral asymptotics formulae for the differences between the inverses of the operators A_0 , A_ν , $A_{\delta,\alpha}$ and $A_{\delta',\beta}$ introduced in Section 2. These asymptotics refine some spectral estimates for resolvent differences found in [6, 7]. The proofs are based on the Krein-type resolvent formulae proved in Section 4, spectral asymptotics for ψ do's on smooth manifolds without boundary and some elements of the ψ dbo calculus.

In all theorems of this section we suppose that the assumptions at the beginning of Section 2 hold. Moreover, let \underline{a}_{nn} and κ_0 be defined as in (3.2) and (3.4). The next theorem contains one of our main results: the spectral asymptotics of the difference between the inverses of $A_{\delta,\alpha}$ and A_0 .

Theorem 5.1. *Let $\alpha \in C^\infty(\Sigma)$ be real-valued, let $A_{\delta,\alpha}$ be the self-adjoint operator in (2.5) and let A_0 be the free operator in (2.3). Then*

$$A_{\delta,\alpha}^{-1} - A_0^{-1}$$

is a compact operator in $L_2(\mathbb{R}^n)$ and the following two statements hold.

(i) *The singular values s_k of $A_{\delta,\alpha}^{-1} - A_0^{-1}$ satisfy*

$$s_k = C_{\delta,\alpha} k^{-\frac{3}{n-1}} + o(k^{-\frac{3}{n-1}}), \quad k \rightarrow \infty,$$

with the constant $C_{\delta,\alpha} = (C'_{\delta,\alpha})^{\frac{3}{n-1}}$,

$$C'_{\delta,\alpha} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_\Sigma \int_{|\xi'|=1} \left(\frac{\underline{a}_{nn}(x') |\alpha(x')|}{4(\kappa_0(x', \xi'))^3} \right)^{\frac{n-1}{3}} d\omega(\xi') d\sigma(x').$$

(ii) If $\alpha(x') \neq 0$ for all $x' \in \Sigma$, then the singular values s_k of $A_{\delta,\alpha}^{-1} - A_0^{-1}$ satisfy

$$s_k = C_{\delta,\alpha} k^{-\frac{3}{n-1}} + O(k^{-\frac{4}{n-1}}), \quad k \rightarrow \infty.$$

Proof. (i) Let us set

$$G_{\delta,\alpha} := A_{\delta,\alpha}^{-1} - A_0^{-1} \quad \text{and} \quad S_\alpha := (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1} \alpha (P_{\gamma,\nu}^+ + P_{\gamma,\nu}^-)^{-1},$$

where $P_{\gamma,\nu}^+$ and $P_{\gamma,\nu}^-$ are defined in (3.9), and let \tilde{K}_γ be as in (4.15). It follows from Corollary 4.5 that

$$G_{\delta,\alpha} = \tilde{K}_\gamma S_\alpha \tilde{K}_\gamma^*, \quad (5.1)$$

which is a bounded self-adjoint operator in $L_2(\mathbb{R}^n)$. We also make use of the operator

$$R_\gamma := \tilde{K}_\gamma^* \tilde{K}_\gamma = (K_\gamma^+)^* K_\gamma^+ + (K_\gamma^-)^* K_\gamma^-,$$

which is a ψ do on Σ of order -1 with the principal symbol $\underline{a}_{nn}/\kappa_0$ according to Lemma 3.2 (iii). Note that for a non-trivial $f \in L_2(\Sigma)$ one has

$$((K_\gamma^\pm)^* K_\gamma^\pm f, f)_{L_2(\Sigma)} = \|K_\gamma^\pm f\|_{L_2(\Omega_\pm)}^2 > 0.$$

Thus the non-negative self-adjoint operators $(K_\gamma^\pm)^* K_\gamma^\pm$ are invertible; hence so is R_γ .

In the following we use that in the pseudodifferential boundary operator calculus, \tilde{K}_γ , \tilde{K}_γ^* and R_γ extend to continuous operators

$$\begin{aligned} \tilde{K}_\gamma &: H^s(\Sigma) \rightarrow H^{s+\frac{1}{2}}(\Omega_+) \times H^{s+\frac{1}{2}}(\Omega_-) && \text{for } s \in \mathbb{R}, \\ \tilde{K}_\gamma^* &: H^t(\Omega_+) \times H^t(\Omega_-) \rightarrow H^{t+\frac{1}{2}}(\Sigma) && \text{for } t > -\frac{1}{2}, \\ R_\gamma &: H^s(\Sigma) \rightarrow H^{s+1}(\Sigma) && \text{for } s \in \mathbb{R} \end{aligned}$$

