Nonlinear Predictive GMV State-Dependent Control

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Abstract

A *Nonlinear Predictive Generalized Minimum Variance* control algorithm is introduced for the control of *nonlinear discrete-time state-dependent multivariable systems.* The process model includes two different types of subsystems to provide a variety of means of modelling the system and inferential control of certain outputs is available. A state-dependent output model is driven from an *unstructured* nonlinear input subsystem which can include explicit transport-delays. A multi-step predictive control cost-function is to be minimised involving weighted error, and either absolute or incremental control signal costing terms. Different patterns of a reduced number of future controls can be used to limit the computational demands.

Keywords: optimal, state-dependent, predictive, nonlinear, minimum-variance.

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1. Introduction

The objective is to design an industrial controller for *nonlinear and state-dependent, or linear parameter varying systems*, which has some of the advantages of the popular *Generalised Predictive Control* (*GPC*) algorithms. The control strategy builds upon previous results on *Nonlinear Generalized Minimum Variance* (*NGMV*) control [1]. The assumption was made that the plant model could be decomposed into a set of delay terms, a very general nonlinear subsystem that had to be stable and a linear subsystem. The plant description used here will be assumed to be similar, however the output subsystem is assumed to be represented in state-dependent, possibly unstable, form.

The multi-step predictive control cost-function to be minimised involves both weighted error and control costing terms, which can be used with different error and control horizons. Two alternative types of control signal input to the plant model are considered. The first is the traditional control signal input and it is this signal which is also penalized in the predictive control criterion. However, as is well know it is sometimes desirable to augment the plant model with an integrator to provide a simple way of introducing integral action. In the augmented system the new system input is the change of control action or increment, and in this case this is the signal which should be penalized in the criterion. The results will apply to both cases and a parameter change between $\beta = 0$ and $\beta = 1$ will provide the necessary switch. The cost includes dynamic weightings on both error and control signals.

There is a rich history of research on nonlinear predictive control ([2] to [7]), but the development proposed is somewhat different, since it is closer in spirit to that of a model based fixed-structure controller for a time-varying system. Part of the plant model can be represented by a very general nonlinear operator and the plant can also include a state-dependent (or linear parameter varying) output sub-system model, rather than a LTI model, as in previous work.

For equivalent linear systems, stability is ensured when the combination of a control weighting function and an error weighted plant model is strictly minimum phase. For nonlinear systems it is shown that a related operator equation is required to have a stable inverse. The dynamic cost-function weightings are chosen to satisfy performance and stability/robustness requirements and a simple method is proposed for obtaining initial values for the weightings.

2. Non-linear Operator and State-Dependent System

The plant model can be nonlinear, dynamic and may have a very general structure. The output subsystem and disturbance model is represented by a so-called *state-dependent* sub-system in Fig. 1. The plant involves two nonlinear subsystems and the first is of a very general nonlinear operator form and written as follows:

$$(\mathcal{W}_1 u)(t) = z^{-k}(\mathcal{W}_{1k} u)(t)$$

The second subsystem is a *state-dependent non-linear form,* which is similar to a time-varying linear system. It is assumed to be *point-wise stabilizable* and *detectable,* and is represented by the operator $\mathcal{W}_0$ written as follows:

$$(\mathcal{W}_0 u_0)(t) = (\mathcal{W}_{0k} z^{-k} u_0)(t)$$

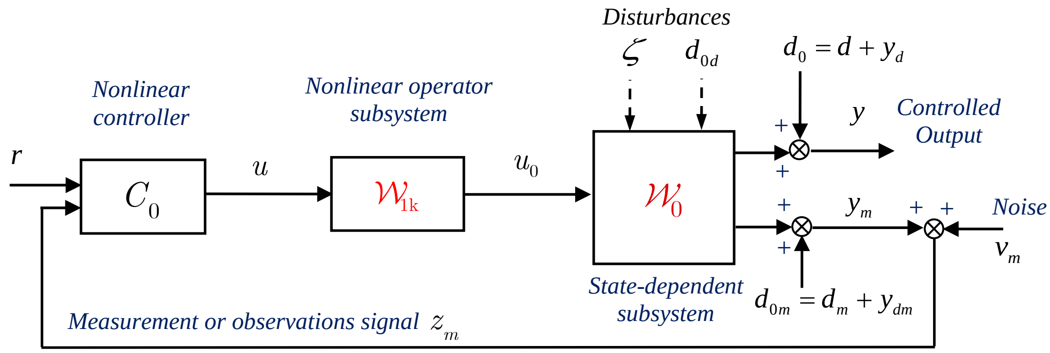

Fig. 1: **Feedback Control with Inferred or Controlled Outputs**

2.1 Signal Definitions

The output of the system to be controlled $y(t)$ may be different to that measured, as shown in Fig. 1, and this output includes deterministic $d(t)$ and stochastic $y_d(t)$ components of the disturbances. The measured output $y_m(t)$ also includes deterministic $d_m(t)$ and stochastic $y_{dm}(t)$ components of the disturbances. The stochastic component is modelled by a disturbance model, driven by zero mean white noise $\{\zeta_0(t)\}$. The measurement noise $\{v_m(t)\}$ is assumed to be zero-mean white noise with covariance matrix $R_f = R_f^T \geq 0$. There is no loss of generality in assuming that $\{\zeta_0(t)\}$ has an identity covariance matrix. The *controlled output* must follow a reference $r(t)$, which is assumed to be known.

2.2 State-Dependent Sub-System Models

The second or output subsystem is in a *state-dependent/LPV* form, which includes the plant and the error weighting models (see [8]). This is assumed to include a common *k-steps transport delay*, and has the *state-equation*:

$$x_0(t+1) = \mathcal{A}_0(x, u_0, p)x_0(t) + \mathcal{B}_0(x, u_0, p)u_0(t-k) + \mathcal{D}_0(x,u_0,p)\,\zeta_0(t) + \mathcal{G}_0(x,u_0,p)\,d_{0d}(t) \quad (1)$$

where the vector p is a vector of known variables like speed of an engine, or altitude of an aircraft that change with operating conditions. The *controlled output* and *measured outputs* (without measurement noise):

$$y(t) = d_0(t) + \mathcal{C}_0(x,u_0,p)\,x_0(t) + \mathcal{E}_0(x,u_0,p)\,u_0(t-k) \quad (2)$$

$$y_m(t) = d_{0m}(t) + \mathcal{C}_{0m}(x_0,u_0,p)\,x_0(t) + \mathcal{E}_{0m}(x_0,u_0,p)\,u_0(t-k) \quad (3)$$

where $x_0(t) \in R^{n_0}$. This model can be a function of the states, inputs and parameters $(x(t), u_0(t-k), p(t))$. The deterministic component of the input disturbance is $d_{0d}(t)$ and the disturbance on the output to be controlled $d_0(t) = d(t) + y_d(t)$ includes a known deterministic component $d(t)$ and a stochastic component $y_d(t)$. The disturbance on the measured output $d_{0m}(t) = d_m(t) + y_{dm}(t)$, where $d_m(t)$ is deterministic and $y_{dm}(t)$ is stochastic. The plant includes a disturbance model on the output, driven by zero mean white noise $\omega(t)$:

$$x_d(t+1) = \mathcal{A}_d x_d(t) + \mathcal{D}_d \omega(t), \qquad x_d(t) \in R^{n_d} \tag{4}$$

$$y_d(t) = \mathcal{C}_d x_d(t) \quad \text{and} \quad y_{dm}(t) = \mathcal{C}_{dm} x_d(t) \tag{5}$$

The signals of interest include the error on the output to be controlled and the measured output:

Error signal: $$e(t) = r(t) - y(t) \tag{6}$$

Observations signal: $$z_m(t) = y_m(t) + v_m(t) \tag{7}$$

The signal to be controlled will involve the weighted tracking error in the system:

$$x_p(t+1) = \mathcal{A}_p x_p(t) + \mathcal{B}_p \left(r(t) - y(t) \right), \qquad x_p(t) \in R^{n_p} \tag{8}$$

$$e_p(t) = \mathcal{C}_p x_p(t) + \mathcal{E}_p \left(r(t) - y(t) \right) \tag{9}$$

The traditional method of introducing integral action in predictive controls is to augment the system input by adding an integrator using the input sub-system:

$$x_i(t+1) = \beta x_i(t) + \Delta u_0(t-k), \qquad x_i(t) \in R^{n_i} \tag{10}$$

$$u_0(t-k) = \beta x_i(t) + \Delta u_0(t-k) = (1 - \beta z^{-1})^{-1} \Delta u_0(t-k) \tag{11}$$

The $\Delta = (1 - \beta z^{-1})$, for $\beta = 1$ and the transfer (11) is an integrator without additional delay, and if $\beta = 0$, then $u_0(t-k) = \Delta u_0(t-k)$. The results can therefore apply to systems using control input or rate of change of control.

2.3 Total Augmented System

The state-space model, for the $r \times m$ multivariable system to be controlled is now defined in augmented system form. Combining the plant, disturbance, integral and weighting equations, the augmented state-vector becomes:

$$x(t) = \left[x_0^T(t) \quad x_d^T(t) \quad x_i^T(t) \quad x_p^T(t) \right]^T$$

To simplify notation write $\mathcal{A}_t = \mathcal{A}\left(x(t), u_0(t-k), p(t)\right)$ and similarly for the time-varying matrices $\mathcal{B}_t$, $\mathcal{C}_t$, $\mathcal{D}_t$ and $\mathcal{E}_t$, with state $x(t) \in R^n$. The augmented system equations may be written as follows:

$$x(t+1) = \mathcal{A}_t x(t) + \mathcal{B}_t \Delta u_0(t-k) + \mathcal{D}_t \xi(t) + d_d(t) \tag{12}$$

$$y(t) = d(t) + \mathcal{C}_t x(t) + \mathcal{E}_t \Delta u_0(t-k) \tag{13}$$

$$y_m(t) = d_m(t) + \mathcal{C}_t^m x(t) + \mathcal{E}_t^m \Delta u_0(t-k) \tag{14}$$

$$z_m(t) = v_m(t) + d_m(t) + \mathcal{C}_t^m x(t) + \mathcal{E}_t^m \Delta u_0(t-k) \tag{15}$$

$$e_p(t) = d_p(t) + \mathcal{C}_{p\,t} x(t) + \mathcal{E}_{p\,t} \Delta u_0(t-k) \tag{16}$$

The augmented system has an input $\Delta u_0(t)$ and the change in actual control is denoted $\Delta u(t)$ (these are related as $\Delta u_0(t) = \mathcal{W}_{1k}(.,.)\Delta u(t)$).

