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Stochastic delay Lotka–Volterra model

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Abstract

We reveal in this paper that the environmental noise will not only suppress a potential population explosion in the stochastic delay Lotka–Volterra model but will also make the solutions to be stochastically ultimately bounded. To reveal these interesting facts, we stochastically perturb the delay Lotka–Volterra model $\dot{x}(t) = \mathrm{diag}(x_1(t), \dots, x_n(t))[b + Ax(t-\tau)]$ into the Itô form $dx(t) = \mathrm{diag}(x_1(t), \dots, x_n(t))[(b + Ax(t-\tau))\,dt + \sigma x(t)\,dw(t)]$, and show that although the solution to the original delay equation may explode to infinity in a finite time, with probability one that of the associated stochastic delay equation does not. We also show that the solution of the stochastic equation will be stochastically ultimately bounded without any additional condition on the matrix A.
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1. Introduction

The delay differential equation

$$\frac{dx(t)}{dt} = x(t)\big[b - ax(t-\tau)\big]$$

has been used to model the population growth of certain species and is known as the delay Lotka–Volterra model or the delay logistic equation. The delay Lotka–Volterra model for n interacting species is described by the n-dimensional delay differential equation

$$\frac{dx(t)}{dt} = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\big[b + Ax(t-\tau)\big], \tag{1.1}$$

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where

$$x = (x_1, \ldots, x_n)^T, \qquad b = (b_1, \ldots, b_n)^T, \qquad A = (a_{ij})_{n\times n}.$$

There is an extensive literature concerned with the dynamics of this delay model and we here only mention Ahmad and Rao [1], Bereketoglu and Gyori [2], Freedman and Ruan [3], He and Gopalsamy [5], Kuang and Smith [8], Teng and Yu [13] among many others. In particular, the books by Gopalsamy [4], Kolmanovskii and Myshkis [6] as well as Kuang [7] are good references in this area.

In Eq. (1.1), the vector $x = (x_1, \ldots, x_n)^T$ denotes the population sizes of the n species. We are therefore not only interested in the positive solutions but also require the solutions not to explode at a finite time. To guarantee the positive solutions without explosion (i.e., the global positive solutions), some conditions are in general needed to impose on the matrix A. For example, He and Gopalsamy [5] discussed the following delay Lotka–Volterra model for 2 interacting species

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[b_1 - a_{11}x_1(t-\tau) + a_{12}x_2(t-\tau)], \\ \frac{dx_2(t)}{dt} = x_2(t)[b_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t-\tau)], \end{cases} \tag{1.2}$$

where all the parameters b_i and a_{ij} are positive. The type of ecological interaction corresponding to this equation is known as facultative mutualism; that is, each species enhances the average growth rate of the other although each species can survive in the absence of the other. To avoid explosion in this model, He and Gopalsamy [5] imposed the condition that $a_{11}a_{22} > a_{12}a_{21}$. The conditions for nonexplosion in the delay mutualise systems for multi interacting species become much more complicated (see, e.g., [4,6,7]) and the research in this area is still going on.

On the other hand, population systems are often subject to environmental noise. Moreover, Mao et al. [12] has recently revealed an important fact that the environmental noise can suppress a potential population explosion. We therefore wonder if the explosion problem for the delay Lotka–Volterra model (1.1) can be avoided by taking the environmental noise into account instead of imposing conditions on the matrix A. One of the aims of this paper is to show that the presence of even a tiny amount of noise can suppress a potential population explosion in this delay model.

To reveal this interesting fact, we stochastically perturb the delay Lotka–Volterra model (1.1) into the Itô stochastic differential delay equation

$$dx(t) = \operatorname{diag}\big(x_1(t), \ldots, x_n(t)\big)\big[\big(b + Ax(t-\tau)\big)\,dt + \sigma x(t)\,dw(t)\big], \tag{1.3}$$

where $w(t)$ is a Brownian motion and $\sigma = (\sigma_{ij})_{n\times n}$ is a matrix representing the intensity of noise. To highlight the presence of noise, it is natural to assume that

$$\sigma_{ii} > 0 \quad \text{if } 1 \leqslant i \leqslant n \quad \text{whilst} \quad \sigma_{ij} \geqslant 0 \quad \text{if } i \neq j. \tag{1.4}$$

In this paper we shall show that under this simple hypothesis on noise, for any $b \in R^n$ and $A \in R^{n\times n}$, the solution of the stochastic differential delay equation (1.4) will remain in the positive cone R^n_+ with probability 1 and, in particular, it will not explode to infinity in a finite time. This makes the stochastic delay Lotka–Volterra model (1.3) significantly different from its original delay equation (1.1) in the sense we need no longer to impose any condition on the matrix A to avoid the explosion. For example, consider the corresponding stochastic version of Eq. (1.2),

$$\begin{cases} dx_1(t) = x_1(t)\{[b_1 - a_{11}x_1(t-\tau) + a_{12}x_2(t-\tau)]\,dt + \varepsilon x_1(t)\,dw(t)\}, \\ dx_2(t) = x_2(t)\{[b_2 + a_{21}x_1(t-\tau) - a_{22}x_2(t-\tau)]\,dt + \varepsilon x_2(t)\,dw(t)\}. \end{cases} \tag{1.5}$$

Our theory shows that for arbitrary parameters a_{ij}, the solution of Eq. (1.5) will not explode to infinity in a finite time with probability one as long as there is a noise (i.e., $\varepsilon > 0$) but its original equation (1.2) will require some conditions on the parameters, e.g., $a_{11}a_{22} > a_{12}a_{21}$.

