The Truncated Euler–Maruyama Method for Stochastic Differential Equations

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Abstract

Influenced by Higham, Mao and Stuart [10], several numerical methods have been developed to study the strong convergence of the numerical solutions to stochastic differential equations (SDEs) under the local Lipschitz condition. These numerical methods include the tamed Euler–Maruyama (EM) method, the tamed Milstein method, the stopped EM, the backward EM, the backward forward EM, etc. In this paper we will develop a new explicit method, called the truncated EM method, for the nonlinear SDE $dx(t) = f(x(t))dt + g(x(t))dB(t)$ and establish the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition $x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K(1+|x|^2)$. The type of convergence specifically addressed in this paper is strong-L^q convergence for $2 \le q < p$, and p is a parameter in the Khasminskii-type condition.

Key words: Stochastic differential equation, local Lipschitz condition, Khasminskii-type condition, truncated Euler-Maruyama method, strong convergence.

1 Introduction

Up to 2002, most of the existing strong convergence theory for numerical methods requires the coefficients of the SDEs to be globally Lipschitz continuous (see, e.g., [18, 21, 26]). However, most SDE models in real life do not obey the global Lipschitz condition. It was in this spirit that Higham, Mao and Stuart in 2002 published a very influential paper [10] (Google citation 286) which opened a new chapter in the study of numerical solutions of SDEs—to study the strong convergence question for numerical approximations under the local Lipschitz condition. Of course, the local Lipschitz condition is not enough to guarantee the existence of the global solution. The additional known condition for the global solution is the linear growth condition, or more generally, the Khasminskii-type conditions (see, e.g., [15, 21, 31]). Instead of imposing these known conditions, Higham, Mao and Stuart [10] proposed the bounded condition on the *pth* moments of both exact solution and numerical solution to the underlying SDE and proved the strong convergence theory. Their theory turns the problem of the strong convergence into the verification of the boundedness of the pth moments of the exact and numerical solutions under the local Lipschitz condition. They showed that under the linear growth condition, both exact and

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numerical solutions by either the Euler-Maruyama (EM) or the stochastic theta method satisfy the moment bounded condition, and hence they proved that the numerical solutions converge to the exact solution in the strong sense under the Local Lipschitz condition and the linear growth condition.

However, the linear growth condition is still too restrictive. The authors in [10] pointed out that in general, it is not clear when such moment bounds can be expected to hold for the EM method even when both drift coefficient and the diffusion coefficient are C^1 (unbounded derivatives of course). More recently, the authors in [13] answered the question negatively by proving that the moment of the explicit EM method will diverge in finite time for those SDEs with either the drift coefficient or the diffusion coefficient being superlinear. Implicit methods have therefore naturally been used to study the numerical solutions to SDEs without the linear growth condition recently, for example, in [23, 33, 34]. For the background on the implicit methods, we refer the reader to the papers [2, 4, 10, 11, 17, 25, 30] and the book [18]. Methods with variable stepsize also attract a lot of attention [5, 28, 35, 37]. Other weak forms of convergence, say weak convergence, convergence in probability and pathwise convergence, are discussed in [1, 7, 16, 18, 22, 24, 27, 36], just to mention a few.

Since the classical explicit EM method has its simple algebraic structure, cheap computational cost and acceptable convergence rate under the global Lipschitz condition, it has been attracting lots of attention [8]. Although the authors in [13] showed the strong and weak divergence in finite time of the EM method for SDEs with non-globally Lipschitz continuous coefficients, some modified EM methods have recently been developed for the nonlinear SDEs without the linear growth condition. For example, the tamed EM method was developed in [14] to approximate SDEs with one-sided Lipschitz drift coefficient and the linear growth diffusion coefficient. This method was further developed in [32] while the tamed Milstein method was developed in [6]. Moreover, the stopped EM method was developed in [20] for nonlinear SDEs as well. These new explicit EM methods have shown their abilities to approximate the solutions of nonlinear SDEs.

In this paper, we will develop another new explicit method for nonlinear SDEs. We will call it the *truncated EM method*. A different method referred to as the "Drift-truncated Euler scheme" was introduced in Hutzenthaler & Jentzen [12] and we hope very much that our chosen scheme name "truncated EM method" will not cause a confusion.

To motivate our new method, we consider a d-dimensional SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{1.1}$$

with the local Lipschitz condition but without the linear growth condition. To guarantee the global solution, we impose a Khasminskii-type condition (see, e.g., [15])

$$2x^T f(x) + |g(x)|^2 \le K(1 + |x|^2). \tag{1.2}$$

The classical method to prove the existence of the global solution under this condition is the truncated method (see, e.g., [15, 21, 29]). That is, for each integer $n \ge 1$, define the truncated functions

$$f_n(x) = f\Big((|x| \wedge n)\frac{x}{|x|}\Big) \quad \text{and} \quad g_n(x) = g\Big((|x| \wedge n)\frac{x}{|x|}\Big). \tag{1.3}$$

Then both f_n and g_n are globally Lipschiz so the following SDE

$$dx_n(t) = f_n(x_n(t))dt + g_n(x_n(t))dB(t) \tag{1.4}$$

has a unique global solution $x_n(t)$ on $t \geq 0$. By the Khasminskii-type condition (1.2), we can then show that $x_n(t)$ will converge to a stochastic process $x(t)$ in probability and this $x(t)$ is the solution to the SDE (1.1). Let us now apply the EM method with stepsize Δ to the SDE (1.4) to obtain the EM approximate solution $x_{\Delta,n}(t)$. It is well known that $x_{\Delta,n}(t)$ will converge to $x_n(t)$ in the strong sense, say L^2 (actually in L^p for any $p > 0$) as $\Delta \to 0$. But $x_n(t)$ will converge to $x(t)$ in probability as $n \to \infty$. It is therefore not difficult to show that for each n, one can choose a stepsize Δ_n such that $x_{\Delta_n,n}(t)$ will converge to $x(t)$ in probability as $n \to \infty$. We can also do so in the other way, that is, for each Δ, choose $n = n_\Delta$ to show that $x_{\Delta,n_\Delta}(t)$ will converge to $x(t)$ in probability as $\Delta \to 0$. Surprisingly, we will see in this paper that choosing $n = n_\Delta$ cleverly (namely $n = \mu^{-1}(h(\Delta))$ in our definition of the truncated EM method below), we can show that $x_{\Delta,n_\Delta}(t)$ will converge to $x(t)$ in the strong sense as $\Delta \to 0$. The type of convergence specifically addressed in this paper is strong-L^q convergence for $2 \leq q < p$, and p is a parameter in the Khasminskii-type condition. Let us begin to develop this new truncated EM method.

2 The Truncated EM Method

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be an m-dimensional Brownian motion defined on the space. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in R^d$, then $|x|$ is the Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. Moreover, for two real numbers a and b, we use $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. If G is a set, its indicator function is denoted by I_G, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise.

Consider a d-dimensional SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{2.1}$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^d$, where

$$f : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad g : \mathbb{R}^d \to \mathbb{R}^{d\times m}.$$

We impose two standing hypotheses in this paper.

Assumption 2.1 *Assume that the coefficients f and g satisfy the local Lipschitz condition: For any $R > 0$, there is a $K_R > 0$ such that*

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq K_R|x - y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$.

Assumption 2.2 *We also assume that the coefficients satisfy the Khasminskii-type condition: There is a pair of constants $p > 2$ and $K > 0$ such that*

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \leq K(1 + |x|^2) \tag{2.3}$$

for all $x \in \mathbb{R}^d$.

We state a known result (see, e.g., [21, 22, 31]) as a lemma for the use of this paper.