2.4 Definition of the Augmented System Matrices

The equations in §2.2 can be combined with a little manipulation to obtain the augmented system matrices. That is the total state-equation model may be written in terms of the augmented system matrices, as follows:

$$x(t+1) = \mathcal{A}_t x(t) + \mathcal{B}_t \Delta u_0(t-k) + \mathcal{D}_t \xi(t) + d_d(t) \tag{17}$$

where the matrices in this equation are defined from the combined model equations:

$$\begin{bmatrix} x_0(t+1) \\ x_d(t+1) \\ x_i(t+1) \\ x_p(t+1) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_0 & 0 & \beta\mathcal{B}_0 & 0 \\ 0 & \mathcal{A}_d & 0 & 0 \\ 0 & 0 & \beta I & 0 \\ -\mathcal{B}_p\mathcal{C}_0 & -\mathcal{B}_p\mathcal{C}_d & -\beta\mathcal{B}_p\mathcal{E}_0 & \mathcal{A}_p \end{bmatrix} \begin{bmatrix} x_0(t) \\ x_d(t) \\ x_i(t) \\ x_p(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_0 \\ 0 \\ I \\ -\mathcal{B}_p\mathcal{E}_0 \end{bmatrix} \Delta u_0(t-k)$$

$$+ \begin{bmatrix} \mathcal{D}_0 & 0 \\ 0 & \mathcal{D}_d \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_0(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \mathcal{G}_0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \mathcal{B}_p \end{bmatrix} \begin{bmatrix} d_{0d}(t) \\ (r(t) - d(t)) \end{bmatrix} \tag{18}$$

The output to be controlled may be written in terms of augmented system model in (13). That is:

$$y(t) = d(t) + \mathcal{C}_0 x_0(t) + \mathcal{C}_d x_d(t) + \mathcal{E}_0 \beta x_i(t) + \mathcal{E}_0 \Delta u_0(t-k)$$

$$= d(t) + \mathcal{C}_t x(t) + \mathcal{E}_t \Delta u_0(t-k) \qquad (19)$$

where $\mathcal{C}_t = \begin{bmatrix} \mathcal{C}_0 & \mathcal{C}_d & \mathcal{E}_0\beta & 0 \end{bmatrix}$ and $\mathcal{E}_t = \mathcal{E}_0$

Similarly from (3) and (5), the measured output may be written in the augmented system as follows:

$$y_m(t) = d_m(t) + \mathcal{C}_t^m x(t) + \mathcal{E}_t^m \Delta u_0(t-k) \qquad (20)$$

where $\mathcal{C}_t^m = \begin{bmatrix} \mathcal{C}_{0m} & \mathcal{C}_{dm} & \mathcal{E}_{0m}\beta & 0 \end{bmatrix}$ and $\mathcal{E}_t^m = \mathcal{E}_{0m}$

Also from (2) and (9), the weighted tracking error to be minimised may be written as:

$$e_p(t) = d_p(t) + \mathcal{C}_{p\,t} x(t) + \mathcal{E}_{p\,t} \Delta u_0(t-k) \qquad (21)$$

where $d_p(t) = \mathcal{E}_p(r(t) - d(t))$, $\mathcal{C}_{p\,t} = \begin{bmatrix} -\mathcal{E}_p\mathcal{C}_0 & -\mathcal{E}_p\mathcal{C}_d & -\beta\mathcal{E}_p\mathcal{E}_0 & \mathcal{C}_p \end{bmatrix}$ and $\mathcal{E}_{p\,t} = -\mathcal{E}_p\mathcal{E}_0$. The subscript *t* on the state matrices here is used for the augmented system and in a slight abuse of notation it also indicates that these matrices are evaluated at time *t*, so that the system matrix at *t+1* is written as $\mathcal{A}_{t+1}$.

3. State-Dependent Future State and Error Models

A state-dependent model prediction equation is required and later an estimator for the state-dependent models. The future values of the states and outputs may be obtained by repeated use of (12) assuming that the future values of the disturbance are known. Introduce the notation:

$$\mathcal{A}_{t+m}^{i-m} = \mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\cdots\mathcal{A}_{t+m} \quad \text{for } i > m\text{, where } \mathcal{A}_{t+m}^{0} = I \text{ for } i = m$$

$$\mathcal{A}_t^i = \mathcal{A}_{t+i-1}\mathcal{A}_{t+i-2}\cdots\mathcal{A}_t \quad \text{for } i > 0\text{, where } \mathcal{A}_t^0 = I \text{ for } i = 0 \qquad (22)$$

Future states: Generalising this result obtain, for $i \geq 1$, the state, at any future time $t+i$, may be written as:

$$x(t+i) = \mathcal{A}_t^i x(t) + \sum_{j=1}^{i} \mathcal{A}_{t+j}^{i-j} \left(\mathcal{B}_{t+j-1} \Delta u_0(t+j-1-k) + \mathcal{D}_{t+j-1}\xi(t+j-1) \right) + d_{dd}(t+i-1) \qquad (23)$$

where
$$d_{dd}(t+i-1) = \sum_{j=1}^{i} \mathcal{A}_{t+j}^{i-j} d_d(t+j-1) \qquad (24)$$

These equations (23) and (24) are valid for $i \geq 0$ if the summation terms are defined as null for i = 0. Noting (16) the weighted error or output signal $e_p(t)$ to be regulated at future times (for $i \geq 0$):

$$e_p(t+i) = d_p(t+i) + \mathcal{C}_{p\,t+i} x(t+i) + \mathcal{E}_{p\,t+i} \Delta u_0(t+i-k)$$

$$= d_{pd}(t+i) + \mathcal{C}_{p\,t+i} \mathcal{A}_t^{\,i} x(t) + \mathcal{C}_{p\,t+i} \sum_{j=1}^{i} \mathcal{A}_{t+j}^{i-j} \mathcal{B}_{t+j-1} \Delta u_0(t+j-1-k)$$

$$+ \mathcal{C}_{p\,t+i} \sum_{j=1}^{i} \mathcal{A}_{t+j}^{i-j} \mathcal{D}_{t+j-1} \xi(t+j-1) + \mathcal{E}_{p\,t+i} \Delta u_0(t+i-k) \qquad (25)$$

where $d_p(t) = \mathcal{E}_p(r(t) - d(t))$ and the deterministic signals:

$$d_{pd}(t+i) = d_p(t+i) + \mathcal{C}_{p\,t+i} d_{dd}(t+i-1) \qquad (26)$$

3.1 State Estimates Using State-Dependent Prediction Models

The *i-steps* prediction of the state for $i \geq 0$ and the output signals may be defined, noting (23), as:

$$\hat{x}(t+i\,|\,t) = \mathcal{A}_t^{\,i} \hat{x}(t\,|\,t) + \sum_{j=1}^{i} \mathcal{A}_{t+j}^{i-j} \mathcal{B}_{t+j-1} \Delta u_0(t+j-1-k) + d_{dd}(t+i-1) \qquad (27)$$

where $\mathcal{A}_{t+j}^{i-j} = \mathcal{A}_{t+i-1} \mathcal{A}_{t+i-2} \ldots \mathcal{A}_{t+j}$ and $d_{dd}(t+i-1) = \sum_{j=1}^{i} \mathcal{A}_{t+j}^{i-j} d_d(t+j-1)$, and for i = 0 the $d_{dd}(t-1) = 0$. The predicted output:

$$\hat{y}(t+i\,|\,t) = d(t+i) + \mathcal{C}_{t+i} \hat{x}(t+i\,|\,t) + \mathcal{E}_{t+i}\, \Delta u_0(t-k+i) \qquad (28)$$

The weighted prediction error for $i \geq 0$:

$$\hat{e}_p(t+i\,|\,t) = d_p(t+i) + \mathcal{C}_{p\,t+i} \hat{x}(t+i\,|\,t) + \mathcal{E}_{p\,t+i} \Delta u_0(t+i-k) \qquad (29)$$

The expression for the future predicted states and error signals may be obtained by changing the prediction time in (27) $t \to t+k$. Then, for $i \geq 0$:

$$\hat{x}(t+k+i\,|\,t) = \mathcal{A}_{t+k}^{i} \hat{x}(t+k\,|\,t) + \sum_{j=1}^{i} \mathcal{A}_{t+k+j}^{i-j} \mathcal{B}_{t+k+j-1} \Delta u_0(t+j-1) + d_{dd}(t+k+i-1) \qquad (30)$$

Predicted weighted output error: Substituting in (29) and simplifying, for $i \to i+k$, and $i \geq 0$, obtain:

$$\hat{e}_p(t+i+k\,|\,t) = d_{pd}(t+i+k) + \mathcal{E}_{p\,t+i+k}\Delta u_0(t+i) + \mathcal{C}_{pt+i+k}\mathcal{A}^{i}_{t+k}\hat{x}(t+k\,|\,t)$$

$$+\mathcal{C}_{pt+i+k}\sum_{j=1}^{i}\mathcal{A}^{i-j}_{t+k+j}\mathcal{B}_{t+k+j-1}\Delta u_0(t+j-1) \qquad (31)$$

and $\hat{e}_p(t+i\,|\,t) = d_{pd}(t+i) + \mathcal{E}_{p\ t+i}\Delta u_0(t+i-k) + \mathcal{C}_{p\ t+i}\mathcal{A}^{i}_{t+k}\hat{x}(t\,|\,t) + \mathcal{C}_{p\ t+i}\sum_{j=1}^{i}\mathcal{A}^{i-j}_{t+j}\mathcal{B}_{t+j-1}\Delta u_0(t+j-1-k)$ (32)

The deterministic signals in this equation:

$$d_{pd}(t+i+k) = d_p(t+i+k) + \sum_{j=1}^{i}\mathcal{C}_{p\,t+i+k}\mathcal{A}^{i-j}_{t+k+j}d_d(t+k+j-1) \qquad (33)$$

and for $i = 0$ the term $d_{pd}(t+k) = d_p(t+k)$.