(we use the same notation for the extended/restricted operators). By Proposition 4.4 (i) and Lemma 3.2 (ii) the ψ do's $(P_{\gamma,\nu}^+ + P_{\gamma,\nu}^-)^{-1}$ and $(P_{\gamma,\nu}^+ + P_{\gamma,\nu}^- - \alpha)^{-1}$ are both of order -1 and have the principal symbol $1/(2\kappa_0)$. Hence S_α is of order -2 and has the principal symbol $\alpha/(4\kappa_0^2)$. It extends to a mapping

$$S_\alpha: H^s(\Sigma) \rightarrow H^{s+2}(\Sigma) \quad \text{for } s \in \mathbb{R}.$$

It now follows from (5.1) that

$$\begin{aligned} G_{\delta,\alpha}^* G_{\delta,\alpha} &= (\tilde{K}_\gamma S_\alpha \tilde{K}_\gamma^*)^* \tilde{K}_\gamma S_\alpha \tilde{K}_\gamma^* \\ &= \tilde{K}_\gamma S_\alpha^* \tilde{K}_\gamma^* \tilde{K}_\gamma S_\alpha \tilde{K}_\gamma^* \\ &= \tilde{K}_\gamma S_\alpha R_\gamma S_\alpha \tilde{K}_\gamma^*, \end{aligned}$$

where we used the symmetry of S_α , which follows from (4.18). Therefore (here $s_k(T)$ denotes the k th singular value and $\lambda_k(T)$ the k th positive eigenvalue of an operator T)

$$\begin{aligned}
(s_k(G_{\delta,\alpha}))^2 &= \lambda_k(G_{\delta,\alpha}^* G_{\delta,\alpha}) \\
&= \lambda_k(\tilde{K}_\gamma S_\alpha R_\gamma S_\alpha \tilde{K}_\gamma^*) \\
&= \lambda_k(S_\alpha R_\gamma S_\alpha \tilde{K}_\gamma^* \tilde{K}_\gamma) \\
&= \lambda_k(S_\alpha R_\gamma S_\alpha R_\gamma) \\
&= \lambda_k(R_\gamma^{1/2} S_\alpha R_\gamma^{1/2} R_\gamma^{1/2} S_\alpha R_\gamma^{1/2}) \\
&= (s_k(R_\gamma^{1/2} S_\alpha R_\gamma^{1/2}))^2,
\end{aligned} \tag{5.2}$$

that is, the singular values s_k of $A_{\delta,\alpha}^{-1} - A_0^{-1}$ are

$$s_k(G_{\delta,\alpha}) = s_k(R_\gamma^{1/2} S_\alpha R_\gamma^{1/2}).$$

Here since R_γ is non-negative and invertible, the operator $R_\gamma^{1/2}$ is well defined as an elliptic ψ do of order $-1/2$ with principal symbol $(\underline{a}_{nn}/\kappa_0)^{1/2}$, by [47]. Moreover, S_α is a ψ do of order -2 with principal symbol $\alpha/(4\kappa_0^2)$; so standard rules of the ψ do calculus yield that the ψ do

$$P_{\delta,\alpha} := R_\gamma^{1/2} S_\alpha R_\gamma^{1/2} \tag{5.3}$$

is of order -3 and with the principal symbol $(\underline{a}_{nn}\alpha)/(4\kappa_0^3)$. Now Theorem 3.1 (i) implies that

$$s_k(P_{\delta,\alpha}) = C_{\delta,\alpha} k^{-\frac{3}{n-1}} + o(k^{-\frac{3}{n-1}}), \quad k \rightarrow \infty,$$

with $C_{\delta,\alpha}$ as in the formulation of the theorem.

(ii) Recall that $\underline{a}_{nn}(x') > 0$ and $\kappa_0(x', \xi') > 0$ for $\xi' \neq 0$; cf. Section 3. Hence the assumption $\alpha(x') \neq 0$ for all $x' \in \Sigma$ implies that the ψ do $P_{\delta,\alpha}$ in (5.3) is elliptic. Moreover, since S_α is then invertible and $R_\gamma^{1/2}$ is invertible, the operator $P_{\delta,\alpha}$ in (5.3) is invertible. Now Theorem 3.1 (ii) implies the assertion. \square

The next theorem contains our second main result: the spectral asymptotics of the difference between the inverses of $A_{\delta',\beta}$ and A_0 . It turns out that the principal term in the asymptotics is independent of β . The proof of Theorem 5.2 is very similar to the proof of Theorem 5.1.