In a population dynamical system, the nonexplosion property is often not good enough but the property of ultimate boundedness is more desired. The conditions for the ultimate boundedness are much more complicated than the conditions for the nonexplosion (see, e.g., [3–5,7,13]) and lots of research are still going on. Naturally, when we study the stochastic delay Lotka–Volterra model (1.3) we would like to find out under what conditions the solutions will be stochastically ultimately bounded. In the first instance, one may feel that we will need some additional conditions on the matrices A and σ. However, in this paper we shall show that the simple hypothesis (1.4) on the noise is enough to guarantee the stochastically ultimate boundedness of the solutions of the stochastic delay Lotka–Volterra model (1.3).

In summary, we reveal in this paper that the environmental noise will not only suppress a potential population explosion in the stochastic delay Lotka–Volterra model (1.3) but will also make the solutions to be stochastically ultimately bounded. The significant contributions of this paper are therefore clear.

2. Positive and global solutions

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). Let $w(t)$ denote one-dimensional Brownian motion defined on this probability space. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|Ax|\colon |x| = 1\}$. We also denote by R^n_+ the positive cone in R^n, that is $R^n_+ = \{x \in R^n\colon x_i > 0 \text{ for all } 1 \leqslant i \leqslant n\}$. Moreover, let $\tau > 0$ and denote by $C([-\tau, 0]; R^n_+)$ the family of continuous functions from $[-\tau, 0]$ to R^n_+.

Consider the delay Lotka–Volterra model for a system with n interacting species, namely

$$\dot{x}_i(t) = x_i(t)\left(b_i + \sum_{j=1}^n a_{ij}x_j(t-\tau)\right) \quad (1 \leqslant i \leqslant n)$$

on $t \geqslant 0$. This takes the matrix form

$$\dot{x}(t) = \text{diag}\big(x_1(t), \ldots, x_n(t)\big)\big[b + Ax(t-\tau)\big], \tag{2.1}$$

where

$$x = (x_1, \dots, x_n)^T, \qquad b = (b_1, \dots, b_n)^T, \qquad A = (a_{ij})_{n\times n}.$$

Given initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, Eq. (2.1) will have a unique local or global solution dependent on the parameters b and A.

In this classical model, the parameter b_i represents the intrinsic growth rate of species i. If one replaces this rate by an average value plus a random fluctuation term

$$b_i + \sum_{j=1}^{n} \sigma_{ij} x_j(t)\dot{w}(t),$$

then Eq. (2.1) becomes a stochastic differential delay equation

$$dx(t) = \operatorname{diag}\big(x_1(t), \dots, x_n(t)\big)\big[\big(b + Ax(t-\tau)\big)\,dt + \sigma x(t)\,dw(t)\big], \tag{2.2}$$

where $\sigma = (\sigma_{ij})_{n\times n}$. Since the purpose of this paper is to discover the effect of environmental noise, we naturally impose the following simple hypothesis on the noise intensities:

(H) $\sigma_{ii} > 0$ if $1 \leqslant i \leqslant n$ whilst $\sigma_{ij} \geqslant 0$ if $i \neq j$.

As the ith state $x_i(t)$ of Eq. (2.2) is the size of the ith species in the system, it should be nonnegative. Moreover, in order for a stochastic differential delay equation to have a unique global (i.e., no explosion in a finite time) solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Mao [9,11]). However, the coefficients of Eq. (2.2) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of Eq. (2.2) may explode at a finite time. In this section we shall show that under simple hypothesis (H) the solution of Eq. (2.2) is not only positive but will also not explode to infinity at any finite time.

Theorem 2.1. *Under hypothesis* (H), *for any system parameters $b \in R^n$ and $A \in R^{n\times n}$, and any given initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, there is a unique solution $x(t)$ to Eq.* (2.2) *on $t \geqslant -\tau$ and the solution will remain in R^n_+ with probability* 1, *namely $x(t) \in R^n_+$ for all $t \geqslant -\tau$ almost surely.*

Before we prove this theorem, let us emphasize the important feature of this theorem. It is well known that Eq. (2.1) may explode to infinity at a finite time for some system parameters $b \in R^n$ and $A \in R^{n\times n}$. However, the explosion will no longer happen as long as there is a noise. In other words, this result reveals the important property that the environmental noise suppresses the explosion for the delay equation.

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$ there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time (cf. Mao [10, p. 95]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau\leqslant t\leqslant 0} \big|x(t)\big| \leqslant \max_{-\tau\leqslant t\leqslant 0} \big|x(t)\big| < k_0.$$

For each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\bigl\{t \in [0, \tau_e):\ x_i(t) \notin (1/k, k) \text{ for some } i = 1, \dots, n\bigr\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+^n$ a.s. for all $t \geqslant 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. To show this statement, let us define a C^2-function $V : R_+^n \to R_+$ by

$$V(x) = \sum_{i=1}^{n} \bigl[\sqrt{x_i} - 1 - 0.5\log(x_i)\bigr].$$

The nonnegativity of this function can be seen from

$$\sqrt{u} - 1 - 0.5\log(u) \geqslant 0 \quad \text{on } u > 0.$$

Let $k \geqslant k_0$ and $T > 0$ be arbitrary. For $0 \leqslant t \leqslant \tau_k \wedge T$, we can apply the Itô formula to $\int_{t-\tau}^{t} |x(s)|^2\, ds + V(x(t))$ to obtain that