3.2 Vector Matrix Form of Equations

The predicted errors or outputs may be computed for controls in a future interval $\tau\ \varepsilon\ [t,t+N]$ for $N \geq 1$. These weighted error signals may be collected in the following $N+1$ vector form:

$$\begin{bmatrix} \hat{e}_p(t+k) \\ \hat{e}_p(t+1+k) \\ \hat{e}_p(t+2+k) \\ \vdots \\ \hat{e}_p(t+N+k) \end{bmatrix} = \begin{bmatrix} d_{pd}(t+k) \\ d_{pd}(t+k+1) \\ d_{pd}(t+k+2) \\ \vdots \\ d_{pd}(t+k+N) \end{bmatrix} + \begin{bmatrix} \mathcal{C}_{pt+k}I \\ \mathcal{C}_{pt+1+k}\mathcal{A}^{1}_{t+k} \\ \mathcal{C}_{pt+2+k}\mathcal{A}^{2}_{t+k} \\ \vdots \\ \mathcal{C}_{pt+N+k}\mathcal{A}^{N}_{t+k} \end{bmatrix}\hat{x}(t+k\,|\,t) + \begin{bmatrix} \mathcal{E}_{pt+k}\Delta u_0(t) \\ \mathcal{E}_{pt+k+1}\Delta u_0(t+1) \\ \mathcal{E}_{pt+k+2}\Delta u_0(t+2) \\ \vdots \\ \mathcal{E}_{pt+k+N}\Delta u_0(t+N) \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathcal{C}_{pt+k+1}\mathcal{B}_{t+k} & 0 & \ddots & 0 & 0 \\ \mathcal{C}_{pt+k+2}\mathcal{A}^{1}_{t+k+1}\mathcal{B}_{t+k} & \mathcal{C}_{pt+k+2}\mathcal{B}_{t+k+1} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ \mathcal{C}_{pt+k+N}\mathcal{A}^{N-1}_{t+k+1}\mathcal{B}_{t+k} & \mathcal{C}_{pt+k+N}\mathcal{A}^{N-2}_{t+k+2}\mathcal{B}_{t+k+1} & \cdots & \mathcal{C}_{pt+k+N}\mathcal{B}_{t+k+N-1} & 0 \end{bmatrix}\begin{bmatrix} \Delta u_0(t) \\ \Delta u_0(t+1) \\ \vdots \\ \Delta u_0(t+N-1) \\ \Delta u_0(t+N) \end{bmatrix} \qquad (34)$$

Future error and predicted error: With an obvious definition of terms this equation may be written as:

$$\hat{E}_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t) + (\mathcal{C}_{Pt+k,N}\mathcal{B}_{t+k,N} + \mathcal{E}_{Pt+k,N})\Delta U^{0}_{t,N} \qquad (35)$$

Define the time-varying matrix: $\mathcal{V}_{Pt+k,N} = \mathcal{C}_{Pt+k,N}\mathcal{B}_{t+k,N} + \mathcal{E}_{Pt+k,N}$ (36)

so that, $\hat{E}_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t) + \mathcal{V}_{Pt+k,N}\Delta U^{0}_{t,N}$ (37)

Similarly the weighted future errors may be written, including $\Xi_{t+k,N}$, as:

$$E_{P t+k,N} = D_{P t+k,N} + \mathcal{C}_{P t+k,N} \mathcal{A}_{t+k,N} x(t+k) + \mathcal{V}_{P t+k,N} \Delta U^{0}_{t,N} + \mathcal{C}_{P t+k,N} \mathcal{D}_{t+k,N} \Xi_{t+k,N} \tag{38}$$

Block matrices: Noting (34) the vectors and block matrices, for the general case of $N \geq 1$, may be defined as:

$$\mathcal{C}_{P t+k,N} = diag\{\mathcal{C}_{p t+k}, \mathcal{C}_{p t+1+k}, \mathcal{C}_{p t+2+k}, \ldots, \mathcal{C}_{p t+N+k}\}$$

$$\mathcal{E}_{P t+k,N} = diag\{\mathcal{E}_{p t+k}, \mathcal{E}_{p t+1+k}, \ldots, \mathcal{E}_{p t+N+k}\} \tag{39}$$

$$\mathcal{A}_{t+k,N} = \begin{bmatrix} I \\ \mathcal{A}^{1}_{t+k} \\ \mathcal{A}^{2}_{t+k} \\ \vdots \\ \mathcal{A}^{N}_{t+k} \end{bmatrix}, \qquad \mathcal{B}_{t+k,N} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathcal{B}_{t+k} & 0 & \cdots & \vdots & 0 \\ \mathcal{A}^{1}_{t+k+1}\mathcal{B}_{t+k} & \mathcal{B}_{t+k+1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ \mathcal{A}^{N-1}_{t+k+1}\mathcal{B}_{t+k} & \mathcal{A}^{N-2}_{t+k+2}\mathcal{B}_{t+k+1} & \cdots & \mathcal{B}_{t+k+N-1} & 0 \end{bmatrix},$$

$$\mathcal{D}_{t+k,N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{D}_{t+k} & 0 & \cdots & \vdots \\ \mathcal{A}_{t+k+1}\mathcal{D}_{t+k} & \mathcal{D}_{t+k+1} & \ddots & \\ \vdots & \vdots & \ddots & 0 \\ \mathcal{A}^{N-1}_{t+k+1}\mathcal{D}_{t+k} & \mathcal{A}^{N-2}_{t+k+2}\mathcal{D}_{t+k+1} & \cdots & \mathcal{D}_{t+k+N-1} \end{bmatrix}, \qquad \Xi_{t,N} = \begin{bmatrix} \xi(t) \\ \xi(t+1) \\ \vdots \\ \xi(t+N-1) \end{bmatrix},$$

$$\hat{E}_{P t+k,N} = \begin{bmatrix} \hat{e}_p(t+k) \\ \hat{e}_p(t+1+k) \\ \hat{e}_p(t+2+k) \\ \vdots \\ \hat{e}_p(t+N+k) \end{bmatrix}, \qquad U^{0}_{t,N} = \begin{bmatrix} \Delta u_0(t) \\ \Delta u_0(t+1) \\ \Delta u_0(t+2) \\ \vdots \\ \Delta u_0(t+N) \end{bmatrix}, \qquad D_{P t,N} = \begin{bmatrix} d_{pd}(t) \\ d_{pd}(t+1) \\ d_{pd}(t+2) \\ \vdots \\ d_{pd}(t+N) \end{bmatrix}$$

The signal $\Delta U^{0}_{t,N}$ denotes a block vector of future input signals. Note that the block vector $D_{P\,t,N}$ denotes a vector of future reference minus known disturbance signal components. The above system matrices $\mathcal{A}_{t+k,N}, \mathcal{B}_{t+k,N}, \mathcal{D}_{t+k,N}$ are of course all functions of future states and the assumption is made that the state dependent signal $x(t)$ is calculable (if $\{\xi(t)\}$ is null $\hat{x}(t\,|\,t) = x(t)$ can be calculated from the model). From (36) the matrix $\mathcal{V}_{P t+k,N} = (\mathcal{C}_{P t+k,N}\mathcal{B}_{t+k,N} + \mathcal{E}_{P t+k,N})$ can be assumed to be full-rank (determined by the weightings).

3.3 Predicted Tracking Error

Noting (38) the *k-steps-ahead tracking error*:

$$E_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}x(t+k) + \mathcal{V}_{Pt+k,N}\Delta U^0_{t,N} + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N} \tag{40}$$

The weighted inferred output is assumed to have the same dimension as the control signal and $\mathcal{V}_{P\,t+k,N}$ used in (40) and defined below, for $N \geq 1$, is square:

$$\mathcal{V}_{Pt,N} = \underbrace{\begin{bmatrix} \mathcal{E}_{pt} & 0 & \cdots & 0 & 0 \\ \mathcal{C}_{pt+1}\mathcal{B}_t & \mathcal{E}_{pt+1} & \cdots & \vdots & 0 \\ \mathcal{C}_{pt+2}\mathcal{A}^1_{t+1}\mathcal{B}_t & \mathcal{C}_{pt+2}\mathcal{B}_{t+1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \mathcal{E}_{pt+N-1} & 0 \\ \mathcal{C}_{pt+N}\mathcal{A}^{N-1}_{t+1}\mathcal{B}_t & \mathcal{C}_{pt+N}\mathcal{A}^{N-2}_{t+2}\mathcal{B}_{t+1} & \cdots & \mathcal{C}_{pt+N}\mathcal{B}_{t+N-1} & \mathcal{E}_{pt+N} \end{bmatrix}}_{\mathcal{V}_{Pt,N}=\mathcal{C}_{Pt,N}\mathcal{B}_{t,N}+\mathcal{E}_{Pt,N}} \tag{41}$$

Based on (35) and (38) the prediction error ($\tilde{E}_{P\,t+k,N} = E_{P\,t+k,N} - \hat{E}_{P\,t+k,N}$):

$$\tilde{E}_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}x(t+k) + \mathcal{V}_{Pt+k,N}\Delta U^0_{t,N} + \mathcal{C}_{Pt+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N}$$

$$-(D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t) + \mathcal{V}_{Pt+k,N}\Delta U^0_{t,N}) \tag{42}$$

Thence, the *inferred output estimation error*:

$$\tilde{E}_{Pt+k,N} = \mathcal{C}_{P\,t+k,N}\mathcal{A}_{t+k,N}\tilde{x}(t+k\,\big|\,t) + \mathcal{C}_{P\,t+k,N}\mathcal{D}_{t+k,N}\Xi_{t+k,N} \tag{43}$$

where the *state estimation error* $\tilde{x}(t+k|t) = x(t+k) - \hat{x}(t+k\,|\,t)$ is independent of the choice of control action. Also recall $\hat{x}(t+k\,|\,t)$ and $\tilde{x}(t+k\,|\,t)$ are orthogonal and the expectation of the product of the future values of the control action (assumed known in deriving the prediction equation), and the zero-mean white noise driving signals, is null. It follows that $\hat{E}_{Pt+k,N}$ in (35) and the prediction error $\tilde{E}_{Pt+k,N}$ are orthogonal.