Lemma 2.3 *Under Assumptions 2.1 and 2.2, the SDE (2.1) has a unique global solution $x(t)$ and, moreover,*

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^p < \infty, \quad \forall T > 0. \tag{2.4}$$

Assumptions 2.1 and 2.2 cover many nonlinear SDEs, for example, the scalar SDE in financial mathematics (see, e.g., [19])

$$dx(t) = (\mu - \alpha x^\beta(t))dt + \sigma x^\theta(t)dB(t), \quad \beta, \theta > 1, \ \mu, \alpha, \sigma > 0, \tag{2.5}$$

and the stochastic population system (see, e.g., [3])

$$dx(t) = \text{diag}(x_1(t), x_2(t), ..., x_d(t))[(b + Ax^2(t))dt + Cx(t)dB(t)], \tag{2.6}$$

where $B(t)$ is a scalar Brownian motion, $b = (b_1, \cdots, b_d)^T$, $x^2 = (x_1^2, \cdots, x_d^2)^T$, $C = (C_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$ and $A = (A_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$ is such that $\lambda_{\max}(A + A^T) < 0$. (See Section 5 below for a further detailed discussion.) It has been shown (see, e.g., [22]) that under Assumptions 2.1 and 2.2, the EM numerical solutions will converge to the true solution in probability but, in general, not in L^2. In this paper, we will develop a new numerical method, called the truncated EM method, and show that the numerical solutions will converge to the true solution in L^p.

To define the truncated EM numerical solutions, we first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(r) \to \infty$ as $r \to \infty$ and

$$\sup_{|x|\le r} \big(|f(x)| \vee |g(x)|\big) \le \mu(r), \quad \forall r \ge 0. \tag{2.7}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(0), \infty)$ to $\mathbb{R}_+$. We also choose a number $\Delta^* \in (0, 1]$ and a strictly decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that

$$h(\Delta^*) \ge \mu(2), \quad \lim_{\Delta\to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4}h(\Delta) \le 1, \quad \forall \Delta \in (0, 1). \tag{2.8}$$

For a given stepsize $\Delta \in (0, 1)$, let us define the truncated functions

$$f_\Delta(x) = f\Big((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\Big) \quad \text{and} \quad g_\Delta(x) = g\Big((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\Big) \tag{2.9}$$

for $x \in \mathbb{R}^d$, where we set $x/|x| = 0$ when $x = 0$. It is easy to see that

$$|f_\Delta(x)| \vee |g_\Delta(x)| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta) \quad \forall x \in \mathbb{R}^d. \tag{2.10}$$

That is, both truncated functions f_Δ and g_Δ are bounded although both f and g may not. Moreover, these truncated functions preserve the Khasminskii-type condition for all $\Delta \in (0, \Delta^*]$ as described in the following lemma.

Lemma 2.4 *Let Assumption 2.2 hold. Then, for all $\Delta \in (0, \Delta^*]$, we have*

$$x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 \le 2K(1 + |x|^2), \quad \forall x \in \mathbb{R}^d. \tag{2.11}$$

Proof. We first observe from (2.8) that

$$\mu^{-1}(h(\Delta)) \ge \mu^{-1}(h(\Delta^*)) \ge 2, \quad \forall \Delta \in (0, \Delta^*].$$

Fix any $\Delta \in (0, \Delta^*]$. For $x \in \mathbb{R}^d$ with $|x| \le \mu^{-1}(h(\Delta))$, we have, by (2.3),

$$x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 = x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K(1+|x|^2)$$

so the required assertion (2.11) holds. For $x \in \mathbb{R}^d$ with $|x| > \mu^{-1}(h(\Delta))$, we have

$$\begin{aligned}
& x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 \\
= \; & x^T f\Big(\mu^{-1}(h(\Delta))\frac{x}{|x|}\Big) + \frac{p-1}{2}\Big|g\Big(\mu^{-1}(h(\Delta))\frac{x}{|x|}\Big)\Big|^2 \\
= \; & \mu^{-1}(h(\Delta))\frac{x^T}{|x|} f\Big(\mu^{-1}(h(\Delta))\frac{x}{|x|}\Big) + \frac{p-1}{2}\Big|g\Big(\mu^{-1}(h(\Delta))\frac{x}{|x|}\Big)\Big|^2 \\
+ \; & \Big(\frac{|x|}{\mu^{-1}(h(\Delta))} - 1\Big)\mu^{-1}(h(\Delta))\frac{x^T}{|x|} f\Big(\mu^{-1}(h(\Delta))\frac{x}{|x|}\Big) \\
\le \; & K(1+[\mu^{-1}(h(\Delta))]^2) + \Big(\frac{|x|}{\mu^{-1}(h(\Delta))} - 1\Big)\mu^{-1}(h(\Delta))\frac{x^T}{|x|} f\Big(\mu^{-1}(h(\Delta))\frac{x}{|x|}\Big),
\end{aligned}$$

where (2.3) has been used. But once again we see from (2.3) that $x^T f(x) \le K(1+|x|^2)$ for any $x \in \mathbb{R}^d$. We therefore have

$$\begin{aligned}
& x^T f_\Delta(x) + \frac{p-1}{2}|g_\Delta(x)|^2 \\
\le \; & K(1+[\mu^{-1}(h(\Delta))]^2) + \Big(\frac{|x|}{\mu^{-1}(h(\Delta))} - 1\Big)K(1+[\mu^{-1}(h(\Delta))]^2) \\
= \; & \frac{|x|}{\mu^{-1}(h(\Delta))}K(1+[\mu^{-1}(h(\Delta))]^2) \\
\le \; & K|x|(0.5+\mu^{-1}(h(\Delta))) \le K|x|(0.5+|x|) \\
\le \; & K(1+|x|)^2 \le 2K(1+|x|^2)
\end{aligned}$$

as required. The proof is complete. □

We can now form the discrete-time truncated EM numerical solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ by setting $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta + g_\Delta(X_\Delta(t_k))\Delta B_k, \tag{2.12}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Let us now form two versions of the continuous-time truncated EM solutions. The first one is defined by

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad t \ge 0. \tag{2.13}$$

This is a simple step process so its sample paths are not continuous. We will refer this as the continuous-time step-process truncated EM solution. The other one is defined by

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta(\bar{x}_\Delta(s))ds + \int_0^t g_\Delta(\bar{x}_\Delta(s))dB(s) \tag{2.14}$$

for $t \ge 0$. We will refer this as the continuous-time continuous-sample truncated EM solution. We observe that $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$ for all $k \ge 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = f_\Delta(\bar{x}_\Delta(t))dt + g_\Delta(\bar{x}_\Delta(t))dB(t). \tag{2.15}$$

3 Convergence of the Truncated EM Solutions

3.1 Moment bound of the truncated EM solutions

By (2.10), it is obvious that

$$\sup_{0\le t\le T} \mathbb{E}|x_\Delta(t)|^p < \infty, \quad \forall T>0.$$

However, it is not so obvious to see that

$$\sup_{0<\Delta\le\Delta^*}\ \sup_{0\le t\le T} \mathbb{E}|x_\Delta(t)|^p < \infty, \quad \forall T>0$$

and this is what we are going to establish in this subsection. Let us first present a lemma which shows that $x_\Delta(t)$ and $\bar{x}_\Delta(t)$ are close to each other in the sense of L^p.

Lemma 3.1 *For any $\Delta \in (0,\Delta^*]$, we have*

$$\mathbb{E}|x_\Delta(t)-\bar{x}_\Delta(t)|^p \le c_p \Delta^{p/2}(h(\Delta))^p, \quad \forall t\ge 0, \tag{3.1}$$

where c_p is a positive constant dependent only on p. Consequently

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(t)-\bar{x}_\Delta(t)|^p = 0, \quad \forall t\ge 0. \tag{3.2}$$

Proof. In what follows, we will use c_p to stand for generic positive real constants dependent only on p and its values may change between occurrences. Fix any $\Delta\in(0,\Delta^*]$ and $t\ge 0$. There is a unique integer $k\ge 0$ such that $t_k\le t\le t_{k+1}$. By (2.10) and the properties of the Itô integral (see, e.g., [21]), we then derive from (2.14) that

$$\begin{aligned}
&\mathbb{E}|x_\Delta(t)-\bar{x}_\Delta(t)|^p = \mathbb{E}|x_\Delta(t)-x_\Delta(t_k)|^p\\
\le\ & c_p\Big(\mathbb{E}\Big|\int_{t_k}^t f_\Delta(\bar{x}_\Delta(s))ds\Big|^p + \mathbb{E}\Big|\int_{t_k}^t g_\Delta(\bar{x}_\Delta(s))dB(s)\Big|^p\Big)\\
\le\ & c_p\Big(\Delta^{p-1}\mathbb{E}\int_{t_k}^t |f_\Delta(\bar{x}_\Delta(s))|^p ds + \Delta^{(p-2)/2}\mathbb{E}\int_{t_k}^t |g_\Delta(\bar{x}_\Delta(s))|^p ds\Big)\\
\le\ & c_p\Delta^{p/2}(h(\Delta))^p,
\end{aligned}$$

which is (3.1). Noting from (2.8) that $\Delta^{p/2}(h(\Delta))^p \le \Delta^{p/4}$, we obtain (3.2) from (3.1) immediately. □

Lemma 3.2 *Let Assumptions 2.1 and 2.2 hold. Then*

$$\sup_{0<\Delta\le\Delta^*}\ \sup_{0\le t\le T} \mathbb{E}|x_\Delta(t)|^p \le C, \quad \forall T>0, \tag{3.3}$$

where, and from now on, C stands for generic positive real constants dependent on T,p,K,x_0 (and $\bar{K}$ as well in the next section) but independent of Δ and its values may change between occurrences.