$$\begin{aligned}
d\Biggl[\int_{t-\tau}^{t} \bigl|x(s)\bigr|^2\, ds + V\bigl(x(t)\bigr)\Biggr]
&= \bigl[\bigl|x(t)\bigr|^2 - \bigl|x(t-\tau)\bigr|^2\bigr]\, dt + \sum_{i=1}^{n} \Biggl\{0.5\bigl[x_i^{-0.5}(t) - x_i^{-1}(t)\bigr] x_i(t) \\
&\quad \times \Biggl[\Biggl(b_i + \sum_{j=1}^{n} a_{ij} x_j(t-\tau)\Biggr) dt + \sum_{j=1}^{n} \sigma_{ij} x_j(t)\, dw(t)\Biggr] \\
&\quad + 0.5\bigl[-0.25 x_i^{-1.5}(t) + 0.5 x_i^{-2}(t)\bigr] x_i^2(t) \Biggl[\sum_{j=1}^{n} \sigma_{ij} x_j(t)\Biggr]^2 dt\Biggr\} \\
&= \Biggl\{\bigl|x(t)\bigr|^2 - \bigl|x(t-\tau)\bigr|^2 + \sum_{i=1}^{n} 0.5\bigl[x_i^{0.5}(t) - 1\bigr]\Biggl(b_i + \sum_{j=1}^{n} a_{ij} x_j(t-\tau)\Biggr) \\
&\quad + \sum_{i=1}^{n} \bigl[0.25 - 0.125 x_i^{0.5}(t)\bigr]\Biggl[\sum_{j=1}^{n} \sigma_{ij} x_j(t)\Biggr]^2\Biggr\} dt \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} 0.5\bigl[x_i^{0.5}(t) - 1\bigr] \sigma_{ij} x_j(t)\, dw(t). \qquad (2.3)
\end{aligned}$$

Compute

$$\sum_{i=1}^{n} 0.5\bigl[x_i^{0.5}(t) - 1\bigr]\Biggl(b_i + \sum_{j=1}^{n} a_{ij} x_j(t-\tau)\Biggr)$$

$$\leqslant \sum_{i=1}^{n} 0.5b_i\big[x_i^{0.5}(t) - 1\big] + \sum_{i=1}^{n} \sum_{j=1}^{n} \bigg[\frac{n}{16} a_{ij}^2 \big[x_i^{0.5}(t) - 1\big]^2 + \frac{1}{n} x_j^2(t - \tau)\bigg]$$
$$= \sum_{i=1}^{n} 0.5b_i\big[x_i^{0.5}(t) - 1\big] + \frac{n}{16} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \big[x_i^{0.5}(t) - 1\big]^2 + \big|x(t - \tau)\big|^2$$

and

$$\sum_{i=1}^{n} \Bigg[\sum_{j=1}^{n} \sigma_{ij} x_j(t)\Bigg]^2 \leqslant \sum_{i=1}^{n} \Bigg[\sum_{j=1}^{n} \sigma_{ij}^2 \sum_{j=1}^{n} x_j^2(t)\Bigg] = |\sigma|^2 \big|x(t)\big|^2.$$

Moreover, by hypothesis (H),

$$\sum_{i=1}^{n} x_i^{0.5}(t) \Bigg[\sum_{j=1}^{n} \sigma_{ij} x_j(t)\Bigg]^2 \geqslant \sum_{i=1}^{n} \sigma_{ii}^2 x_i^{2.5}(t).$$

Substituting these into (2.3) yields

$$d\Bigg[\int_{t-\tau}^{t} \big|x(s)\big|^2\, ds + V\big(x(t)\big)\Bigg]$$
$$\leqslant F\big(x(t)\big)\, dt + \sum_{i=1}^{n} \sum_{j=1}^{n} 0.5\big[x_i^{0.5}(t) - 1\big] \sigma_{ij} x_j(t)\, dw(t), \tag{2.4}$$

where

$$F(x) = \big(1 + 0.25|\sigma|^2\big)|x|^2 + \sum_{i=1}^{n} 0.5b_i\big[x_i^{0.5} - 1\big]$$
$$+ \frac{n}{16} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \big[x_i^{0.5} - 1\big]^2 - \frac{1}{8} \sum_{i=1}^{n} \sigma_{ii}^2 x_i^{2.5}. \tag{2.5}$$

It is straightforward to see that $F(x)$ is bounded, say by K, in R_+^n. We therefore obtain that

$$d\Bigg[\int_{t-\tau}^{t} \big|x(s)\big|^2\, ds + V\big(x(t)\big)\Bigg] \leqslant K\, dt + \sum_{i=1}^{n} \sum_{j=1}^{n} 0.5\big[x_i^{0.5}(t) - 1\big] \sigma_{ij} x_j(t)\, dw(t).$$

Integrating both sides from 0 to $\tau_k \wedge T$, and then taking expectations, yields

$$E\Bigg[\int_{\tau_k \wedge T - \tau}^{\tau_k \wedge T} \big|x(s)\big|^2\, ds + V\big(x(\tau_k \wedge T)\big)\Bigg] \leqslant \int_{-\tau}^{0} \big|x(s)\big|^2\, ds + V\big(x(0)\big) + KE(\tau_k \wedge T).$$

Consequently,

$$EV\big(x(\tau_k \wedge T)\big) \leqslant \int_{-\tau}^{0} \big|x(s)\big|^2\, ds + V\big(x(0)\big) + KT. \tag{2.6}$$