3.4 Time-Varying Kalman Estimator in Predictor Corrector Form

The state estimate $\hat{x}(t+k\,|\,t)$ may be obtained, *k steps ahead*, from a *Kalman filter* [9]. These are well known, but the result below accommodates the *delays on input channels* and *through terms* [9]. The estimates can be computed using:

$$\hat{x}(t+1|t) = \mathcal{A}_t \hat{x}(t|t) + \mathcal{B}_t \Delta u_0(t-k) + d_d(t)$$

$$\hat{x}(t+1|t+1) = \hat{x}(t+1|t) + \mathcal{K}_{f\,t+1}\left(z_m(t+1) - \hat{z}_m(t+1|t)\right)$$

where $$\hat{z}_m(t+1|t) = d_m(t+1) + \mathcal{C}^m_{t+1}\hat{x}(t+1|t) + \mathcal{E}^m_{t+1}\Delta u_0(t+1-k)$$

The state estimate $\hat{x}(t+k|t)$ may be obtained, *k steps-ahead*, in a computationally efficient form from [9], where the number of states in the filter is not increased by the number of the delay elements *k*. From (27) the *k-steps* prediction is given as:

$$\hat{x}(t+k|t) = \mathcal{A}_t^k \hat{x}(t|t) + \mathcal{T}(k, z^{-1})\Delta u_0(t) + d_{dd}(t+k-1) \tag{44}$$

The *finite pulse response model* term:

$$\mathcal{T}(k, z^{-1}) = \sum_{j=1}^{k} \mathcal{A}_{t+j}^{k-j} \mathcal{B}_{t+j-1} z^{j-1-k} \tag{45}$$

where the summation terms in (45) are assumed null for $k = 0$ so that $\mathcal{T}(0, z^{-1}) = 0$, $d_{dd}(t-1) = 0$, and $d_{dd}(t+k-1) = \sum_{j=1}^{k} \mathcal{A}_{t+j}^{k-j} d_d(t+j-1)$.

4. Generalized Predictive Control for State-Dependent Systems

A brief derivation of a *GPC* controller is provided below for a state-dependent system with input $u_0(t)$. This is the first step in the solution of the *NPGMV* control solution derived subsequently. The *GPC performance index*:

$$J = E\left\{\sum_{j=0}^{N} e_p(t+j+k)^T e_p(t+j+k) + \sum_{j=0}^{N_u} \lambda_j^2 (\Delta u_0(t+j))^T \Delta u_0(t+j) \middle| t\right\} \tag{46}$$

where $E\{.|t\}$ denotes the conditional expectation, conditioned on measurements up to time *t* and λ_j denotes a scalar control signal weighting factor. In this definition note that the error minimized is *k*-steps ahead of the control signal, since $u_0(t)$ affects the error $e_p(t+k)$ after *k-steps*. By suitable definition of the augmented system the cost can include dynamic error, input and state-costing terms. The future optimal control signal is to be calculated for the interval $\tau \in [t, t+N_u]$, which depends on the number of steps $(N_u + 1)$ in the control signal costing term in (46). If the states are not available for feedback then the Kalman estimator must be introduced. Also recall from (43) the weighted tracking error $E_{Pt+k,N} = \tilde{E}_{Pt+k,N} + \hat{E}_{Pt+k,N}$. The multi-step cost-function:

$$J = E\{J_t\} = E\left\{E^T_{Pt+k,N} E_{Pt+k,N} + \Delta U^{0T}_{t,N_u} \Lambda^2_{N_u} \Delta U^0_{t,N_u} \mid t\right\} \tag{47}$$

Assuming the *Kalman filter* is introduced, from (47),

$$J = E\left\{(\hat{E}_{Pt+k,N} + \tilde{E}_{Pt+k,N})^T (\hat{E}_{Pt+k,N} + \tilde{E}_{Pt+k,N}) + \Delta U^{0T}_{t,N_u} \Lambda^2_{N_u} \Delta U^0_{t,N_u} \mid t\right\} \tag{48}$$

Here the cost-function weightings on inputs $\Delta u_0(t)$ at future times are written as $\Lambda^2_{N_u} = diag\{\lambda_0^2, \lambda_1^2, ..., \lambda_{N_u}^2\}$.

The terms in the cost-index can then be simplified, noting $\hat{E}_{Pt+k,N}$ is orthogonal to the estimation error $\tilde{E}_{Pt+k,N}$:

$$J = \hat{E}^T_{Pt+k,N} \hat{E}_{Pt+k,N} + \Delta U^{0T}_{t,N_u} \Lambda^2_{N_u} \Delta U^0_{t,N_u} + J_0 \tag{49}$$

where $J_0 = E\{\tilde{E}^T_{Pt+k,N} \tilde{E}_{Pt+k,N} \mid t\}$ is independent of control action.

4.1 Connection Matrix and Control Profile

Instead of a single control horizon number N_u a control profile can be defined of the form:

$$row\{P_u\} = [\text{lengths of intervals in samples} \quad \text{number of repetitions}]$$

For example, letting P_u = [1 3; 2 2; 3 1] represents 3 different initial controls for each sample, then 2 samples with the same control used but this is repeated again, and finally 3 samples with the same control used. This enables a control trajectory to be defined where initially the control changes every sample instant and then it only changes every two sample instants and finally it remains fixed for 3 sample intervals. Based on a control profile, it is easy to specify the transformation matrix T_u, relating the control moves to be optimized (say vector V) to the full control vector (U), that is, $U = T_u \times V$. For the above example, the connection matrix can be defined:

$$T_u = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix} \quad \text{and} \quad U = T_u V \Rightarrow \begin{bmatrix} u(t)\\ u(t+1)\\ u(t+2)\\ u(t+3)\\ u(t+4)=u(t+3)\\ u(t+5)\\ u(t+6)=u(t+5)\\ u(t+7)\\ u(t+8)=u(t+7)\\ u(t+9)=u(t+7) \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6 \end{bmatrix}$$

In the case of the incremental control formulation, the connection matrix:

$$T_{\Delta u} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \quad \text{and} \quad \Delta U = T_{\Delta u} \Delta V \Rightarrow \begin{bmatrix} \Delta u(t)\\ \Delta u(t+1)\\ \Delta u(t+2)\\ \Delta u(t+3)\\ \Delta u(t+4)=0\\ \Delta u(t+5)\\ \Delta u(t+6)=0\\ \Delta u(t+7)\\ \Delta u(t+8)=0\\ \Delta u(t+9)=0 \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \Delta v_1\\ \Delta v_2\\ \Delta v_3\\ \Delta v_4\\ \Delta v_5\\ \Delta v_6 \end{bmatrix}$$

Clearly, this represents a situation with $N_u = 3+2+1 = 6$ control moves and involves a total of $N = 3\times1+2\times2+1\times3$ sample points. There are 4 control moves that have not been calculated in this example, representing a substantial computational saving. For simplicity the same symbol will be used to represent the connection matrix for the control and incremental control cases (T_u) but when using it should be recalled that different definitions will be needed. The control horizon may be less than the error horizon and we may define the future control changes $\Delta U^0_{t,N}$ as $\Delta U^0_{t,N} = T_u \, \Delta U^0_{t,N_u}$.

4.2 State Dependent GPC Solution

To compute the vector of future weighted error signals note:

$$\mathcal{V}_{P\,t+k,N} \Delta U^0_{t,N} = \mathcal{V}_{P\,t+k,N} T_u \, \Delta U^0_{t,N_u} \tag{50}$$

Then from (37) and (50):

$$\hat{E}_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t) + \mathcal{V}_{Pt+k,N}\Delta U^0_{t,N} = \tilde{D}_{Pt+k,N} + \mathcal{V}_{Pt+k,N}T_u\,\Delta U^0_{t,N_u} \qquad (51)$$

where $\tilde{D}_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t)$. Noting (36) and substituting from (35) for the vector of state-estimates:

$$J = (\tilde{D}_{Pt+k,N} + \mathcal{V}_{Pt+k,N}T_u\,\Delta U^0_{t,N_u})^T(\tilde{D}_{Pt+k,N} + \mathcal{V}_{Pt+k,N}T_u\,\Delta U^0_{t,N_u}) + \Delta U^{0T}_{t,N_u}\Lambda^2_{N_u}\Delta U^0_{t,N_u} + J_0$$

$$= \tilde{D}^T_{Pt+k,N}\tilde{D}_{Pt+k,N} + \Delta U^{0T}_{t,N_u}T_u^T\mathcal{V}^T_{Pt+k,N}\tilde{D}_{Pt+k,N} + \tilde{D}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N}T_u\,\Delta U^0_{t,N_u} + \Delta U^{0T}_{t,N_u}\mathcal{X}_{t+k,N_u}\Delta U^0_{t,N_u} + J_0 \qquad (52)$$

where $\mathcal{X}_{t+k,N_u} = T_u^T\mathcal{V}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N}T_u + \Lambda^2_{N_u}$. From a perturbation and gradient calculation [9], noting that the J_0 term is independent of the control action, the vector of *GPC future optimal control signals*:

$$\Delta U^0_{t,N_u} = -\mathcal{X}^{-1}_{t+k,N_u}T_u^T\mathcal{V}^T_{Pt+k,N}\tilde{D}_{Pt+k,N} = -\mathcal{X}^{-1}_{t+k,N_u}T_u^T\mathcal{V}^T_{Pt+k,N}\left(D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t)\right) \qquad (53)$$

where $\Delta U^0_{t,N_u} = \begin{bmatrix} \Delta u_0(t) \\ \Delta u_0(t+1) \\ \Delta u_0(t+2) \\ \vdots \\ \Delta u_0(t+N_u) \end{bmatrix}$ and $D_{Pt,N} = \begin{bmatrix} d_{pd}(t) \\ d_{pd}(t+1) \\ d_{pd}(t+2) \\ \vdots \\ d_{pd}(t+N) \end{bmatrix}$.

The *GPC* optimal control signal at time *t* is defined from this vector based on the *receding horizon principle* [10] and is taken as the first element in the vector of future control increments $\Delta U^0_{t,N_u}$.