Proof. Fix any $\Delta \in (0, \Delta^*]$ and $T \geq 0$. By the Itô formula, we derive from (2.14) that, for $0 \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}|x_\Delta(t)|^p &\leq |x_0|^p + \mathbb{E}\int_0^t p|x_\Delta(s)|^{p-2}\Big(x_\Delta^T(s) f_\Delta(\bar{x}_\Delta(s)) + \frac{p-1}{2}|g_\Delta(\bar{x}_\Delta(s))|^2\Big)ds \\
&= |x_0|^p + \mathbb{E}\int_0^t p|x_\Delta(s)|^{p-2}\Big(\bar{x}_\Delta^T(s) f_\Delta(\bar{x}_\Delta(s)) + \frac{p-1}{2}|g_\Delta(\bar{x}_\Delta(s))|^2\Big)ds \\
&+ \mathbb{E}\int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - \bar{x}_\Delta(s))^T f_\Delta(\bar{x}_\Delta(s))ds.
\end{aligned}$$

By Lemma 2.4 and the Young inequality

$$a^{p-2}b \leq \frac{p-2}{p}\, a^p + \frac{2}{p}\, b^{p/2}, \quad \forall a, b \geq 0,$$

we then have

$$\begin{aligned}
\mathbb{E}|x_\Delta(t)|^p &\leq |x_0|^p + \mathbb{E}\int_0^t Kp|x_\Delta(s)|^{p-2}(1 + |\bar{x}_\Delta(s)|^2)ds \\
&+ (p-2)\mathbb{E}\int_0^t |x_\Delta(s)|^p ds + 2\mathbb{E}\int_0^t |x_\Delta(s) - \bar{x}_\Delta(s))|^{p/2}|f_\Delta(\bar{x}_\Delta(s))|^{p/2}ds \\
&\leq C_1 + C_2\int_0^t \Big(\mathbb{E}|x_\Delta(s)|^p + \mathbb{E}|\bar{x}_\Delta(s)|^p\Big)ds \\
&+ 2\mathbb{E}\int_0^T |x_\Delta(s) - \bar{x}_\Delta(s))|^{p/2}|f_\Delta(\bar{x}_\Delta(s))|^{p/2}ds.
\end{aligned}$$

where C_1 and C_2 are generic and may change throughout this proof (we do not want to use a single C in a single inequality but use C_1 and C_2 to indicate these two constants differ). By Lemma 3.1 and inequalities (2.10) and (2.8), we have

$$\begin{aligned}
&\mathbb{E}\int_0^T |x_\Delta(s) - \bar{x}_\Delta(s))|^{p/2}|f_\Delta(\bar{x}_\Delta(s))|^{p/2}ds \\
\leq\ & (h(\Delta))^{p/2}\int_0^T \mathbb{E}(|x_\Delta(s) - \bar{x}_\Delta(s))|^{p/2})ds \\
\leq\ & (h(\Delta))^{p/2}\int_0^T (\mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s))|^p)^{1/2}ds \\
\leq\ & c_p T(h(\Delta))^p \Delta^{p/4} \leq c_p T.
\end{aligned} \tag{3.4}$$

We therefore have

$$\begin{aligned}
\mathbb{E}|x_\Delta(t)|^p &\leq C_1 + C_2\int_0^t \Big(\mathbb{E}|x_\Delta(s)|^p + \mathbb{E}|\bar{x}_\Delta(s)|^p\Big)ds \\
&\leq C_1 + C_2\int_0^t \Big(\sup_{0\leq u\leq s}\mathbb{E}|x_\Delta(u)|^p\Big)ds.
\end{aligned}$$

As this holds for any $t \in [0, T]$ while the right-hand side is non-decreasing in t, we then see

$$\sup_{0\leq u\leq t}\mathbb{E}|x_\Delta(u)|^p \leq C_1 + C_2\int_0^t \Big(\sup_{0\leq u\leq s}\mathbb{E}|x_\Delta(u)|^p\Big)ds.$$

The well-known Gronwall inequality yields that

$$\sup_{0\le u\le T} \mathbb{E}|x_\Delta(u)|^p \le C.$$

As this holds for any $\Delta \in (0, \Delta^*]$ while C is independent of Δ, we see the required assertion (3.3). $\Box$

3.2 Strong convergence

In this subsection we will show that

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(T) - x(T)|^q = 0 \quad \text{and} \quad \lim_{\Delta\to 0} \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^q = 0$$

for any $T > 0$ and $2 \le q < p$. In the remaining of this subsection, we fix $T > 0$ arbitrarily.

Lemma 3.3 *Let Assumptions 2.1 and 2.2 hold. For any real number $R > |x_0|$, define the stopping time*

$$\tau_R = \inf\{t \ge 0 : |x(t)| \ge R\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (and as usual $\emptyset$ denotes the empty set). Then

$$\mathbb{P}(\tau_R \le T) \le \frac{C}{R^2}. \tag{3.5}$$

(Recall that C stands for generic positive real constants dependent on T, p, K, x_0 so C here is independent of R.)

Proof. By the Itô formula and Assumption 2.2, we derive that

$$\begin{aligned} \mathbb{E}|x(t\wedge\tau_R)|^2 &\le |x_0|^2 + \mathbb{E}\int_0^{t\wedge\tau_R} 2K(1 + |x(s)|^2)ds \\ &\le |x_0|^2 + 2KT + 2K\int_0^t \mathbb{E}|x(s\wedge\tau_R)|^p ds \end{aligned}$$

for any $0 \le t \le T$. The Gronwall inequality shows

$$\mathbb{E}|x(T\wedge\tau_R)|^2 \le C.$$

This implies

$$R^2\mathbb{P}(\tau_R \le T) \le C$$

and the assertion follows. $\Box$

Lemma 3.4 *Let Assumptions 2.1 and 2.2 hold. For any real number $R > |x_0|$ and $\Delta \in (0, \Delta^*)$, define the stopping time*

$$\rho_{\Delta,R} = \inf\{t \ge 0 : |x_\Delta(t)| \ge R\}.$$

Then

$$\mathbb{P}(\rho_{\Delta,R} \le T) \le \frac{C}{R^2}. \tag{3.6}$$

(Please recall that C is independent of Δ and R.)