Note that for every $\omega \in \{\tau_k \leqslant T\}$, there is some i such that $x_i(\tau_k, \omega)$ equals either k or $1/k$, and hence $V(x(\tau_k, \omega))$ is no less than either

$$\sqrt{k} - 1 - 0.5\log(k)$$

or

$$\sqrt{1/k} - 1 - 0.5\log(1/k) = \sqrt{1/k} - 1 + 0.5\log(k).$$

Consequently,

$$V\big(x(\tau_k, \omega)\big) \geqslant \big[\sqrt{k} - 1 - 0.5\log(k)\big] \wedge \big[0.5\log(k) - 1 + \sqrt{1/k}\,\big].$$

It then follows from (2.6) that

$$\int_{-\tau}^{0} \big|x(s)\big|^2\,ds + V\big(x(0)\big) + KT \geqslant E\big[1_{\{\tau_k \leqslant T\}}(\omega) V\big(x(\tau_k, \omega)\big)\big]$$
$$\geqslant P\{\tau_k \leqslant T\}\big(\big[\sqrt{k} - 1 - 0.5\log(k)\big] \wedge \big[0.5\log(k) - 1 + \sqrt{1/k}\,\big]\big),$$

where $1_{\{\tau_k \leqslant T\}}$ is the indicator function of $\{\tau_k \leqslant T\}$. Letting $k \to \infty$ gives

$$\lim_{k\to\infty} P\{\tau_k \leqslant T\} = 0$$

and hence

$$P\{\tau_\infty \leqslant T\} = 0.$$

Since $T > 0$ is arbitrary, we must have

$$P\{\tau_\infty < \infty\} = 0,$$

so $P\{\tau_\infty = \infty\} = 1$ as required. □

3. Stochastically ultimate boundedness

Theorem 2.1 shows that under the simple hypothesis (H) the solutions of Eq. (2.2) will remain in the positive cone R_+^n. This nice positive property provides us with a great opportunity to construct other types of Lyapunov functions to discuss how the solutions vary in R_+^n in more detail.

As mentioned in Section 1, the nonexplosion property in a population dynamical system is often not good enough but the property of ultimate boundedness is more desired. Let us now give the definition of stochastically ultimate boundedness.

Definition 3.1. Equation (2.2) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0, 1)$, there is a positive constant $H = H(\varepsilon)$ such that for any initial data $\{x(t){:}\ -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R_+^n)$, the solution $x(t)$ of Eq. (2.2) has the property that

$$\limsup_{T\to\infty} P\big\{\big|x(t)\big| \leqslant H\big\} \geqslant 1 - \varepsilon. \tag{3.1}$$

Let us present a useful lemma from which the stochastically ultimate boundedness will follow directly.

Lemma 3.2. *Let hypothesis* (H) *hold and* $\theta \in (0,1)$. *Then there is a positive constant* $K = K(\theta)$, *which is independent of the initial data* $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau,0]; R_+^n)$, *such that the solution* $x(t)$ *of Eq.* (2.2) *has the property that*

$$\limsup_{t\to\infty} E\big|x(t)\big|^{\theta} \leqslant K. \tag{3.2}$$

Proof. Define

$$V(x) = \sum_{i=1}^n x_i^{\theta} \quad \text{for } x \in R_+^n.$$

By the Itô formula, we have

$$dV\big(x(t)\big) = LV\big(x(t), x(t-\tau)\big)\,dt + \left(\sum_{i=1}^n \theta x_i^{\theta}(t) \sum_{j=1}^n \sigma_{ij} x_j(t)\right) dw(t), \tag{3.3}$$

where $LV : R_+^n \times R_+^n \to R$ is defined by

$$LV(x,y) = \sum_{i=1}^n \theta x_i^{\theta}\left[b_i + \sum_{j=1}^n a_{ij} y_j\right] - \frac{\theta(1-\theta)}{2} \sum_{i=1}^n x_i^{\theta}\left[\sum_{j=1}^n \sigma_{ij} x_j\right]^2.$$

Compute

$$\begin{aligned} LV(x,y) &\leqslant \sum_{i=1}^n \theta b_i x_i^{\theta} + \sum_{i=1}^n \sum_{j=1}^n \left[\frac{n}{4}\theta^2 a_{ij}^2 x_i^{2\theta} + \frac{1}{n} y_j^2\right] - \frac{\theta(1-\theta)}{2} \sum_{i=1}^n \sigma_{ii}^2 x_i^{2+\theta} \\ &= \sum_{i=1}^n \theta b_i x_i^{\theta} + \frac{n}{4}\theta^2 \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 x_i^{2\theta} - \frac{\theta(1-\theta)}{2} \sum_{i=1}^n \sigma_{ii}^2 x_i^{2+\theta} + |y|^2 \\ &= F(x) - V(x) - e^{\tau}|x|^2 + |y|^2, \end{aligned}$$

where

$$F(x) = e^{\tau}|x|^2 + \sum_{i=1}^n (1+\theta b_i) x_i^{\theta} + \frac{n}{4}\theta^2 \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 x_i^{2\theta} - \frac{\theta(1-\theta)}{2} \sum_{i=1}^n \sigma_{ii}^2 x_i^{2+\theta}.$$

Note that $F(x)$ is bounded in R_+^n, namely

$$K_1 := \sup_{x \in R_+^n} F(x) < \infty.$$

We therefore have

$$LV(x,y) \leqslant K_1 - V(x) - e^{\tau}|x|^2 + |y|^2.$$

Substituting this into (3.3) gives

$$dV\big(x(t)\big) = \big[K_1 - V\big(x(t)\big) - e^{\tau}\big|x(t)\big|^2 + \big|x(t-\tau)\big|^2\big]\,dt + \left(\sum_{i=1}^{n}\theta x_i^{\theta}(t)\sum_{j=1}^{n}\sigma_{ij}x_j(t)\right)dw(t). \tag{3.4}$$