4.3 Equivalent Cost Optimization Problem

The above is equivalent to a special cost-minimisation control problem which is needed to motivate the *NPGMV* problem. Let $\mathcal{X}_{t+k,N_u} = T_u^T\mathcal{V}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N}T_u + \Lambda^2_{N_u}$, that enters (53), be factorised as:

$$\mathcal{Y}^T_{t+k,N_u}\mathcal{Y}_{t+k,N_u} = \mathcal{X}_{t+k,N_u} = T_u^T\mathcal{V}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N}T_u + \Lambda^2_{N_u} \qquad (54)$$

Then by *completing the squares* in (52) the cost becomes:

$$J = \left(\tilde{D}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N}T_u\mathcal{Y}^{-1}_{t+k,N_u} + \Delta U^{0T}_{t,N_u}\mathcal{Y}^T_{t+k,N_u}\right)\left(\mathcal{Y}^{-T}_{t+k,N_u}T_u^T\mathcal{V}^T_{Pt+k,N}\tilde{D}_{Pt+k,N} + \mathcal{Y}_{t+k,N_u}\Delta U^0_{t,N_u}\right)$$

$$+\tilde{D}^T_{Pt+k,N}(I - \mathcal{V}_{Pt+k,N}T_u\,\mathcal{Y}^{-1}_{t+k,N_u}\mathcal{Y}^{-T}_{t+k,N_u}T_u^T\mathcal{V}^T_{Pt+k,N})\tilde{D}_{Pt+k,N} + J_0 \qquad (55)$$

By comparison with (55), the cost-function may be written as:

$$J = \hat{\varPsi}^{0T}_{t+k,N_u} \hat{\varPsi}^{0}_{t+k,N_u} + J_{10}(t) \tag{56}$$

where the "squared" term in (55):

$$\hat{\varPsi}^{0}_{t+k,N_u} = \mathcal{Y}^{-T}_{t+k,N_u} T_u^T \mathcal{V}^T_{Pt+k,N} \tilde{D}_{Pt+k,N} + \mathcal{Y}_{t+k,N_u} \Delta U^0_{t,N_u}$$

$$= \mathcal{Y}^{-T}_{t+k,N_u} T_u^T \mathcal{V}^T_{Pt+k,N} \left(D_{Pt+k,N} + \mathcal{C}_{Pt+k,N} \mathcal{A}_{t+k,N} \hat{x}(t+k \mid t) \right) + \mathcal{Y}_{t+k,N_u} \Delta U^0_{t,N_u} \tag{57}$$

The cost-terms that are independent of the control action $J_{10}(t) = J_0 + J_1(t)$ where,

$$J_1(t) = \tilde{D}^T_{Pt+k,N} (I - \mathcal{V}_{Pt+k,N} T_u \mathcal{Y}^{-1}_{t+k,N_u} \mathcal{Y}^{-T}_{t+k,N_u} T_u^T \mathcal{V}^T_{Pt+k,N}) \tilde{D}_{Pt+k,N} \tag{58}$$

The optimal control is found by setting the first term to zero, that is $\hat{\varPsi}^{0}_{t+k,N_u} = 0$. This gives the same optimal control as (53). It follows that the *GPC* optimal controller is the same as the controller to minimise the norm of the signal $\hat{\varPsi}^{0}_{t+k,N_u}$, defined in (57). The vector of optimal future controls:

$$\Delta U^0_{t,N_u} = -\mathcal{X}^{-1}_{t+k,N_u} T_u^T \mathcal{V}^T_{Pt+k,N} \tilde{D}_{Pt+k,N} = -\mathcal{X}^{-1}_{t+k,N_u} T_u^T \mathcal{V}^T_{Pt+k,N} \left(D_{Pt+k,N} + \mathcal{C}_{Pt+k,N} \mathcal{A}_{t+k,N} \hat{x}(t+k \mid t) \right) \tag{59}$$

4.4 Modified Cost-Function Generating GPC Controller

The above discussion motivates the definition of a new *multi-step minimum variance cost* problem that is similar to the minimisation problem (56) but where the link to *NGMV* design can be established. The signal to be minimised in the *GMV* problem involves a *weighted sum of error and input signals* [11]. The vector of future values, for a *multi-step criterion*:

$$\Phi_{t+k,N} = P_{CN,t} E_{Pt+k,N} + F^0_{CN,t} \Delta U^0_{t,N_u} \tag{60}$$

where the cost-function weightings $P_{CN,t} = T_u^T \mathcal{V}^T_{Pt+k,N}$ and $F^0_{CN,t} = \Lambda^2_{N_u}$. These are based on the *GPC* weightings in (47) and are justified later in *Theorem* 1 below. Now define a minimum-variance *multi-step* cost-function, using a vector of signals:

$$\tilde{J} = E\{\tilde{J}_t\} = E\{\Phi^T_{t+k,N} \Phi_{t+k,N} \mid t\} \tag{61}$$

Predicting forward *k-steps*:

$$\Phi_{t+k,N} = P_{CN,t} E_{Pt+k,N} + F^0_{CN,t} \Delta U^0_{t,N_u} \tag{62}$$

Now consider the signal Φ_{t+k,N_u} and substitute for $E_{Pt+k,N} = \hat{E}_{Pt+k,N} + \tilde{E}_{Pt+k,N}$:

$$\Phi_{t+k,N} = P_{CN,t}(\hat{E}_{Pt+k,N} + \tilde{E}_{Pt+k,N}) + F^0_{CN,t}\Delta U^0_{t,N} = \left(P_{CN,t}\hat{E}_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u}\right) + P_{CN,t}\tilde{E}_{Pt+k,N} \tag{63}$$

This may be written as:

$$\Phi_{t+k,N} = \hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N} \tag{64}$$

where the *predicted signal* $\hat{\Phi}_{t+k,N} = \left(P_{CN,t}\hat{E}_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u}\right)$ and the *prediction error* $\tilde{\Phi}_{t+k,N_u} = P_{CN,t}\tilde{E}_{Pt+k,N}$. The performance index (61) may therefore be simplified, recalling $\hat{E}_{Pt+k,N}$ and $\tilde{E}_{Pt+k,N}$ are orthogonal, as follows:

$$\tilde{J}(t) = E\{\tilde{J}_t\} = E\{\Phi^T_{t+k,N}\Phi_{t+k,N} \mid t\} = E\{(\hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N})^T(\hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N}) \mid t\}$$

$$= \hat{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} + E\{\tilde{\Phi}^T_{t+k,N}\tilde{\Phi}_{t+k,N} \mid t\} = \hat{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} + \tilde{J}_1(t) \tag{65}$$

where $\tilde{J}_1(t) = E\{\tilde{\Phi}^T_{t+k,N}\tilde{\Phi}_{t+k,N} \mid t\} = E\{\tilde{E}^T_{Pt+k,N}P^T_{CN,t}P_{CN,t}\tilde{E}_{Pt+k,N} \mid t\}$. The prediction $\hat{\Phi}_{t+k,N}$ may be simplified as follows:

$$\hat{\Phi}_{t+k,N} = P_{CN,t}\hat{E}_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u} = P_{CN,t}(\tilde{D}_{Pt+k,N} + \mathcal{V}_{Pt+k,N}T_u\,\Delta U^0_{t,N_u}) + F^0_{CN,t}\Delta U^0_{t,N_u}$$

By substituting from (54) (noting $P_{CN,t}\mathcal{V}_{Pt+k,N}T_u + F^0_{CN,t} = \mathcal{X}_{t+k,N_u}$),

$$\hat{\Phi}_{t+k,N} = P_{CN,t}\tilde{D}_{Pt+k,N} + \mathcal{X}_{t+k,N_u}\Delta U^0_{t,N_u} \tag{66}$$

Recall the weightings are assumed to be chosen so that $\mathcal{X}_{t+k,N_u}$ is non-singular. From a similar argument to that in the previous section the *predictive control* sets the first *squared term* in (65) to zero $\hat{\Phi}_{t+k,N} = 0$ and this expression is the same as the vector of future *GPC* controls.

Theorem 1: Equivalent Minimum Variance Cost Problem

Consider the minimisation of the *GPC* cost index (46) for the system and assumptions introduced in §2, where the nonlinear subsystem $\mathcal{W}_{1k} = I$ and the vector of optimal *GPC* controls is given by (53). Assume that the cost index is redefined to have a multi-step minimum variance form (61):

$$\tilde{J}(t) = E\{\Phi^T_{t+k,N}\Phi_{t+k,N} \mid t\}, \quad \text{where} \quad \Phi_{t+k,N_u} = P_{CN,t}E_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u} \tag{67}$$

Let the cost-function weightings be defined relative to the original *GPC* cost-index as:

$$P_{CN,t} = T_u^T\mathcal{V}^T_{Pt+k,N} \qquad \text{and} \qquad F^0_{CN,t} = \Lambda^2_{N_u}$$

The vector of future optimal controls that minimize (67) follows as:

$$\Delta U^0_{t,N_u} = -\mathcal{X}^{-1}_{t+k,N_u} T_u^T \mathcal{V}^T_{Pt+k,N}\left(D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t)\right) \tag{68}$$

where $\mathcal{X}_{t+k,N_u} = T_u^T \mathcal{V}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N} T_u + \Lambda^2_{N_u}$. This optimal control (68) is identical to the vector of *GPC* controls.