Proof. We simply write $\rho_{\Delta,R} = \rho$. By the Itô formula, we have that for $0 \le t \le T$,

$$\begin{aligned}
\mathbb{E}|x_\Delta(t\wedge\rho)|^2 &= |x_0|^2 + \mathbb{E}\int_0^{t\wedge\rho}\Big(2x_\Delta^T(s)f_\Delta(\bar{x}_\Delta(s)) + |g_\Delta(\bar{x}_\Delta(s))|^2\Big)ds \\
&= |x_0|^2 + \mathbb{E}\int_0^{t\wedge\rho}\Big(2\bar{x}_\Delta^T(s)f_\Delta(\bar{x}_\Delta(s)) + |g_\Delta(\bar{x}_\Delta(s))|^2\Big)ds \\
&+ \mathbb{E}\int_0^{t\wedge\rho} 2(x_\Delta(s) - \bar{x}_\Delta(s))^T f_\Delta(\bar{x}_\Delta(s))ds.
\end{aligned}$$

By Lemma 2.4, we then derive that,

$$\begin{aligned}
\mathbb{E}|x_\Delta(t\wedge\rho)|^2 &\le |x_0|^2 + \mathbb{E}\int_0^{t\wedge\rho} 2K(1+|\bar{x}_\Delta(s)|^2)ds \\
&+ \mathbb{E}\int_0^{t\wedge\rho} 2|x_\Delta(s) - \bar{x}_\Delta(s)||f_\Delta(\bar{x}_\Delta(s))|ds \\
&\le |x_0|^2 + 2KT + 4K\int_0^t \mathbb{E}|x_\Delta(s\wedge\rho)|^2 ds \\
&+ 4K\int_0^T \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^2 ds \\
&+ 2\mathbb{E}\int_0^T |x_\Delta(s) - \bar{x}_\Delta(s)||f_\Delta(\bar{x}_\Delta(s))|ds.
\end{aligned}$$

But, by Lemma 3.1, we have

$$\int_0^T \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^2 ds \le C,$$

while by Lemma 3.1 and inequalities (2.10) and (2.8), we derive

$$\begin{aligned}
\mathbb{E}\int_0^T |x_\Delta(s) - \bar{x}_\Delta(s)||f_\Delta(\bar{x}_\Delta(s))|ds &\le h(\Delta)\mathbb{E}\int_0^T \Big(\mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^p\Big)^{1/p} ds \\
&\le Th(\Delta)\Big(c_p\Delta^{p/2}(h(\Delta))^p\Big)^{1/p} \le C(h(\Delta)\Delta^{1/4})^2 \le C.
\end{aligned}$$

We hence have

$$\mathbb{E}|x_\Delta(t\wedge\rho)|^2 \le C + 4K\int_0^t \mathbb{E}|x_\Delta(s\wedge\rho)|^2 ds.$$

The Gronwall inequality shows

$$\mathbb{E}|x_\Delta(T\wedge\rho)|^2 \le C.$$

This implies the required assertion (3.6) easily. □

Theorem 3.5 *Let Assumptions 2.1 and 2.2 hold. Then, for any $q \in [2, p)$,*

$$\lim_{\Delta\to 0}\mathbb{E}|x_\Delta(T) - x(T)|^q = 0 \quad \textit{and} \quad \lim_{\Delta\to 0}\mathbb{E}|\bar{x}_\Delta(T) - x(T)|^q = 0. \tag{3.7}$$

Proof. Let τ_R and $\rho_{\Delta,R}$ be the same as before. Set

$$\theta_{\Delta,R} = \tau_R \wedge \rho_{\Delta,R} \quad \text{and} \quad e_\Delta(T) = x_\Delta(T) - x(T).$$

Using the Young inequality, we derive that for any $\delta > 0$,

$$\begin{aligned}\mathbb{E}|e_\Delta(T)|^q &= \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\Big) + \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}\le T\}}\Big) \\ &\le \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\Big) + \frac{q\delta}{p}\mathbb{E}|e_\Delta(T)|^p + \frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}(\theta_{\Delta,R}\le T). \quad (3.8)\end{aligned}$$

By Lemmas 2.3 and 3.2, we have

$$\mathbb{E}|e_\Delta(T)|^p \le C$$

while by Lemmas 3.3 and 3.4,

$$\mathbb{P}(\theta_{\Delta,R}\le T) \le \mathbb{P}(\tau_R \le T) + \mathbb{P}(\rho_{\Delta,R}\le T) \le \frac{C}{R^2}.$$

We hence have

$$\mathbb{E}|e_\Delta(T)|^q \le \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\Big) + \frac{Cq\delta}{p} + \frac{C(p-q)}{pR^2\delta^{q/(p-q)}}. \quad (3.9)$$

Now, let $\varepsilon > 0$ be arbitrary. Choose δ sufficiently small for $Cq\delta/p \le \varepsilon/3$ and then choose R sufficiently large for

$$\frac{C(p-q)}{pR^2\delta^{q/(p-q)}} \le \frac{\varepsilon}{3}.$$

We then see from (3.9) that for this particularly chosen R,

$$\mathbb{E}|e_\Delta(T)|^q \le \mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\Big) + \frac{2\varepsilon}{3}. \quad (3.10)$$

If we can show that for all sufficiently small Δ,

$$\mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\Big) \le \frac{\varepsilon}{3}, \quad (3.11)$$

we then have

$$\lim_{\Delta\to 0}\mathbb{E}|e_\Delta(T)|^q = 0$$

and then by Lemma 3.1, we also have

$$\lim_{\Delta\to 0}\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^q = 0.$$

In other words, to complete our proof, all we need is to show (3.11). For this purpose, we define the truncated functions

$$F_R(x) = f\Big((|x|\wedge R)\frac{x}{|x|}\Big) \quad \text{and} \quad G_R(x) = g\Big((|x|\wedge R)\frac{x}{|x|}\Big), \quad x\in\mathbb{R}^d.$$

Without loss of any generality, we may assume that Δ^* is already sufficiently small for $\mu^{-1}(h(\Delta^*)) \ge R$. Hence, for all $\Delta \in (0,\Delta^*]$, we have that

$$f_\Delta(x) = F_R(x) \quad \text{and} \quad g_\Delta(x) = G_R(x)$$

for all $x\in\mathbb{R}^d$ with $|x|\le R$. Consider the SDE

$$dy(t) = F_R(y(t))dt + G_R(y(t))dB(t) \quad (3.12)$$

on $t \geq 0$ with the initial value $y(0) = x_0$. By Assumption 2.1, we see that both $F_R(x)$ and $G_R(x)$ are globally Lipschitz continuous with the Lipschitz constant K_R. So the SDE (3.12) has a unique global solution $y(t)$ on $t \geq 0$. It is straightforward to see that

$$x(t \wedge \tau_R) = y(t \wedge \tau_R) \quad a.s. \quad \text{for all } t \geq 0. \tag{3.13}$$

On the other hand, for each stepsize $\Delta \in (0, \Delta^*]$, we can apply the EM method to the SDE (3.12) and we denote by $y_\Delta(t)$ the continuous-time continuous-sample EM solution. It is again straightforward to see that

$$x_\Delta(t \wedge \rho_{\Delta,R}) = y_\Delta(t \wedge \rho_{\Delta,R}) \quad a.s. \quad \text{for all } t \geq 0. \tag{3.14}$$

However, it is well known (see, e.g., [18, 21]) that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |y(t) - y_\Delta(t)|^q\Big) \leq H\Delta^{q/2}, \tag{3.15}$$

where H is a positive constant dependent on K_R, T, x_0, q. Consequently,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |y(t \wedge \theta_{\Delta,R}) - y_\Delta(t \wedge \theta_{\Delta,R})|^q\Big) \leq H\Delta^{q/2}.$$

Using (3.13) and (3.14), we then have

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |x(t \wedge \theta_{\Delta,R}) - x_\Delta(t \wedge \theta_{\Delta,R})|^q\Big) \leq H\Delta^{q/2}, \tag{3.16}$$

which implies

$$\mathbb{E}\Big(|x(T \wedge \theta_{\Delta,R}) - x_\Delta(T \wedge \theta_{\Delta,R})|^q\Big) \leq H\Delta^{q/2}.$$

Finally

$$\begin{aligned}\mathbb{E}\Big(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R}>T\}}\Big) &= \mathbb{E}\Big(|e_\Delta(T \wedge \theta_{\Delta,R})|^q I_{\{\theta_{\Delta,R}>T\}}\Big) \\ &\leq \mathbb{E}\Big(|x(T \wedge \theta_{\Delta,R}) - x_\Delta(T \wedge \theta_{\Delta,R})|^q\Big) \leq H\Delta^{q/2}.\end{aligned}$$

This implies (3.11) as desired. The proof is therefore complete. $\Box$

4 Stronger Results with an Additional Condition

In the previous section, we showed that both truncated EM solutions $x_\Delta(T)$ and $\bar{x}_\Delta(T)$ will converge to the true solution $x(T)$ in L^q for any $T > 0$. This is sufficient for some applications e.g. when we need to approximate the European put or call option value (see, e.g., [9]). However, we sometimes need to approximate quantities that are path-dependent, for example, the European barrier option value. In these situations, we will need a stronger convergence result like

$$\lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\leq t\leq T} |x_\Delta(t) - x(t)|^q\Big) = 0. \tag{4.1}$$

For this purpose, let us impose an additional condition.