Once again by the Itô formula we have

$$\begin{aligned} d\big[e^t V\big(x(t)\big)\big] &= e^t\big[V\big(x(t)\big)\,dt + dV\big(x(t)\big)\big] \\ &\leqslant e^t\big[K_1 - e^{\tau}\big|x(t)\big|^2 + \big|x(t-\tau)\big|^2\big]\,dt \\ &\quad + e^t\left(\sum_{i=1}^{n}\theta x_i^{\theta}(t)\sum_{j=1}^{n}\sigma_{ij}x_j(t)\right)dw(t). \end{aligned}$$

We hence derive that

$$\begin{aligned} e^t EV\big(x(t)\big) &\leqslant V\big(x(0)\big) + K_1 e^t - E\int_0^t e^{s+\tau}\big|x(s)\big|^2\,ds + E\int_0^t e^{s}\big|x(s-\tau)\big|^2\,ds \\ &= V\big(x(0)\big) + K_1 e^t - E\int_0^t e^{s+\tau}\big|x(s)\big|^2\,ds + E\int_{-\tau}^{t-\tau} e^{s+\tau}\big|x(s)\big|^2\,ds \\ &\leqslant V\big(x(0)\big) + K_1 e^t + \int_{-\tau}^{0}\big|x(s)\big|^2\,ds. \end{aligned}$$

This implies immediately that

$$\limsup_{t\to\infty} EV\big(x(t)\big) \leqslant K_1.$$

On the other hand, we have

$$|x|^2 \leqslant n \max_{1\leqslant i\leqslant n} x_i^2$$

so

$$|x|^{\theta} \leqslant n^{\theta/2}\max_{1\leqslant i\leqslant n} x_i^{\theta} \leqslant n^{\theta/2}V(x).$$

We therefore finally have

$$\limsup_{t\to\infty} E\big|x(t)\big|^{\theta} \leqslant n^{\theta/2}K_1$$

and the assertion (3.2) follows by setting $K = n^{\theta/2}K_1$. □

Theorem 3.3. *Under hypothesis* (H), *Eq.* (2.2) *is stochastically ultimately bounded.*

Proof. By Lemma 3.2, there is $K > 0$ such that

$$\limsup_{t\to\infty} E\Big(\sqrt{\big|x(t)\big|}\Big) \leqslant K.$$

Now, for any $\varepsilon > 0$, let $H = K^2/\varepsilon^2$. Then by Chebyshev's inequality,

$$P\{|x(t)| > H\} \leqslant \frac{E(\sqrt{|x(t)|})}{\sqrt{H}}.$$

Hence

$$\limsup_{t\to\infty} P\{|x(t)| > H\} \leqslant \frac{K}{\sqrt{H}} = \varepsilon.$$

This implies

$$\limsup_{t\to\infty} P\{|x(t)| \leqslant H\} \geqslant 1 - \varepsilon$$

as required. □

4. Moment average in time

The result in the previous section shows that the solutions of Eq. (2.2) will be stochastically ultimately bounded. That is, the solutions will be ultimately bounded with large probability. The following result shows that the average in time of the second moment of the solutions will be bounded.

Theorem 4.1. *Under hypothesis* (H), *there is a positive constant K, which is independent of the initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, such that the solution $x(t)$ of Eq.* (2.2) *has the property that*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T E|x(t)|^2\,dt \leqslant K. \tag{4.1}$$

Proof. We use the same notations as in the proof of Theorem 2.1. Write (2.5) as

$$F(x) = F_1(x) - |x|^2$$

with

$$F_1(x) = \big(2 + 0.25|\sigma|^2\big)|x|^2 + \sum_{i=1}^n 0.5b_i\big[x_i^{0.5} - 1\big] + \frac{n}{16}\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2\big[x_i^{0.5} - 1\big]^2 - \frac{1}{8}\sum_{i=1}^n \sigma_{ii}^2 x_i^{2.5}. \tag{4.2}$$

Clearly, F_1 is bounded in R^n_+, namely

$$K = \max_{x\in R^n_+} F_1(x) < \infty.$$

So

$$F(x) \leqslant K - |x|^2.$$

Using this estimation, integrating both sides of (2.4) from 0 to $\tau_k \wedge T$, and then taking expectations, we obtain that

$$0 \leqslant \int_{-\tau}^{0} |x(s)|^2 \, ds + V(x(0)) + KE(\tau_k \wedge T) - E \int_{0}^{\tau_k \wedge T} |x(t)|^2 \, dt.$$

Letting $k \to \infty$ yields

$$E \int_{0}^{T} |x(t)|^2 \, dt \leqslant \int_{-\tau}^{0} |x(s)|^2 \, ds + V(x(0)) + KT.$$

Dividing both sides by T and then letting $T \to \infty$ we get

$$\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} E|x(t)|^2 \, dt \leqslant K$$

as required. □

5. Asymptotic pathwise estimation

In the previous sections we have discussed how the solutions vary in R_+^n in probability or in moment. In this section we will discuss the solutions pathwisely.