■

Solution: The proof follows by collecting the results above. ■

5. Nonlinear Predictive GMV Optimal Control

The aim of the nonlinear control design approach is to ensure certain input-output maps are finite-gain m_2 stable and the cost-index is minimized. Recall that the input to the system is the control signal $u(t)$, shown in Fig. 1, rather than the input to the state-dependent sub-system $u_0(t)$. The cost-function for the nonlinear control problem must therefore include an additional control costing term, although the costing on the intermediate signal $u_0(t)$ can be retained. If the smallest delay in each output of the plant is of *k-steps* the control signal *t* affects the output *k-steps* later. For *NGMV* the signal costing $\left(\mathcal{F}_c\Delta u\right)(t) = \left(\mathcal{F}_{ck} z^{-k}\Delta u\right)(t)$. Typically this weighting on the nonlinear sub-system input will be a linear dynamic operator [12], assumed to be full rank and invertible. In analogy with the *GPC* problem a *multi-step cost index* may be defined that is an extension of (61):

$$J_p = E\{\Phi^{0T}_{t+k,N}\Phi^0_{t+k,N}\,|\,t\} \tag{69}$$

Thus, consider a signal whose variance is to be minimised, involving a weighted sum of error, input and control signals ([11], [13]):

$$\Phi^0_{t+k,N} = P_{CN,t}E_{Pt+k,N} + F^0_{CN,t}\,\Delta U^0_{t,N_u} + \mathcal{F}_{\mathrm{c}k,N_u}\Delta U_{t,N_u} \tag{70}$$

The non-linear function $\mathcal{F}_{\mathrm{c}k,N_u}\Delta U_{t,N_u}$ will normally be defined to have a simple block diagonal form:

$$(\mathcal{F}_{\mathrm{c}k,N_u}\Delta U_{t,N_u}) = diag\{\left(\mathcal{F}_{ck}\Delta u\right)(t),\left(\mathcal{F}_{ck}\Delta u\right)(t+1),...,\left(\mathcal{F}_{ck}\Delta u\right)(t+N_u)\} \tag{71}$$

Note the vector of changes at the input of the state-dependent sub-system:

$$\Delta U^0_{t,N_u} = (\mathcal{W}_{1\mathrm{k},\mathrm{N}_u}\Delta U_{t,N_u}) \tag{72}$$

This is, the output of the nonlinear input-subsystem $\mathcal{W}_{1\mathrm{k},\mathrm{N}_u}$, which also has a block diagonal matrix form:

$$(\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}) = diag\{\mathcal{W}_{1k}, \mathcal{W}_{1k}, \dots, \mathcal{W}_{1k}\}\Delta U_{t,N_u} = [(\mathcal{W}_{1k}\Delta u)(t)^T, \dots, (\mathcal{W}_{1k}\Delta u)(t+N_u)^T]^T \tag{73}$$

5.1 The NPGMV Control Solution

Note the *state estimation error* is independent of the choice of control action. Also recall that the optimal $\hat{x}(t+k\,|\,t)$ and $\tilde{x}(t+k\,|\,t)$ are orthogonal and the expectation of the product of the future values of the control action (assumed known in deriving the prediction equation), and the zero-mean white noise driving signals, is null. It follows that $\hat{E}_{Pt+k,N}$ and the prediction error $\tilde{E}_{Pt+k,N}$ are orthogonal. The solution of the *NPGMV* control problem follows from similar steps to those in §3.3. Observe from (62) that $\Phi_{t+k,N} = P_{CN,t}E_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u}$ and $\Phi^0_{t+k,N} = \hat{\Phi}^0_{t+k,N} + \tilde{\Phi}^0_{t+k,N}$. It follows from (70) that the predicted signal:

$$\hat{\Phi}^0_{t+k,N} = \hat{\Phi}_{t+k,N} + (\mathcal{F}_{ck,N_u}\Delta U_{t,N_u}) = P_{CN,t}\hat{E}_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u} + (\mathcal{F}_{ck,N_u}\Delta U_{t,N_u}) \tag{74}$$

and the estimation error:

$$\tilde{\Phi}^0_{t+k,N} = \tilde{\Phi}_{t+k,N} = P_{CN,t}\tilde{E}_{Pt+k,N} = T_u^T \mathcal{V}^T_{Pt+k,N}\tilde{E}_{Pt+k,N} \tag{75}$$

The future predicted values of the signal $\hat{\Phi}^0_{t+k,N}$ involve the estimated vector of weighted errors $P_{CN,t}\hat{E}_{Pt+k,N}$, which are orthogonal to $P_{CN,t}\tilde{E}_{Pt+k,N}$. The estimation error is zero-mean and the expected value of the product with any known signal is null. The multi-step cost index may therefore be written as:

$$\tilde{J}(t) = \hat{\Phi}^{0T}_{t+k,N}\hat{\Phi}^0_{t+k,N} + \tilde{J}_1(t) \tag{76}$$

The *condition for optimality* $\hat{\Phi}^0_{t+k,N} = 0$ now becomes:

$$P_{CN,t}\hat{E}_{Pt+k,N} + F^0_{CN,t}\Delta U^0_{t,N_u} + \mathcal{F}_{ck,N_u}\Delta U_{t,N_u} = 0 \tag{77}$$

5.2 NPGMV Optimal Control

The vector of future optimal control signals, to minimise (76), follows from the condition for optimality in (77)

$$P_{CN,t}\hat{E}_{Pt+k,N} + \Lambda^2_{N_u}\mathcal{W}_{1k,N_u}\Delta U_{t,N_u} + \mathcal{F}_{ck,N_u}\Delta U_{t,N_u} = 0$$

$$\Delta U_{t,N_u} = (\mathcal{F}_{ck,N_u} + \Lambda^2_{N_u}\mathcal{W}_{1k,N_u})^{-1}(-P_{CN}\hat{E}_{Pt+k,N}) \tag{78}$$

An alternative solution of (77), gives:

$$\Delta U_{t,N_u} = \mathcal{F}_{ck,N_u}^{-1}\left(-T_u^T \mathcal{V}_{Pt+k,N}^T \hat{E}_{Pt+k,N} - \Lambda_{N_u}^2 \mathcal{W}_{1k,N_u} \Delta U_{t,N_u}\right) \tag{79}$$

Further simplification by noting the *condition for optimality* $\hat{\Phi}_{t+k,N}^0 = 0$ may be written, from (51), (54), (72) and (74) as $P_{CN,t}\hat{E}_{Pt+k,N} + F_{CN,t}^0 \Delta U_{t,N_u}^0 + (\mathcal{F}_{ck,N_u}\Delta U_{t,N_u}) = 0$, and becomes:

$$P_{CN,t}\tilde{D}_{Pt+k,N} + \left(\mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u} + \mathcal{F}_{ck,N_u}\right)\Delta U_{t,N_u} = 0 \tag{80}$$

where $\tilde{D}_{Pt+k,N} = D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t)$. The *vector of future optimal control* becomes:

$$\Delta U_{t,N_u} = \left(\mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u} + \mathcal{F}_{ck,N_u}\right)^{-1}\left(-P_{CN,t}D_{Pt+k_0,N} - \mathcal{C}_{\phi t}\hat{x}(t+k\,|\,t)\right) \tag{81}$$

where from $P_{CN,t} = T_u^T \mathcal{V}_{Pt+k,N}^T$ and $\mathcal{C}_{\phi t}$ is defined as:

$$\mathcal{C}_{\phi t} = P_{CN,t}\mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N} = T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N} \tag{82}$$

An alternative useful solution follows from (80) as:

$$\Delta U_{t,N_u} = \mathcal{F}_{ck,N_u}^{-1}\left(-P_{CN,t}\tilde{D}_{Pt+k,N} - \mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}\right)$$

$$= \mathcal{F}_{ck,N_u}^{-1}\left(-P_{CN,t}D_{Pt+k,N} - \mathcal{C}_{\phi t}\hat{x}(t+k\,|\,t) - \mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}\right)$$

The control law is to be implemented using a *receding horizon philosophy*. Let $C_{I0} = [I, 0, ..., 0]$ and $C_{0I} = \begin{bmatrix} 0 & I_N \end{bmatrix}$ so that the current and future controls are $\Delta u(t) = [I, 0, ..., 0]\Delta U_{t,N}$ and $\Delta U_{t,N}^f = C_{0I}\Delta U_{t,N}$.

Theorem 2: NPGMV State-Dependent Optimal Control

Consider the linear components of the plant, disturbance and output weighting models put in augmented state equation form (12), with input from the nonlinear finite gain stable plant dynamics $\mathcal{W}_{1k}$. Assume that the *multi-step predictive controls* cost-function to be minimised, involves a sum of future cost terms, and is defined in vector form as:

$$J_p = E\{\Phi_{t+k,N}^{0T}\Phi_{t+k,N}^0\,|\,t\} \tag{83}$$

where the signal $\Phi_{t+k_0,N}^0$ depends upon future error, input and nonlinear control signal costing terms:

$$\Phi_{t+k,N}^0 = P_{CN,t}E_{Pt+k,N} + F_{CN,t}^0\Delta U_{t,N_u}^0 + \mathcal{F}_{ck,N_u}\Delta U_{t,N_u} \tag{84}$$

Assume the error and input cost-function weightings are introduced as in the *GPC* problem (46) and these are used to define the block matrix cost weightings $P_{CN,t} = T_u^T \mathcal{V}_{Pt+k,N}^T$ and $F_{CN,t}^0 = \Lambda_{N_u}^2$. Also assume that the control signal cost weighting is nonlinear and is of the form $(\mathcal{F}_c \Delta u)(t) = (\mathcal{F}_{ck} \Delta u)(t-k)$, where $\mathcal{F}_{ck}$ is full rank and invertible operator. Then the *NPGMV* optimal control law to minimize the variance (83) is given as:

$$U_{t,N} = \mathcal{F}_{ck,N_u}^{-1} \left(-P_{CN,t} D_{Pt+k,N} - \mathcal{C}_{\phi t} \hat{x}(t+k \mid t) - \mathcal{X}_{t+k,N_u} \mathcal{W}_{1k,N_u} \Delta U_{t,N_u} \right) \qquad (85)$$

where $\mathcal{X}_{t+k,N_u} = T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{V}_{Pt+k,N} T_u + \Lambda_{N_u}^2$ and $\mathcal{C}_{\phi t} = T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{C}_{Pt+k,N} \mathcal{A}_{t+k,N}$. The current control can be computed using the *receding horizon principle* from the first component in the *vector of future optimal controls.* ■

Solution: The proof of the optimal control was given before the Theorem. The assumption to ensure closed-loop stability is explained in the stability analysis that follows below. ■

Remarks: The expressions for the *NPGMV* control (81) and (85) lead to alternative structures for implementation but the second in Fig. 2, is more suitable for implementation. Inspection of the cost term (84) when the input costing F_{CN}^0 is null gives $\Phi_{t+k,N}^0 = P_{CN,t} E_{Pt+k,N} + \mathcal{F}_{ck,N} U_{t,N}$ and the limiting case of the *NPGMV* controller is related to an *NGMV* controller [12].