Assumption 4.1 *Assume that there is a pair of constants $r \geq 2$ and $\bar{K} > 0$ such that*

$$|g(x)|^2 \leq \bar{K}(1 + |x|^r), \quad \forall x \in \mathbb{R}^d. \tag{4.2}$$

Of course, when $r = 2$, this is the linear growth condition on g. However, our assumption allows $r > 2$. That is, we allow the diffusion coefficient g to grow faster than linearly. With this additional condition, we will be able to show that (4.1) holds for all $q < 2 + p - r$ when $p > r$. To show this, let us present a number of lemmas. Once again, we fix $T > 0$ arbitrarily in this section.

Lemma 4.2 *Let Assumptions 2.1, 2.2 and 4.1 hold and assume that $p > r$. Set $\bar{p} = 2 + p - r$. Then*

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} |x(t)|^{\bar{p}} \Big) \leq C. \tag{4.3}$$

Proof. By the Itô formula and Assumption 2.2, we can show that

$$|x(t)|^{\bar{p}} \leq |x_0|^{\bar{p}} + \int_0^t pK|x(s)|^{\bar{p}-2}(1 + x(s)|^2)ds + \int_0^t p|x(s)|^{\bar{p}-2}x^T(s)g(x(s))dB(s)$$

for all $t \in [0, T]$. Hence, by Lemma 2.3,

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} |x(t)|^{\bar{p}} \Big) \leq C + \mathbb{E}\Big(\sup_{0 \leq t \leq T} \Big| \int_0^t p|x(s)|^{\bar{p}-2}x^T(s)g(x(s))dB(s) \Big| \Big).$$

By the Burkholder–Davis–Gundy inequality (see, e.g., [21]) and Assumption 4.1 as well as Lemma 2.3, we then derive that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0 \leq t \leq T} |x(t)|^{\bar{p}} \Big) &\leq C + 4\sqrt{2}p\mathbb{E}\Big(\Big[\int_0^T |x(t)|^{2\bar{p}-2}|g(x(t))|^2 dt \Big]^{1/2} \Big) \\
&\leq C + 4\sqrt{2}p\mathbb{E}\Big(\Big[\Big(\sup_{0 \leq t \leq T} |x(t)|^{\bar{p}} \Big) \int_0^T |x(t)|^{\bar{p}-2}|g(x(t))|^2 dt \Big]^{1/2} \Big) \\
&\leq C + \frac{1}{2}\mathbb{E}\Big(\sup_{0 \leq t \leq T} |x(t)|^{\bar{p}} \Big) + 16p^2\bar{K}\mathbb{E} \int_0^T |x(t)|^{\bar{p}-2}(1 + |x(t)|^r)dt.
\end{aligned}$$

This implies

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} |x(t)|^{\bar{p}} \Big) \leq C + 32p^2\bar{K}\mathbb{E} \int_0^T |x(t)|^{\bar{p}-2}(1 + |x(t)|^r)dt.$$

Noting that $\bar{p} - 2 + r = p$, we can apply the Young inequality and then Lemma 2.3 to get

$$\mathbb{E} \int_0^T |x(t)|^{\bar{p}-2}(1 + |x(t)|^r)dt \leq C,$$

and hence the required assertion (4.3) follows. $\square$

Lemma 4.3 *Let Assumptions 2.1, 2.2 and 4.1 hold and assume that $p > r$. Set $\bar{p} = 2 + p - r$. Then*

$$\sup_{0 < \Delta \leq \Delta^*} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |x_\Delta(t)|^p \Big) \leq C. \tag{4.4}$$

Proof. Fix any $\Delta \in (0, \Delta^*]$. Using the Itô formula and Assumption 2.2, we can show in the same way as Lemma 3.2 was proved that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)|^{\bar{p}}\Big) \le C + \mathbb{E}\Big(\sup_{0\le t\le T}\Big|\int_0^t p|x_\Delta(s)|^{\bar{p}-2}x_\Delta^T(s)g(\bar{x}_\Delta(s))dB(s)\Big|\Big).$$

In the same way as Lemma 4.2 was proved, we can then show that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)|^{\bar{p}}\Big) \le C + 32p^2\mathbb{E}\int_0^T |x_\Delta(t)|^{\bar{p}-2}|g_\Delta(\bar{x}_\Delta(t))|^2dt.$$

But, the truncated function g_Δ preserves the growth condition (4.2), namely,

$$|g_\Delta(x)|^2 \le \bar{K}(1+|x|^r), \quad \forall x \in \mathbb{R}^d.$$

We then have

$$\mathbb{E}\int_0^T |x_\Delta(t)|^{\bar{p}-2}|g_\Delta(\bar{x}_\Delta(t))|^2dt \le \bar{K}\mathbb{E}\int_0^T |x_\Delta(t)|^{\bar{p}-2}(1+|\bar{x}_\Delta(t)|^r)dt.$$

By Lemma 3.2, it is straightforward to show

$$\mathbb{E}\int_0^T |x_\Delta(t)|^{\bar{p}-2}(1+|\bar{x}_\Delta(t)|^r)dt \le C.$$

We hence have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)|^{\bar{p}}\Big) \le C$$

as required. □

Theorem 4.4 *Let Assumptions 2.1, 2.2 and 4.1 hold and assume that $p > r$. Set $\bar{p} = 2 + p - r$. Then, for any $q \in [2, \bar{p})$,*

$$\lim_{\Delta\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t) - x(t)|^q\Big) = 0. \tag{4.5}$$

Proof. We use the same notation as in the proof of Theorem 3.5. Using the Young inequality, we can show that for any $\delta > 0$,

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big)\\ \le\ &\mathbb{E}\Big(I_{\{\theta_{\Delta,R}>T\}}\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) + \frac{q\delta}{p}\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) + \frac{p-q}{p\delta^{q/(p-q)}}\mathbb{P}(\theta_{\Delta,R}\le T). \end{aligned}\tag{4.6}$$

By Lemmas 4.2, 4.3, 3.3 and 3.4, we can then have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) \le \mathbb{E}\Big(I_{\{\theta_{\Delta,R}>T\}}\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) + \frac{Cq\delta}{p} + \frac{C(p-q)}{pR^2\delta^{q/(p-q)}}. \tag{4.7}$$

But, by (3.16),

$$\mathbb{E}\Big(I_{\{\theta_{\Delta,R}>T\}}\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T}|x(t\wedge\theta_{\Delta,R}) - x_\Delta(t\wedge\theta_{\Delta,R})|^q\Big) \le H\Delta^{q/2}.$$

We therefore have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) \le H\Delta^{q/2} + \frac{Cq\delta}{p} + \frac{C(p-q)}{pR^2\delta^{q/(p-q)}}. \tag{4.8}$$

Now, for any $\varepsilon > 0$, we first choose δ sufficiently small for $Cq\delta/p \le \varepsilon/3$ and then choose R sufficiently large for

$$\frac{C(p-q)}{pR^2\delta^{q/(p-q)}} \le \frac{\varepsilon}{3},$$

and further then choose Δ sufficiently small for $H\Delta^{q/2} \le \varepsilon/3$ to get that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|e_\Delta(t)|^q\Big) \le \varepsilon$$

for all sufficiently small Δ. That is, we have proved the required assertion (4.5). □

We observe that it is much easier to compute $\bar{x}_\Delta(t)$ than $x_\Delta(t)$ in practice. It is therefore more desirable to have

$$\lim_{\Delta\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{x}_\Delta(t)-x(t)|^q\Big) = 0.$$

For this purpose, let us present another lemma.