Theorem 5.1. *Let hypothesis* (H) *hold. Then, for any initial data* $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R_+^n)$*, the solution* $x(t)$ *of Eq.* (2.2) *has the property that*

$$\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leqslant 1 \quad a.s. \tag{5.1}$$

Proof. Define

$$V(x) = \sum_{i=1}^{n} x_i \quad \text{for } x \in R_+^n.$$

It is easy to see that

$$dV(x(t)) = x^T(t)\big([b + Ax(t-\tau)]\,dt + \sigma x(t)\,dw(t)\big).$$

Let $\gamma > 0$ be arbitrary. By Itô's formula we can show that

$$\begin{aligned} e^{\gamma t} \log(V(x(t))) &= \log(V(x(0))) \\ &\quad + \int_0^t e^{\gamma s} \left(\frac{x^T(s)}{V(x(s))}[b + Ax(s-\tau)] - \frac{1}{2V^2(x(s))}|x^T(s)\sigma x(s)|^2 \right) ds \\ &\quad + \gamma \int_0^t e^{\gamma s} \log(V(x(s)))\, ds + M(t), \end{aligned} \tag{5.2}$$

where

$$M(t) = \int_0^t \frac{e^{\gamma s}}{V(x(s))} x^T(s)\sigma x(s)\,dw(s)$$

is a real-valued continues local martingale vanishing at $t = 0$ and its quadratic form is given by

$$\langle M(t), M(t)\rangle = \int_0^t \frac{e^{2\gamma s}}{V^2(x(s))} \left|x^T(s)\sigma x(s)\right|^2 ds. \tag{5.3}$$

Let $\varepsilon \in (0, 1)$ and $\theta > 1$ be arbitrary. By the exponential martingale inequality (cf. Mao [10, Theorem 1.2.4]), we can show that for every integer $k \geqslant 1$,

$$P\left\{\sup_{0\leqslant t\leqslant k}\left[M(t) - \frac{\varepsilon}{2}e^{-\gamma k}\langle M(t), M(t)\rangle\right] > \frac{\theta e^{\gamma k}}{\varepsilon}\log k\right\} \leqslant k^{-\theta}.$$

Since the series $\sum_{k=1}^{\infty} k^{-\theta}$ converges, the well-known Borel–Cantelli lemma yields that exists an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ there exists an integer $k_0 = k_0(\omega)$ such that

$$M(t) \leqslant \frac{\varepsilon}{2}e^{-\gamma k}\langle M(t), M(t)\rangle + \frac{\theta e^{\gamma k}}{\varepsilon}\log k$$

for all $0 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$. Substituting this into Eq. (5.2) and then using (5.3) we obtain that

$$\begin{aligned} e^{\gamma t}\log\big(V\big(x(t)\big)\big) &= \log\big(V\big(x(0)\big)\big) \\ &\quad + \int_0^t e^{\gamma s}\left(\frac{x^T(s)}{V(x(s))}\big[b + Ax(s-\tau)\big] - \frac{1-\varepsilon}{2V^2(x(s))}\left|x^T(s)\sigma x(s)\right|^2\right) ds \\ &\quad + \gamma\int_0^t e^{\gamma s}\log\big(V\big(x(s)\big)\big)\,ds + \frac{\theta e^{\gamma k}}{\varepsilon}\log k \end{aligned} \tag{5.4}$$

for $0 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$. By the elementary inequality

$$\frac{V^2(x)}{n} \leqslant |x|^2 \leqslant nV^2(x), \tag{5.5}$$

we have

$$\begin{aligned} \frac{x^T(s)}{V(x(s))}\big[b + Ax(s-\tau)\big] &\leqslant \frac{|x(s)|}{V(x(s))}\big[|b| + \|A\|\big|x(s-\tau)\big|\big] \\ &\leqslant \sqrt{n}\,\big[|b| + \|A\|\big|x(s-\tau)\big|\big] \end{aligned} \tag{5.6}$$

and

$$\frac{1}{V^2(x(s))}\left|x^T(s)\sigma x(s)\right|^2 \geqslant \frac{\hat{\sigma}^2|x|^4}{V^2(x(s))} \geqslant \frac{\hat{\sigma}^2}{n}\left|x(s)\right|^2, \tag{5.7}$$

where $\hat{\sigma} = \min_{1\leqslant i\leqslant n} \sigma_{ii} > 0$. Substituting this into (5.4) gives

$$\begin{aligned} e^{\gamma t}\log\bigl(V\bigl(x(t)\bigr)\bigr) &= \log\bigl(V\bigl(x(0)\bigr)\bigr) \\ &\quad + \int_0^t e^{\gamma s}\biggl(\sqrt{n}\bigl[|b| + \|A\|\bigl|x(s-\tau)\bigr|\bigr] - \frac{(1-\varepsilon)\hat{\sigma}^2}{2n}\bigl|x(s)\bigr|^2\biggr)\,ds \\ &\quad + \gamma\int_0^t e^{\gamma s}\log\bigl(V\bigl(x(s)\bigr)\bigr)\,ds + \frac{\theta e^{\gamma k}}{\varepsilon}\log k, \end{aligned} \tag{5.8}$$

for $0 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$. Noting

$$\int_0^t e^{\gamma s}\bigl|x(s-\tau)\bigr|\,ds \leqslant \int_{-\tau}^{t-\tau} e^{\gamma(s+\tau)}\bigl|x(s)\bigr|\,ds \leqslant \int_{-\tau}^{0} e^{\gamma\tau}\bigl|x(s)\bigr|\,ds + \int_0^t e^{\gamma(s+\tau)}\bigl|x(s)\bigr|\,ds,$$

we can rewrite (5.8) as

$$\begin{aligned} e^{\gamma t}\log\bigl(V\bigl(x(t)\bigr)\bigr) &\leqslant C + \frac{\theta e^{\gamma k}}{\varepsilon}\log k \\ &\quad + \int_0^t e^{\gamma s}\biggl(\gamma\log\bigl(V\bigl(x(s)\bigr)\bigr) + \sqrt{n}\bigl[|b| + e^{\gamma\tau}\|A\|\bigl|x(s)\bigr|\bigr] - \frac{(1-\varepsilon)\hat{\sigma}^2}{2n}\bigl|x(s)\bigr|^2\biggr)\,ds \end{aligned} \tag{5.9}$$

for $0 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$, where

$$C = \log\bigl(V\bigl(x(s)\bigr)\bigr) + \sqrt{n}\,e^{\gamma\tau}\|A\|\int_{-\tau}^{0}\bigl|x(s)\bigr|\,ds.$$