Theorem 5.2. *Let $\beta \in C^\infty(\Sigma)$ be real-valued such that $\beta(x') \neq 0$ for all $x' \in \Sigma$, let $A_{\delta',\beta}$ be the self-adjoint operator in (2.6) and let A_0 be the free operator in (2.3). Then*

$$A_{\delta',\beta}^{-1} - A_0^{-1}$$

is a compact operator in $L_2(\mathbb{R}^n)$, and its singular values satisfy

$$s_k = C_{\delta'} k^{-\frac{2}{n-1}} + O(k^{-\frac{3}{n-1}}), \quad k \rightarrow \infty,$$

with the constant $C_{\delta'} = (C'_{\delta'})^{\frac{2}{n-1}}$,

$$C'_{\delta'} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_\Sigma \int_{|\xi'|=1} \left(\frac{\underline{a}_{nn}(x')}{2(\kappa_0(x', \xi'))^2} \right)^{\frac{n-1}{2}} d\omega(\xi') d\sigma(x').$$

Proof. According to Corollary 4.8 we have

$$A_{\delta',\beta}^{-1} - A_0^{-1} = \tilde{K}_\nu (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \beta (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1} \tilde{K}_\nu^*,$$

where $P_{\nu,\gamma}^+$, $P_{\nu,\gamma}^-$ and \tilde{K}_ν are as in (3.9) and (4.19), respectively. Let us set

$$R_\nu := \tilde{K}_\nu^* \tilde{K}_\nu \quad \text{and} \quad T_\beta := (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-)^{-1} \beta (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1}.$$

Since K_ν^\pm are injective, the non-negative self-adjoint operators $(K_\nu^\pm)^* K_\nu^\pm$ are invertible; hence so is R_ν . As in the proof of Theorem 5.1 we make use of the fact that \tilde{K}_ν , \tilde{K}_ν^* and T_β extend to continuous operators between the respective spaces. Then the same computation as in (5.2) implies

$$s_k(A_{\delta',\beta}^{-1} - A_0^{-1}) = s_k(R_\nu^{1/2} T_\beta R_\nu^{1/2}).$$

The operator $R_\nu^{1/2}$ is well defined as an elliptic ψ do of order $-3/2$ with principal symbol $(\underline{a}_{nn}/\kappa_0^3)^{1/2}$ by [47]. It follows from Lemma 3.2 (ii) and Proposition 4.6 (i) that T_β is a ψ do of order 1 with principal symbol $\kappa_0/2$. By standard rules of the calculus the ψ do

$$P_{\delta',\beta} := R_\nu^{1/2} T_\beta R_\nu^{1/2} \tag{5.4}$$

is of order -2 and has the principal symbol $\underline{a}_{nn}/(2\kappa_0^2)$. Since $\underline{a}_{nn}(x') > 0$ and $\kappa_0(x', \xi') > 0$ for $\xi' \neq 0$, it follows that the operator $P_{\delta',\beta}$ is elliptic. Since T_β and $R_\nu^{1/2}$ are invertible the same true for $P_{\delta',\beta}$ in (5.4). Now Theorem 3.1 (ii) yields the spectral asymptotics

$$s_k(P_{\delta',\beta}) = C_{\delta'} k^{-\frac{2}{n-1}} + O(k^{-\frac{3}{n-1}}), \quad k \rightarrow \infty,$$

with $C_{\delta'}$ as in the formulation of the theorem. \square

Finally we state a third result on spectral asymptotics. In the next theorem the difference of the inverses of $A_{\delta',\beta}$ and the orthogonal sum of the Neumann operators A_ν is considered. The proof is similar to the proofs of the previous theorems. Therefore we just give the main arguments.

Theorem 5.3. *Let $\beta \in C^\infty(\Sigma)$ be real-valued such that $\beta(x') \neq 0$ for all $x' \in \Sigma$, let $A_{\delta',\beta}$ be the self-adjoint operator in (2.6) and let A_ν be as in (2.4). Then*

$$A_{\delta',\beta}^{-1} - A_\nu^{-1}$$

is a compact operator in $L_2(\mathbb{R}^n)$, and its singular values satisfy

$$s_k = C_{\delta',\beta,\nu} k^{-\frac{3}{n-1}} + O(k^{-\frac{4}{n-1}}), \quad k \rightarrow \infty,$$

with the constant $C_{\delta',\beta,\nu} = (C'_{\delta',\beta,\nu})^{\frac{3}{n-1}}$,

$$C'_{\delta',\beta,\nu} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_\Sigma \int_{|\xi'|=1} \left(\frac{\underline{a}_{nn}(x')}{|\beta(x')| (\kappa_0(x', \xi'))^3} \right)^{\frac{n-1}{3}} d\omega(\xi') d\sigma(x').$$