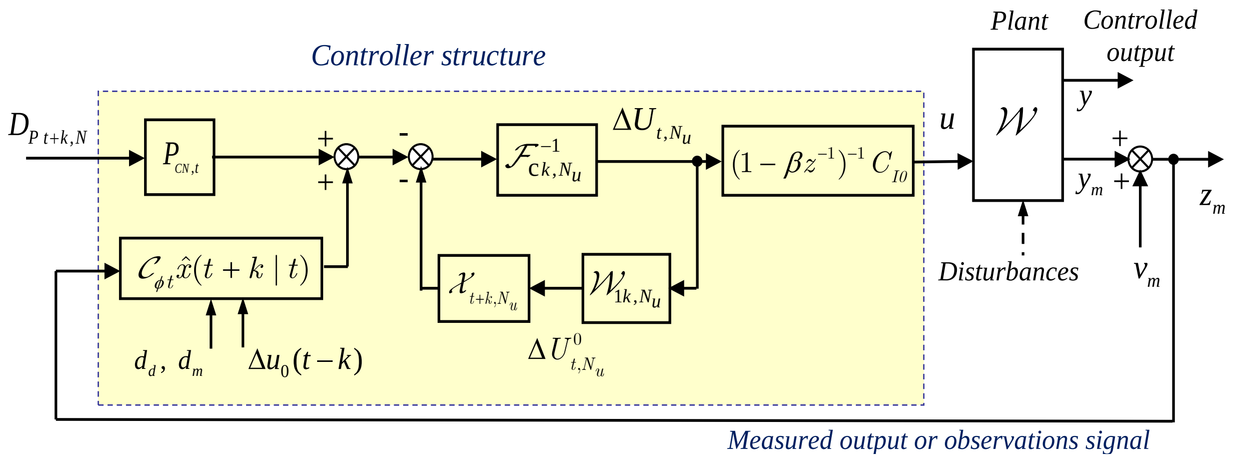

Fig. 2: **Implementation Form of NPGMV State-Dependent Controller Structure**

6. Stability of the Closed-Loop

For linear *GMV* designs stability is ensured when the combination of a control weighting and an error weighted plant model transfer is strictly minimum-phase. For the nonlinear predictive control a nonlinear operator:

$$\left(I+\mathcal{F}_{ck,N_u}^{-1}\left(\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0}+\mathcal{X}_{t+k,N_u}\right)\mathcal{W}_{1k,N_u}\right)$$

must have a stable inverse (shown below). It will be assumed that the stochastic external inputs are null and the only inputs are those due to the deterministic signals. The state:

$$x(t)=(I-z^{-1}\mathcal{A}_t)^{-1}z^{-1}(\mathcal{B}_t u_0(t-k)+d_d(t))=\Phi_t\left(\mathcal{B}_t u_0(t-k)+d_d(t)\right) \tag{86}$$

$$x(t+k)=\Phi_{t+k}(\mathcal{B}_{t+k}u_0(t)+d_d(t+k)) \tag{87}$$

where $\Phi_t=(I-z^{-1}\mathcal{A}_t)^{-1}z^{-1}$. The predicted state $\hat{x}(t+k\,|\,t)=x(t+k)=\Phi_{t+k}(\mathcal{B}_{t+k}u_0(t)+d_d(t+k))$ and from (85):

$$U_{t,N}=\mathcal{F}_{ck,N_u}^{-1}\left(-P_{CN,t}D_{P\,t+k,N}-\mathcal{C}_{\phi t}\hat{x}(t+k\,|\,t)-\mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}\right)$$

$$=\mathcal{F}_{ck,N_u}^{-1}\left(-P_{CN,t}D_{P\,t+k,N}-\mathcal{C}_{\phi t}\Phi_{t+k}d_d(t+k)-\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}u_0(t)-\mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}\right) \tag{88}$$

Assuming the control costing is a linear model the condition for optimality (88):

$$\left(\mathcal{F}_{ck,N_u}+\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0}\mathcal{W}_{1k,N_u}+\mathcal{X}_{t+k,N_u}\mathcal{W}_{1k,N_u}\right)\Delta U_{t,N_u}=-\left(P_{CN,t}D_{P\,t+k,N}+\mathcal{C}_{\phi t}\Phi_{t+k}d_d(t+k)\right)$$

The input nonlinear sub-system can be assumed finite-gain m_2 stable and $\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}$ may be written as $(\mathcal{W}_{1k,N_u}\Delta U_{t,N_u})=[(\mathcal{W}_{1k}\Delta u)(t)^T,...,(\mathcal{W}_{1k}\Delta u)(t+N_u)^T]^T$. The vector of future optimal controls becomes:

$$\Delta U_{t,N_u}=\left(I+\mathcal{F}_{ck,N_u}^{-1}\left(\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0}+\mathcal{X}_{t+k,N_u}\right)\mathcal{W}_{1k,N_u}\right)^{-1}\mathcal{F}_{ck,N_u}^{-1}\left(-P_{CN,t}D_{P\,t+k,N}-\mathcal{C}_{\phi t}\Phi_{t+k}d_d(t+k)\right) \tag{89}$$

The *NL* Subsystem future outputs follows as $\mathcal{W}_{1k,N_u}\Delta U_{t,N_u}$ and the future plant outputs $\mathcal{W}_{k,N_u}U_{t,N}$. It follows a *necessary condition for stability* is that the operator that follows is finite gain stable:

$$\mathcal{H}_{t+k,N_u}=\left(I+\mathcal{F}_{ck,N_u}^{-1}\left(\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0}+\mathcal{X}_{t+k,N_u}\right)\mathcal{W}_{1k,N_u}\right)^{-1} \tag{90}$$

6.1 Sufficient Condition for Stability and Robustness

If the output sub-system were linear time-invariant and not subject to uncertainty, a similar stability argument to that in [14] could be used to argue from (89) that no cancellation of unstable modes could occur if the controller is implemented in its minimal form. The robustness of the solution may be considered and a sufficient condition for stability in the presence of uncertainty can be obtained by first noting the solution can be related to the well-known *Smith Predictor* structure. To establish this equivalence consider the more usual problem, where system outputs controlled are the same as those measured and where absolute control is costed. The algebra is similar to the non-state-dependent problems considered in [13]. The controller, which should not be implemented in this form, is shown in Fig. 3. The $\mathcal{T}_{f1}(z^{-1})$ term in this solution is obtained by writing the *Kalman filter* loop in terms of the operator equations that follow:

Estimator: $\hat{x}(t|t) = \mathcal{T}_{f1}(z^{-1})(z(t) - d(t)) + \mathcal{T}_{f2}(z^{-1})u_0(t-k)$

The transfer operators here: $\mathcal{T}_{f1}(z^{-1}) = (I - z^{-1}(I - \mathcal{K}_{ft}\mathcal{C}_{t+1})\mathcal{A}_t)^{-1}\mathcal{K}_{ft-1}$

$$\mathcal{T}_{f2}(z^{-1}) = (I - z^{-1}(I - \mathcal{K}_{ft}\mathcal{C}_{t+1})\mathcal{A}_t))^{-1}\left((I - \mathcal{K}_{ft-1}\mathcal{C}_t)\mathcal{B}_{t-1}z^{-1} - \mathcal{K}_{ft-1}\mathcal{E}_t\right)$$

Unbiased estimates property: Observe that for the *Kalman filter* to be unbiased:

$$\mathcal{T}_{f1}(z^{-1})(\mathcal{C}_t\Phi_t(z^{-1})\mathcal{B}_t + \mathcal{E}_t) + \mathcal{T}_{f2}(z^{-1}) = \Phi_t(z^{-1})\mathcal{B}_t$$

The parallel paths in Fig. 3, from control input are useful if the plant has an additive uncertainty of the form $\mathcal{W} = \overline{\mathcal{W}} + \Delta\mathcal{W}$. The diagram in Fig. 3 may then be redrawn as shown in Fig. 4.

For the sufficient condition for optimality note that the operator $\mathcal{H}_{t+k,N_u}$ actually represents the internal feedback loop in Fig. 5. Thus the operator S_1 representing the path between φ and u includes this stable sub-system and the *Kalman filter* sub-system. The operator S_1 and uncertainty model $S_2 = \Delta\mathcal{W}$ can both therefore be assumed stable. The *small gain theorem* [15], can now be invoked to provide a sufficient condition for stability. Recall this can be used to establish input-output stability conditions for a feedback system. It provides a sufficient condition for finite gain $\mathcal{L}_p$ stability of the closed-loop system. If two input-output stable systems S_1 and S_2 are connected as shown in a feedback loop, then the closed-loop is input-output stable if the loop gain $\|S_1\|.\|S_2\| < 1$, where the norm used is any induced norm. To deal with unstable signals the space $\mathcal{L}_{p,e}$ (see [16]) is used, where the upper limit of the norm summation is finite. The sufficient condition for stability requires $\|S_1\| < 1/\|\Delta\mathcal{W}\|$ so the gain of the inner feedback loop term should be sufficiently small when the uncertainty is large.