Lemma 4.5 *Let $q \ge 2$ and $\Delta \in (0, \Delta^*]$. Let n be a sufficiently large integer for which*

$$\Big(\frac{2n}{2n-1}\Big)^q (T+1)^{q/2n} \le 2 \quad \textit{and} \quad \frac{n-1}{2n} > \frac{1}{3}. \tag{4.9}$$

We then have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar{x}_\Delta(t)|^q\Big) \le 2^{q+1}n^{q/2}(h(\Delta))^q\Delta^{q(n-1)/2n}. \tag{4.10}$$

Consequently

$$\lim_{\Delta\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar{x}_\Delta(t)|^q\Big) = 0. \tag{4.11}$$

Proof. Let N be the integer part of T/Δ. Then

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar{x}_\Delta(t)|^q\Big)\\
\le\ & \mathbb{E}\Big(\max_{0\le k\le N}\sup_{t_k\le t\le t_{k+1}}|f_\Delta(\bar{x}_\Delta(t_k))(t-t_k) + g_\Delta(\bar{x}_\Delta(t_k))(B(t)-B(t_k))|^q\Big)\\
\le\ & 2^{q-1}\mathbb{E}\Big(\max_{0\le k\le N}\sup_{t_k\le t\le t_{k+1}}|f_\Delta(\bar{x}_\Delta(t_k))|^q(t-t_k)^q + |g_\Delta(\bar{x}_\Delta(t_k))|^q|B(t)-B(t_k)|^q\Big)\\
\le\ & 2^{q-1}(h(\Delta))^q\Big[\Delta^q + \mathbb{E}\Big(\max_{0\le k\le N}\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^q\Big)\Big].
\end{aligned} \tag{4.12}$$

By the Hölder inquality and the Doob martingale inequality, we then derive that

$$\begin{aligned}
&\mathbb{E}\Big(\max_{0\le k\le N}\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^q\Big)\\
\le\ &\Big[\mathbb{E}\Big(\max_{0\le k\le N}\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big)\Big]^{q/2n}\\
\le\ &\Big[\sum_{k=0}^{N}\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big)\Big]^{q/2n}\\
\le\ &\Big[\sum_{k=0}^{N}\Big(\frac{2n}{2n-1}\Big)^{2n}\mathbb{E}|B(t_{k+1})-B(t_k)|^{2n}\Big]^{q/2n}\\
\le\ &\Big[\sum_{k=0}^{N}\Big(\frac{2n}{2n-1}\Big)^{2n}(2n-1)!!\Delta^{n}\Big]^{q/2n}\\
\le\ &\Big[\Big(\frac{2n}{2n-1}\Big)^{2n}(T+1)(2n-1)!!\Delta^{n-1}\Big]^{q/2n},
\end{aligned} \tag{4.13}$$

where $(2n-1)!!=(2n-1)\times(2n-3)\times\cdots\times 3\times 1$. But

$$[(2n-1)!!]^{1/n}\le\frac{1}{n}\sum_{i=1}^{n}(2i-1)=n.$$

Using (4.9), we then derive from (4.13) that

$$\begin{aligned}
\mathbb{E}\Big(\max_{0\le k\le N}\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^q\Big) &\le n^{q/2}\Big(\frac{2n}{2n-1}\Big)^{q}(T+1)^{q/2n}\Delta^{q(n-1)/2n}\\
&\le 2n^{q/2}\Delta^{q(n-1)/2n}.
\end{aligned} \tag{4.14}$$

Substituting this into (4.12) yields the required assertion (4.10). Finally, by (4.9) and (2.8),

$$(h(\Delta))^q\Delta^{q(n-1)/2n}\le(h(\Delta))^q\Delta^{q/3}=\Delta^{q/12}(\Delta^{1/4}h(\Delta))^q\le\Delta^{q/12}.$$

We hence obtain the other assertion (4.11) from (4.10). The proof is complete. □

The following theorem now follows from Theorem 4.4 and Lemma 4.5 immediately.

Theorem 4.6 *Let Assumptions 2.1, 2.2 and 4.1 hold and assume that $p>r$. Set $\bar p=2+p-r$. Then, for any $q\in[2,\bar p)$,*

$$\lim_{\Delta\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|\bar x_\Delta(t)-x(t)|^q\Big)=0. \tag{4.15}$$

5 Corollaries and Examples

In this section we will demonstrate that Assumptions 2.2 and 4.1 cover many SDEs in various branches of science and industry and hence our new truncated EM method is applicable in many areas. Let us first recall the conditions that are frequently used in the study of numerical solutions of SDEs. They are the one-sided linear growth condition on the drift coefficient f and the linear growth condition on the diffusion coefficient g (see, e.g., [10, 13]). To be precise, let us state them as an assumption.

Assumption 5.1 *Assume that there are two positive constants K_1 and K_2 such that*

$$x^T f(x) \le K_1(1+|x|^2) \quad \text{and} \quad |g(x)|^2 \le K_2(1+|x|^2) \tag{5.1}$$

for all $x \in \mathbb{R}^d$.

Under this assumption, for any $p > 2$, we have

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le (K_1 + 0.5(p-1)K_2)(1+|x|^2)$$

for all $x \in \mathbb{R}^d$. We therefore see that Assumptions 2.2 and 4.1 are satisfied with $K = K_1 + 0.5(p-1)K_2$, $\bar{K} = K_2$ and $r = 2$. The following corollary follows from Theorems 4.4 and 4.6 immediately.

Corollary 5.2 *Let Assumptions 2.1 and 5.1 hold. Then, for any $q \ge 2$,*

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \le t \le T} |x_\Delta(t) - x(t)|^q \Big) = 0 \quad \text{and} \quad \lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \le t \le T} |\bar{x}_\Delta(t) - x(t)|^q \Big) = 0. \tag{5.2}$$

We next introduce a condition which covers the SDEs like the scalar equations

$$dx(t) = (x(t) + x^2(t) - x^5(t))dt + x^2(t)dB(t), \tag{5.3}$$

or

$$dx(t) = (x(t) + x^2(t) - 2x^3(t))dt + x^2(t)dB(t). \tag{5.4}$$

Assumption 5.3 *Assume that there are three constants $\rho > 2$ and $K_1, K_2 > 0$ such that*

$$x^T f(x) \le K_1(1+|x|^2) - K_2|x|^\rho, \quad \forall x \in \mathbb{R}^d. \tag{5.5}$$

Under Assumptions 4.1 and 5.3, let us consider two cases:

Case (i) when $\rho > r$. In this case, for any $p > 2$, we have

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K_1(1+|x|^2) - K_2|x|^\rho + \frac{p-1}{2}\ \bar{K}(1+|x|^r).$$

But $-K_2|x|^\rho + \frac{p-1}{2}\ \bar{K}(1+|x|^r)$ is bounded above by a positive constant, say K_3. So

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le (K_1 + K_3)(1+|x|^2).$$

This shows that Assumption 2.2 is satisfied for any $p > 2$.

Case (ii) when $\rho = r$. In this case, we need to assume that $2K_2/\bar{K} > r-1$ additionally. Let $p = 1 + 2K_2/\bar{K}$. Then $p > r$ and, moreover, we have

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K_1(1+|x|^2) + K_2 \le (K_1 + K_2)(1+|x|^2).$$

This shows that Assumption 2.2 is satisfied for $p = 1 + 2K_2/\bar{K}$. The following corollary hence follows from Theorems 4.4 and 4.6 again.

Corollary 5.4 *Let Assumptions 2.1, 4.1 and 5.3 hold.*

(i) If $\rho > r$, then the assertions in (5.2) hold for any $q \geq 2$.

(ii) If $\rho = r$ and $2K_2/\bar{K} > r - 1$, then the assertions in (5.2) hold for any $2 \leq q < 3 + 2K_2/\bar{K} - r$.