It is easy to see that the following polynomial is bounded by a positive constant, say K,

$$\gamma\log\bigl(V(x)\bigr) + \sqrt{n}\bigl[|b| + e^{\gamma\tau}\|A\||x|\bigr] - \frac{(1-\varepsilon)\hat{\sigma}^2}{2n}|x|^2 \leqslant K, \quad \forall x \in R_+^n.$$

It therefore follows from (5.9) that

$$e^{\gamma t}\log\bigl(V\bigl(x(t)\bigr)\bigr) \leqslant C + \frac{\theta e^{\gamma k}}{\varepsilon}\log k + \frac{K}{\gamma}e^{\gamma t} \tag{5.10}$$

for $0 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$, Consequently, for any $\omega \in \Omega_0$, if $(k-1) \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$, we have

$$\frac{\log(V(x(t)))}{\log t} \leqslant \frac{1}{\log(k-1)}\biggl[e^{-\gamma(k-1)}C + \frac{\theta e^{\gamma}}{\varepsilon}\log k + \frac{K}{\gamma}\biggr].$$

This implies

$$\limsup_{t\to\infty}\frac{\log(V(x(t)))}{\log t} \leqslant \frac{\theta e^{\gamma}}{\varepsilon} \quad \text{a.s.}$$

By letting $\varepsilon \to 1$, $\theta \to 1$ and $\gamma \to 0$, we obtain

$$\limsup_{t\to\infty} \frac{\log(V(x(t)))}{\log t} \leqslant 1 \quad \text{a.s.}$$

Recalling the inequality (5.5) we obtain the required assertion (5.1). □

Theorem 5.1 shows that for arbitrarily small $\varepsilon > 0$, there is a positive random variable $T = T(\omega)$ such that, with probability 1,

$$\frac{\log(|x(t)|)}{\log t} \leqslant 1+\varepsilon, \quad \forall t \geqslant T,$$

namely

$$\left|x(t)\right| \leqslant t^{1+\varepsilon}, \quad \forall t \geqslant T.$$

If we set

$$\kappa := \sup_{0\leqslant t\leqslant T} \left|x(t)\right|,$$

which is clearly a finite random variable, then, with probability 1,

$$\left|x(t)\right| \leqslant \kappa + t^{1+\varepsilon}, \quad \forall t \geqslant 0.$$

This means that $|x(t)|$ will grow at most polynomially with order close to 1. The following theorem further shows that such growth can only happen occasionally.

Theorem 5.2. *Under hypothesis* (H), *for any initial data* $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau,0]; R^n_+)$, *the solution* $x(t)$ *of Eq.* (2.2) *has the property that*

$$\limsup_{t\to\infty} \frac{1}{t}\left[\log\left(\left|x(t)\right|\right) + \frac{\hat{\sigma}^2}{4n}\int_0^t \left|x(s)\right|^2 ds\right] \leqslant \sqrt{n}\,|b| + \frac{n^2\|A\|^2}{\hat{\sigma}^2} \quad \text{a.s.}, \tag{5.11}$$

where $\hat{\sigma} = \min_{1\leqslant i\leqslant n} \sigma_{ii} > 0$.

Proof. Let $V(x)$ be the same as defined in the proof of Theorem 5.1, namely $V(x) = \sum_{i=1}^n x_i$. By Itô's formula we have

$$\begin{aligned}\log\bigl(V\bigl(x(t)\bigr)\bigr) &= \log\bigl(V\bigl(x(0)\bigr)\bigr)\\ &\quad+\int_0^t \left(\frac{x^T(s)}{V(x(s))}\bigl[b + Ax(s-\tau)\bigr] - \frac{|x^T(s)\sigma x(s)|^2}{2V^2(x(s))}\right) ds + M(t),\end{aligned} \tag{5.12}$$

where

$$M(t) = \int_0^t \frac{x^T(s)\sigma x(s)}{V(x(s))}\, dw(s)$$

is a real-valued continues local martingale vanishing at $t=0$ and its quadratic form is given by

$$\langle M(t), M(t)\rangle = \int_0^t \frac{|x^T(s)\sigma x(s)|^2}{V^2(x(s))}\,ds. \tag{5.13}$$

Now, let $\varepsilon \in (0, 0.5)$ be arbitrary. By the exponential martingale inequality (cf. Mao [10, Theorem 1.2.4]), we can show that for every integer $k \geqslant 1$,

$$P\left\{\sup_{0\leqslant t\leqslant k}\left[M(t) - \frac{\varepsilon}{2}\langle M(t), M(t)\rangle\right] > \frac{2\log k}{\varepsilon}\right\} \leqslant k^{-2}.$$