Proof. According to Proposition 4.6 (ii) we have

$$A_{\delta',\beta}^{-1} - A_\nu^{-1} = \tilde{K}_\nu (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1} \tilde{K}_\nu^*,$$

where $P_{\nu,\gamma}^+$, $P_{\nu,\gamma}^-$ and \tilde{K}_ν are as in (3.9) and (4.19), respectively. Let

$$R_\nu := \tilde{K}_\nu^* \tilde{K}_\nu \quad \text{and} \quad S_\beta := (\beta - (P_{\nu,\gamma}^+ + P_{\nu,\gamma}^-))^{-1},$$

and observe that \tilde{K}_ν , \tilde{K}_ν^* and S_β extend to continuous operators between the respective spaces. As in the proofs of Theorems 5.1 and 5.2 one verifies

$$s_k(A_{\delta',\beta}^{-1} - A_\nu^{-1}) = s_k(R_\nu^{1/2} S_\beta R_\nu^{1/2}).$$

Furthermore, $R_\nu^{1/2}$ is a ψ do of order $-3/2$ with principal symbol $(\underline{a}_{nn}/\kappa_0^3)^{1/2}$, and by Proposition 4.6 (i) the ψ do S_β is of order 0 with principal symbol β^{-1} . Therefore the ψ do

$$P_{\delta',\beta,\nu} := R_\nu^{1/2} S_\beta R_\nu^{1/2}$$

is of order -3 and has the principal symbol $\underline{a}_{nn}/(\beta\kappa_0^3)$. It also follows that $P_{\delta',\beta,\nu}$ is elliptic and invertible, and hence Theorem 3.1 (ii) yields the spectral asymptotics

$$s_k(P_{\delta',\beta,\nu}) = C_{\delta',\beta,\nu} k^{-\frac{3}{n-1}} + O(k^{-\frac{4}{n-1}}), \quad k \rightarrow \infty,$$

with $C_{\delta',\beta,\nu}$ of the form as stated in the theorem. \square

The constants $C'_{\delta,\alpha}$, $C'_{\delta'}$ and $C'_{\delta',\beta,\nu}$ in the previous theorems can be computed explicitly in the special case $\mathcal{A} = -\Delta$.

Example 5.4. Denote by $|\Sigma|$ the area of Σ and by $|S_{n-1}|$ the area of the $(n-1)$ -dimensional unit sphere (for $n=2$ we have $|S_{n-1}| = 2$). According to Example 3.4 in the special case $\mathcal{A} = -\Delta$ the constant $C'_{\delta,\alpha}$ in Theorem 5.1 is

$$\begin{aligned} C'_{\delta,\alpha} &= \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma} \int_{|\xi'|=1} \left(\frac{1 \cdot |\alpha(x')|}{4|\xi'|^3} \right)^{\frac{n-1}{3}} d\omega(\xi') d\sigma(x') \\ &= \frac{|S_{n-1}|}{(n-1)(2\pi)^{n-1} 4^{(n-1)/3}} \int_{\Sigma} |\alpha(x')|^{\frac{n-1}{3}} d\sigma(x'), \end{aligned}$$

the constant $C'_{\delta'}$ in Theorem 5.2 is

$$\begin{aligned} C'_{\delta'} &= \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma} \int_{|\xi'|=1} \left(\frac{1}{2|\xi'|^2} \right)^{\frac{n-1}{2}} d\omega(\xi') d\sigma(x') \\ &= \frac{|\Sigma| \cdot |S_{n-1}|}{(n-1)(2\pi)^{n-1} 2^{(n-1)/2}}, \end{aligned}$$

and the constant $C'_{\delta',\beta,\nu}$ in Theorem 5.3 is

$$\begin{aligned} C'_{\delta',\beta,\nu} &= \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma} \int_{|\xi'|=1} \left(\frac{1}{|\beta(x')| \cdot |\xi'|^3} \right)^{\frac{n-1}{3}} d\omega(\xi') d\sigma(x') \\ &= \frac{|S_{n-1}|}{(n-1)(2\pi)^{n-1}} \int_{\Sigma} \left(\frac{1}{|\beta(x')|} \right)^{\frac{n-1}{3}} d\sigma(x'). \end{aligned}$$

Remark 5.5. We note that the Krein-type formulae in Section 4 for the differences of the inverses $A_{\delta,\alpha}^{-1} - A_0^{-1}$, $A_{\delta',\beta}^{-1} - A_0^{-1}$ and $A_{\delta',\beta}^{-1} - A_\nu^{-1}$ can be generalized for the differences of the resolvents $(A_{\delta,\alpha} - \lambda)^{-1} - (A_0 - \lambda)^{-1}$, $(A_{\delta',\beta} - \lambda)^{-1} - (A_0 - \lambda)^{-1}$ and $(A_{\delta',\beta} - \lambda)^{-1} - (A_\nu - \lambda)^{-1}$ for all λ in the respective resolvent sets. Making use of such resolvent formulae one

can show that the spectral asymptotics in Theorem 5.1, Theorem 5.2 and Theorem 5.3 remain true for all λ in the respective resolvent sets.

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