Recalling the SDEs (2.5) and (2.6), we see that they are in a similar fashion as the SDEs (5.3) and (5.4). However, the SDEs (2.5) and (2.6) are those that model either the financial quantities or population sizes so their states take nonnegative numbers, that is, these SDEs are in the nonnegative cone $\mathbb{R}^n_+ = \{(x_1, \cdots, x_d)^T \in \mathbb{R}^d : x_i \geq 0,\ 1 \leq i \leq d\}$. On the other hand, both SDEs (5.3) and (5.4) are in the whole real space $\mathbb{R}$. In fact, our theory so far works for SDEs in the whole $\mathbb{R}^d$. We now explain our theory can be applied to the SDEs in $\mathbb{R}^d_+$ as well. We use the SDEs (2.5) and (2.6) as the examples. In the following examples, $B(t)$ will be a scalar Brownian motion.

Example 5.5 We first consider the SDE (2.5) under the condition $\beta + 1 \geq 2\theta$. We claim that for any initial value $x(0) > 0$, there is a unique global solution $x(t)$ to the SDE (2.5) and the solution will remain to be positive with probability one. In fact, define a C^2-function $V : (0, \infty) \to \mathbb{R}_+$ by

$$V(x) = x - 1 - \log(x).$$

It is easy to show that, for $x \in (0, \infty)$,

$$\begin{aligned} & V'(x)(\mu - \alpha x^\beta) + 0.5V''(x)\sigma^2 x^{2\theta} \\ = \; & \mu - \mu x^{-1} - \alpha x^\beta + \alpha x^{\beta-1} + 0.5\sigma^2 x^{2\theta-2}, \end{aligned}$$

which is bounded above by a constant. From here it is almost standard to show what we have just claimed (see the proof of [21, Theorem 2.1] on pages 381–384).

We may therefore write the SDE (2.5) as equation

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{5.6}$$

in $\mathbb{R}$ by extending the definitions of the coefficients f and g from $\mathbb{R}_+$ to $\mathbb{R}$ as follows

$$f(x) = \begin{cases} \mu - \alpha x^\beta & \text{if } x \geq 0, \\ \mu & \text{if } x < 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sigma x^\theta & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Obviously, both f and g are locally Lipschitz continuous in $\mathbb{R}$. Moreover, Assumption 4.1 is satisfied with $\bar{K} = \sigma^2$ and $r = 2\theta$. To verify Assmmption 2.2, we consider two cases:

Case (i) when $\beta + 1 > 2\theta$. In this case, for any $p > 2$, we have that, for $x \geq 0$,

$$xf(x) + \frac{p-1}{2}|g(x)|^2 = \mu x - \beta x^{\beta+1} + \frac{(p-1)\sigma^2}{2} x^{2\theta},$$

which is bounded above by a positive constant, say K. In other words,

$$xf(x) + \frac{p-1}{2}|g(x)|^2 \leq K \quad \text{for } x \geq 0.$$

On the other hand, $xf(x) + \frac{p-1}{2}|g(x)|^2 = \mu x \leq 0$ for $x < 0$. So we always have

$$xf(x) + \frac{p-1}{2}|g(x)|^2 \leq K, \quad \forall x \in \mathbb{R}.$$

This shows that Assumption 2.2 is satisfied for any $p > 2$. By Theorems 4.4 and 4.6, we can therefore conclude that the assertions in (5.2) hold for any $q \geq 2$, where $x(t)$ in (5.2) now means the solution of equation (2.5) and $x_\Delta(t)$ and $\bar{x}_\Delta(t)$ stand for the truncated EM solutions of equation (5.6).

Case (ii) when $\beta + 1 = 2\theta$. In this case, we need to assume that $2\beta > \sigma^2(2\theta - 1)$ additionally. Let $p = 1 + 2\beta/\sigma^2$. Then $p > 2\theta > 2$. Moreover, we can show easily that

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \leq \mu(1 + |x|^2).$$

In other words, Assumption 2.2 is satisfied for $p = 1 + 2\beta/\sigma^2$. By Theorems 4.4 and 4.6, we can therefore conclude that the assertions in (5.2) hold for any $2 \leq q < 3 + 2\beta/\sigma^2 - 2\theta$.

Example 5.6 Let us now consider the SDE (2.6), namely the SDE

$$dx(t) = F(x(t))dt + G(x(t))dB(t) \tag{5.7}$$

in $\mathbb{R}^d_+$, where $F, G : \mathbb{R}^d_+ \to \mathbb{R}^d$ are defined by

$$F(x) = \mathrm{diag}(x_1, x_2, ..., x_d)(b + Ax^2) \quad \text{and} \quad G(x) = \mathrm{diag}(x_1, x_2, ..., x_d)Cx$$

for $x \in \mathbb{R}^d$. It is known (see, e.g., [3]) that for any initial value $x(0) \in \mathbb{R}^d_+$, the solution $x(t)$ of the SDE (5.7) will remain to be in $\mathbb{R}^d_+$ with probability one. We may therefore write the SDE (5.7) as the following equation

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{5.8}$$

in $\mathbb{R}^d$ by extending the definitions of the coefficients from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ as follows

$$f(x) = \begin{cases} F(x) & \text{if } x \in \mathbb{R}^d_+, \\ F(\hat{x}) & \text{if } x \notin \mathbb{R}^d_+, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} G(x) & \text{if } x \in \mathbb{R}^d_+, \\ G(\hat{x}) & \text{if } x \notin \mathbb{R}^d_+, \end{cases}$$

where $\hat{x} = (x_1 \vee 0, x_2 \vee 0, \cdots, x_d \vee 0)$ for $x \notin \mathbb{R}^d_+$. Obviously, both f and g are locally Lipschitz continuous in $\mathbb{R}^d$. Note that for $x \in \mathbb{R}^d_+$,

$$|g(x)|^2 = |G(x)|^2 = x^T C^T \mathrm{diag}(x_1^2, x_2^2, ..., x_d^2)Cx \leq |x|^2 x^T C^T Cx \leq \lambda_{\max}(C^T C)|x|^4,$$

while for $x \notin \mathbb{R}^d_+$,

$$|g(x)|^2 = |G(\hat{x})|^2 \leq \lambda_{\max}(C^T C)|\hat{x}|^4 \leq \lambda_{\max}(C^T C)|x|^4.$$

We hence always have that

$$|g(x)|^2 \leq \lambda_{\max}(C^T C)|x|^4, \quad \forall x \in \mathbb{R}^d. \tag{5.9}$$

This means that Assumption 4.1 is satisfied with $\bar{K} = \lambda_{\max}(C^T C)$ and $r = 4$. To fulfil Assumption 2.2, we assume that

$$-\lambda_{\max}(A + A^T) > 3d\lambda_{\max}(C^T C). \tag{5.10}$$

Letting

$$p = 1 - \frac{\lambda_{\max}(A + A^T)}{d\lambda_{\max}(C^T C)}, \tag{5.11}$$

we have $p > 4$. Recalling the notation $x^2 = (x_1^2, \cdots, x_d^2)^T$ and setting $\bar{b} = \max_{1\le i\le d} |b_i|$, we derive that for $x \in \mathbb{R}_+^d$,

$$x^T f(x) = x^T F(x) = (x^2)^T (b + A)x^2 \le \bar{b}|x|^2 + \frac{1}{2}\lambda_{\max}(A + A^T)|x^2|^2,$$

while for $x \notin \mathbb{R}_+^d$,

$$x^T f(x) = x^T F(\hat{x}) = (\hat{x}^2)^T (b + A)\hat{x}^2 \le \bar{b}|\hat{x}|^2 + \frac{1}{2}\lambda_{\max}(A + A^T)|\hat{x}^2|^2.$$

Observing that $|\hat{x}|^2 \le |x|^2$ and $|\hat{x}^2|^2 \le |x^2|^2$, we therefore see that

$$x^T f(x) \le \bar{b}|x|^2 + \frac{1}{2}\lambda_{\max}(A + A^T)|x^2|^2, \quad \forall x \in \mathbb{R}^d.$$

But it is easy to show that $|x|^4 \le d|x^2|^2$. Conseqnently

$$x^T f(x) \le \bar{b}|x|^2 + \frac{1}{2d}\lambda_{\max}(A + A^T)|x|^4, \quad \forall x \in \mathbb{R}^d. \tag{5.12}$$

Combining (5.9) and (5.12), we get that

$$x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le \bar{b}|x|^2 + \frac{1}{2d}\lambda_{\max}(A + A^T)|x|^4 + \frac{p-1}{2}\lambda_{\max}(C^T C)|x|^4 = \bar{b}|x|^2$$

for all $x \in \mathbb{R}^d$. That is, Assumption 2.2 is satisfied. By Theorems 4.4 and 4.6, we can therefore conclude that the assertions in (5.2) hold for any $2 \le q < p - 2$ if condition (5.10) holds.