Since the series $\sum_{k=1}^{\infty} k^{-2}$ converges, the well-known Borel–Cantelli lemma yields that there is $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$ there exists a random integer $k_0(\omega)$ such that for all $k \geqslant k_0(\omega)$,

$$\sup_{0\leqslant t\leqslant k}\left[M(t) - \frac{\varepsilon}{2}\langle M(t), M(t)\rangle\right] \leqslant \frac{2\log k}{\varepsilon}$$

which implies

$$M(t) \leqslant \frac{\varepsilon}{2}\langle M(t), M(t)\rangle + \frac{2\log k}{\varepsilon} \quad \text{on } 0 \leqslant t \leqslant k.$$

Substituting this into (5.12) and making use of (5.13), (5.6) and (5.7) we derive that

$$\begin{aligned}
\log\bigl(V\bigl(x(t)\bigr)\bigr) &\leqslant \log\bigl(V\bigl(x(0)\bigr)\bigr) \\
&\quad + \int_0^t \left(\frac{x^T(s)}{V(x(s))}\bigl[b + Ax(s-\tau)\bigr] - \frac{(1-\varepsilon)|x^T(s)\sigma x(s)|^2}{2V^2(x(s))}\right) ds + \frac{2\log k}{\varepsilon} \\
&\leqslant \log\bigl(V\bigl(x(0)\bigr)\bigr) + \frac{2\log k}{\varepsilon} \\
&\quad + \int_0^t \left(\sqrt{n}\,\bigl[|b| + \|A\|\bigl|x(s-\tau)\bigr|\bigr] - \frac{(1-\varepsilon)\hat{\sigma}^2}{2n}\bigl|x(s)\bigr|^2\right) ds
\end{aligned}$$

for $0 \leqslant t \leqslant k_0(\omega)$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$. Rearranging the above inequality gives

$$\begin{aligned}
&\log\bigl(V\bigl(x(t)\bigr)\bigr) + \frac{(1-2\varepsilon)\hat{\sigma}^2}{4n}\int_0^t \bigl|x(s)\bigr|^2 ds \\
&\quad \leqslant \log\bigl(V\bigl(x(0)\bigr)\bigr) + \frac{2\log k}{\varepsilon} \\
&\qquad + \int_0^t \left(\sqrt{n}\,\bigl[|b| + \|A\|\bigl|x(s-\tau)\bigr|\bigr] - \frac{\hat{\sigma}^2}{4n}\bigl|x(s)\bigr|^2\right) ds
\end{aligned} \tag{5.14}$$

for $0 \leqslant t \leqslant k_0(\omega)$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$. Note that

$$\int_0^t \left|x(s-\tau)\right| ds \leqslant \int_{-\tau}^{t-\tau} \left|x(s)\right| ds \leqslant \int_{-\tau}^{0} \left|x(s)\right| ds + \int_0^t \left|x(s)\right| ds.$$

It follows from (5.14) that

$$\log\bigl(V\bigl(x(t)\bigr)\bigr) + \frac{(1-2\varepsilon)\hat{\sigma}^2}{4n} \int_0^t \left|x(s)\right|^2 ds$$
$$\leqslant C + \frac{2\log k}{\varepsilon} + \int_0^t \left(\sqrt{n}\left[|b| + \|A\|\left|x(s)\right|\right] - \frac{\hat{\sigma}^2}{4n}\left|x(s)\right|^2\right) ds \tag{5.15}$$

for $0 \leqslant t \leqslant k_0(\omega)$ and $k \geqslant k_0(\omega)$ whenever $\omega \in \Omega_0$, where

$$C = \log\bigl(V\bigl(x(0)\bigr)\bigr) + \sqrt{n}\,\|A\| \int_{-\tau}^{0} \left|x(s)\right| ds.$$

It is easy to see that

$$\sqrt{n}\left[|b| + \|A\||x|\right] - \frac{\hat{\sigma}^2}{4n}|x|^2 \leqslant \sqrt{n}\,|b| + \frac{n^2\|A\|^2}{\hat{\sigma}^2} := K_1, \quad \forall x \in R_+^n.$$

Thus, if $\omega \in \Omega_0$,

$$\log\bigl(V\bigl(x(t)\bigr)\bigr) + \frac{(1-2\varepsilon)\hat{\sigma}^2}{4n} \int_0^t \left|x(s)\right|^2 ds \leqslant C + \frac{2\log k}{\varepsilon} + Kt$$

for $0 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$. Consequently, for any $\omega \in \Omega_0$, if $k-1 \leqslant t \leqslant k$ and $k \geqslant k_0(\omega)$,

$$\frac{1}{t}\left[\log\bigl(V\bigl(x(t)\bigr)\bigr) + \frac{(1-2\varepsilon)\hat{\sigma}^2}{4n} \int_0^t \left|x(s)\right|^2 ds\right] \leqslant \frac{1}{k-1}\left[C + \frac{2\log k}{\varepsilon}\right] + K_1,$$

which implies

$$\limsup_{t\to\infty} \frac{1}{t}\left[\log\bigl(V\bigl(x(t)\bigr)\bigr) + \frac{(1-2\varepsilon)\hat{\sigma}^2}{4n} \int_0^t \left|x(s)\right|^2 ds\right] \leqslant K_1 \quad \text{a.s.}$$

Using (5.5) and letting ε tend to zero yields

$$\limsup_{t\to\infty} \frac{1}{t}\left[\log\bigl(\left|x(t)\right|\bigr) + \frac{\hat{\sigma}^2}{4n} \int_0^t \left|x(s)\right|^2 ds\right] \leqslant K_1 \quad \text{a.s.}$$

which is the required assertion (5.11). □

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