6 Conclusions

In this paper we have developed a new explicit method, called the truncated EM method, for the nonlinear SDE $dx(t) = f(x(t))dt + g(x(t))dB(t)$. For a given stepsize Δ, we define the discrete-time truncated EM numerical solution and then form two versions of the continuous-time truncated EM solutions, namely the continuous-time step-process truncated EM solution $\bar{x}_\Delta(t)$ and the continuous-time continuous-sample truncated EM solution $x_\Delta(t)$. Under the local Lipschitz condition plus the Khasminskii-type condition $x^T f(x) + \frac{p-1}{2}|g(x)|^2 \le K(1 + |x|^2)$ for some $p > 2$, we have successfully shown the strong convergence of both continuous-time truncated EM solutions to the true solution in the sense that

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(T) - x(T)|^q = 0 \quad \text{and} \quad \lim_{\Delta\to 0} \mathbb{E}|\bar{x}_\Delta(T) - x(T)|^q = 0$$

for any $T > 0$ and $2 \le q < p$. Moreover, with another additional condition on the diffusion coefficient, namely $|g(x)|^2 \le \bar{K}(1 + |x|^r)$ for some $r \in [2, p)$, we have shown the stronger convergence results in the sense that

$$\lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t) - x(t)|^q\Big) = 0 \quad \text{and} \quad \lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\le t\le T} |\bar{x}_\Delta(t) - x(t)|^q\Big) = 0$$

for any $T > 0$ and $2 \le q < 2 + p - r$.

It is interesting to show an order of strong-L^q convergence for the truncated EM method under these conditions. However, we will report the results on the convergence rate in another paper due to the page limit here.

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References

[1] Anderson D. and Mattingly J., *A weak trapezoidal method for a class of stochastic differential equations*. Commun. Math. Sci. 9(1) (2011), 301-318.

[2] Appleby J. A.D., Guzowska M., Kelly C. and Rodkina A., *Preserving positivity in solutions of discretised stochastic differential equations*. Appl. Math. Comput. 217 (2010), 763-774.

[3] Bahar A. and Mao X., *Stochastic delay population denamics*. Journal of International Applied Math. 11(4)(2004), 377-400.

[4] Burrage K. and Tian T., *Predictor-corrector methods of Runge-Kutta type for stochastic differential equations*. SIAM J. Numer. Anal. 40(4) (2002), 1516-1537.

[5] Burrage P. M. and Burrage K., *A variable stepsize implementation for stochastic differential equations*. SIAM J. Sci. Comput. 24(3)(2002), 848-864.

[6] Gan S. and Wang X., *The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients*. J. Differ. Equ. Appl., DOI:10.1080/10236198.2012.656617.

[7] Gyöngy I., *A note on Euler's approximations*. Potential Anal. 8(3) (1998), 205-216.

[8] Higham D.J., *Stochastic ordinary differential equations in applied and computational mathematics*. IMA J. Appl. Math. 76 (2011), 449-474.

[9] Higham D.J. and Mao X., *Convergence of Monte Carlo simulations involving the mean-reverting square root process*. Journal of Computational Finance, 8(3) (2005), 35-62.

[10] Higham D.J., Mao X. and Stuart A., *Strong convergence of Euler-type methods for nonlinear stochastic differential equations*. SIAM J. Numer. Anal. 40(3) (2003), 1041-1063.

[11] Hu Y., *Semi-implicit Euler-Maruyama scheme for stiff stochastic equations*. Stochastic Analysis and Related Topics, V (Silivri, 1994), 183-202, Progr. Probab., 38, Birkhäuser Boston, Boston, MA, 1996.

[12] Hutzenthaler M. and Jentzen A., *Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients*, arXiv:1203.5809.

[13] Hutzenthaler M., Jentzen A. and Kloeden P., *Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients*. Proc. R. Soc. A 467 (2011), 1563-1576.

[14] Hutzenthaler M., Jentzen A. and Kloeden P., *Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients*. Ann. Appl. Probab. 22(2) (2012), 1611-1641.

[15] Khasminskii R.Z. , *Stochastic Stability of Differential Equations*, Alphen: Sijtjoff and Noordhoff, 1980. (Translation of the Russian edition, Moscow, Nauka 1969).

[16] Jentzen A., Kloeden P. and Neuenkirch A., *Pathwise approximation of stochastic differential equations on domains: higher order convergence rates without global Lipschitz coefficients*. Numer. Math. 112(1)(2009), 41-64.

[17] Klauder J.R., Petersen W.P., *Numerical integration of multiplicative-noise stochastic differential equations*. SIAM J. Numer. Anal. 22(1985), 1153-1166.

[18] Kloeden P. E. and Platen E., *Numerical Solution of Stochastic Differential Equations*, Springer-Verlog, Berlin, 1992.

[19] Lewis A.L., *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press, 2000.

[20] Liu W. and Mao X. *Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations*. Applied Mathematics and Computation 223 (2013), 389–400.

[21] Mao X., *Stochastic Differential Equations and Applications*, 2nd Edition, Horwood, Chichester, UK, 2007.

[22] Mao X., *Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions*. Appl. Math. Comput. 217(2011), 5512-5524.

[23] Mao X. and Szpruch L., *Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients*, Journal of Computational and Applied Mathematics 238 (2013), 14–28.

[24] Marion G., Mao X. and Renshaw E., *Convergence of the Euler scheme for a class of stochastic differential equations*. Int. Math. J. 1(2002), 9-22.

[25] Milstein G. N., Platen E. and Schurz H., *Balanced implicit methods for stiff stochastic system*. SIAM J. Numer. Anal. 35(1998), 1010-1019.

[26] Milstein G. N. and Tretyakov M. V., *Stochastic Numerics for Mathematical Physics*, Springer-Verlag, Berlin, 2004.

[27] Milstein G. N. and Tretyakov M. V., *Numerical integration of stochastic differential equations with nonglobally Lipschitz coefficients*. SIAM J. Numer. Anal. 43(2005), 1139-1154.

[28] Müller-Gronbach T., *The optimal uniform approximation of systems of stochastic differential equations*. Ann. Appl. Probab. 12(2)(2002), 664-690.

[29] Oksendal B., *Stochastic Differential Equations: an Introduction with Applications*, 6th Edition, Springer, Berlin, 2003.

[30] Saito Y. and Mitsui T., *T-stability of numerical scheme for stochastic differential equations*. World Sci. Ser. Appl. Anal. 2(1993), 333-344.

[31] Song M., Hu L., Mao X. and Zhang L., *Khasminskii-Type theorems for stochastic functional differential equations*. Discrete and Continuous Dynamical Systems B 18(6) (2013), 1697-1714.

[32] Sabanis S., *A note on tamed Euler approximations*. Electron. Commun. Probab. 18(47) (2013), 1-10.

[33] Szpruch L., Mao X., Higham D. and Pan J., *Numerical simulation of a strongly nonlinear Ait-Sahalia type interest rate model*. BIT Numerical Mathematics 51(2011), 405–425.

[34] Tretyakov M.V. and Zhang Z., *A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications*. SIAM Journal of Numerical Analysis 51 (2013), 3135-3162.

[35] Valinejad A. and Hosseini S. M., *A variable step-size control algorithm for the weak approximation of stochastic differential equations*. Numer. Algorithms 55(4)(2010), 429-446.

[36] Wu F., Mao X. and Chen K., *A highly sensitive mean-reverting process in finance and the Euler-Maruyama approximations*. J. Math. Anal. Appl. 348(2008), 540-554.

[37] Werner R. and Renate W., *Stepsize control for mean-square numerical methods for stochastic differential equations with small noise*. SIAM J. Sci. Comput. 28(2)(2006), 604-625.