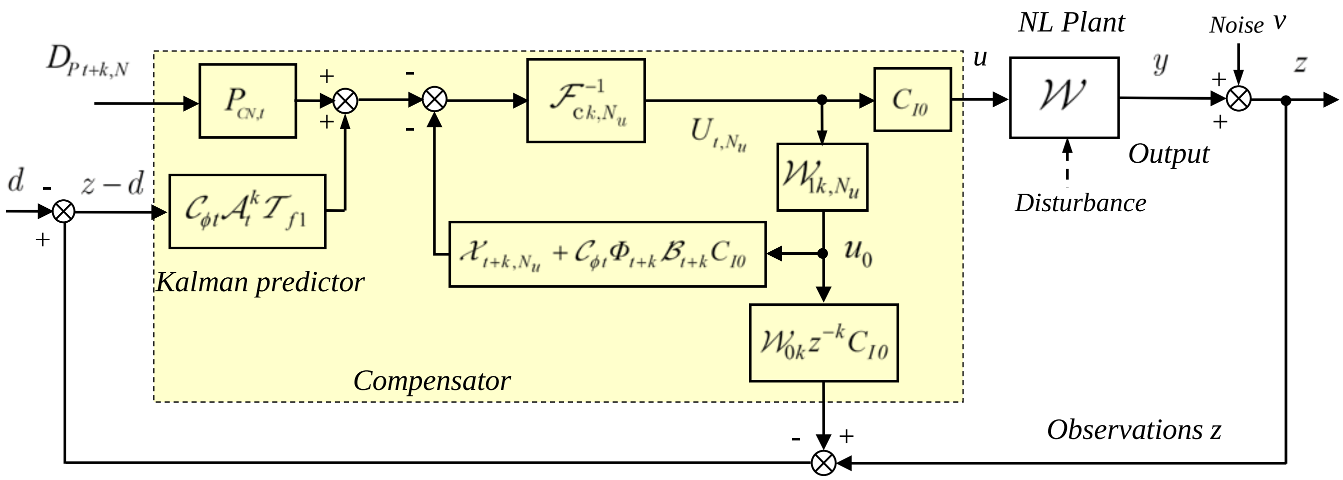

Fig. 3: **Nonlinear Smith Predictor Implied by NPGMV Compensator Structure**

$$\Delta U_{t,N_u} = \left(I + \mathcal{F}^{-1}_{ck,N_u}\left(\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0} + \mathcal{X}_{t+k,N_u}\right)\mathcal{W}_{1k,N_u}\right)^{-1}\mathcal{F}^{-1}_{ck,N_u}\left(-P_{CN,t}D_{Pt+k,N} - \mathcal{C}_{\phi t}\Phi_{t+k}d_d(t+k)\right)$$

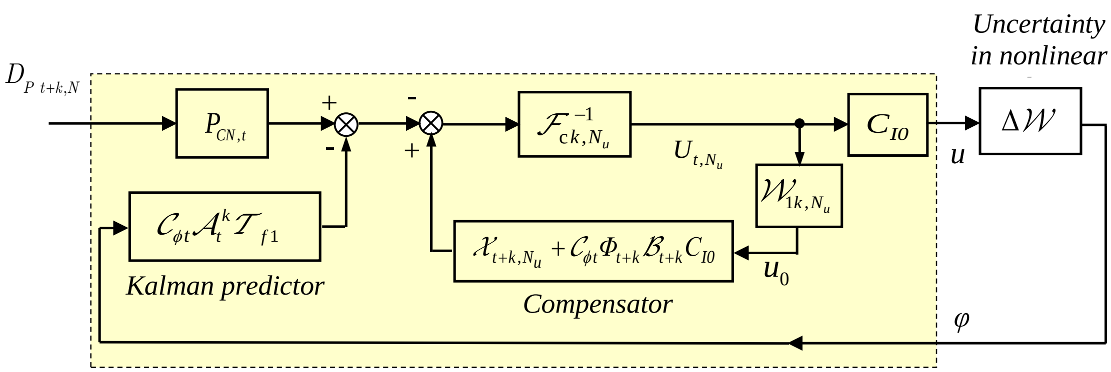

Fig. 4: **Feedback Loop when Additive Uncertainty Included**

6.2 Cost Weightings and Relationship to Stability

Say there exists a *PID* controller that will stabilize the nonlinear system, without transport delay, then a set of cost weightings can be defined to guarantee the existence of this inverse and hence ensure the stability of the closed-loop. A stabilising control law can be found from cost-function weightings derived below. Assume $\Lambda^2_{N_u} \to 0$, then from (54) $\mathcal{X}_{t+k,N_u} \to T_u^T \mathcal{V}^T_{Pt+k,N}\mathcal{V}_{Pt+k,N}T_u$, and from (89):

$$\Delta U_{t,N_u} \rightarrow \left(I + \mathcal{F}_{ck,N_u}^{-1}\left(\mathcal{C}_{\phi t}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0} + T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{V}_{Pt+k,N} T_u\right)\mathcal{W}_{1k,N_u}\right)^{-1} \mathcal{F}_{ck,N_u}^{-1}\left(-P_{CN,t}D_{Pt+k,N} - \mathcal{C}_{\phi t}\Phi_{t+k}d_d(t+k)\right)$$

In the case of a single-step cost with a through term the matrix $\mathcal{V}_{Pt+k,N} = \mathcal{E}_{t+k,N}$ can be assumed square and non-singular. In the case $N = 0$ $\mathcal{V}_{Pt,N} = \mathcal{E}_{pt}$ and $P_{CN,t} = T_u^T \mathcal{V}_{Pt+k,N}^T = \mathcal{V}_{Pt+k,N}^T = \mathcal{E}_{pt+k}^T$, $\mathcal{C}_{\phi t} = \mathcal{E}_{pt+k}^T \mathcal{C}_{pt+k}$. Hence,

$$u(t) \rightarrow \left(I + \mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T\left(\mathcal{C}_{pt+k}\Phi_{t+k}\mathcal{B}_{t+k}C_{I0} + \mathcal{E}_{pt+k}\right)\mathcal{W}_{1k,}\right)^{-1} \mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T\left(-D_{Pt+k,N} - \mathcal{C}_{pt+k}\Phi_{t+k}d_d(t+k)\right)$$

Also assume the dynamic weighting is on the plant outputs $y_p(t) = P_C(z^{-1})y(t)$ then $\mathcal{E}_{pt+k} + \mathcal{C}_{pt+k}\Phi_{t+k}\mathcal{B}_{t+k} = P_C \mathcal{W}_{0k}$,

$$u(t) \rightarrow \left(I + \mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T P_C \mathcal{W}_{0k}\mathcal{W}_{1k}\right)^{-1} \mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T\left(-D_{Pt+k,N} - \mathcal{C}_{pt+k}\Phi_{t+k}d_d(t+k)\right) \tag{91}$$

The term $(I + \mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T P_C \mathcal{W}_{0k}\mathcal{W}_{1k})$ may be interpreted as the return-difference operator for a nonlinear system with delay-free plant $\mathcal{W}_k = \mathcal{W}_{0k}\mathcal{W}_{1k}$. Thus, if the plant has a controller K_{PID} that stabilises this model, the ratio of weightings can be chosen as $\mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T P_C = K_{PID}$.

An extension of this idea is when a set of controllers say $K_i(z^{-1})$ for $i=1,\ldots,n_k$ stabilise the system then a set of weightings can be defined to satisfy $(\mathcal{F}_{ck}^{-1}\mathcal{E}_{pt+k}^T P_C)_i = K_i$. The best robust cost-weightings can then be chosen using a technique like Monte-Carlo simulation covering a range of uncertainty [17].

7. NPGMV Special Simple Form

In some cases the nonlinear system can be represented by the state-dependent model only and the black-box model $\mathcal{W}_{1k}$ can be set equal to the identity $\mathcal{W}_{1k} = I$ (so that $\mathcal{W}_{1k,N} = I_N$). In this case $u_0(t) = u(t)$ and the control weighting involves a combination of the constant Λ_N^2 and dynamic $\mathcal{F}_{ck,N}$ weighting terms. From (80):

$$P_{CN,t}\left(D_{Pt+k,N} + \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}\hat{x}(t+k\,|\,t)\right) + \left(\mathcal{X}_{t+k,N_u} + \mathcal{F}_{ck,N_u}\right)\Delta U_{t,N_u} = 0 \tag{92}$$

The vector of future controls:

$$\Delta U_{t,N} = \left(\mathcal{X}_{t+k,N_u} + \mathcal{F}_{ck,N_u}\right)^{-1}\left(-P_{CN,t}D_{Pt+k,N} - \mathcal{C}_{\phi t}\hat{x}(t+k\,|\,t)\right) \tag{93}$$

where $\mathcal{X}_{t+k,N_u} = T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{V}_{Pt+k,N} T_u + \Lambda_{N_u}^2$, $P_{CN,t} = T_u^T \mathcal{V}_{Pt+k,N}^T$ and $\mathcal{C}_{\phi t} = T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{C}_{Pt+k,N}\mathcal{A}_{t+k,N}$.

7.1 Special Weighting Case

Assume the dynamic control weighting $\mathcal{F}_{ck}(z^{-1})$ is linear, or alternatively, has a nonlinear decomposition into a non-dynamic or constant term $\mathcal{F}_{ck}^{a}$ and an operator term $\mathcal{F}_{ck}^{b}(z^{-1})$, including at least a unit-delay $\mathcal{F}_{ck}(z^{-1}) = \mathcal{F}_{ck}^{a} + \mathcal{F}_{ck}^{b}(z^{-1})$. In this case further simplifications arise and there is no algebraic loop. Note the block version of these functions, involves the decomposition of $\mathcal{F}_{ck,N_u}$ into terms $\mathcal{F}_{ck,N_u}^{a}$ and $\mathcal{F}_{ck,N_u}^{b}(z^{-1})$. Hence the algorithms may be simplified by substituting $\mathcal{F}_{ck,N_u}(z^{-1}) = \mathcal{F}_{ck,N_u}^{a} + \mathcal{F}_{ck,N_u}^{b}(z^{-1})$. From (92)

$$\left(P_{CN,t}D_{Pt+k,N} + \mathcal{C}_{\phi t}\hat{x}(t+k\,|\,t)\right) + \left(\mathcal{X}_{t+k,N_u} + \mathcal{F}_{ck,N_u}^{a}\right)\Delta U_{t,N_u} + \mathcal{F}_{ck,N_u}^{b}(z^{-1})\Delta U_{t,N_u} = 0$$

Thence for a linear control costing:

$$\Delta U_{t,N_u} = \left(\mathcal{X}_{t+k,N_u} + \mathcal{F}_{ck,N_u}^{a}\right)^{-1}\left(-P_{CN,t}D_{Pt+k,N} - \mathcal{C}_{\phi t}\,\hat{x}(t+k\,|\,t) - \mathcal{F}_{ck,N_u}^{b}(z^{-1})\Delta U_{t,N_u}\right) \tag{94}$$

where $\mathcal{C}_{\phi t} = T_u^T \mathcal{V}_{Pt+k,N}^T \mathcal{C}_{Pt+k,N} \mathcal{A}_{t+k,N}$ and $P_{CN,t} = T_u^T \mathcal{V}_{Pt+k,N}^T$. Similar results can be obtained when $\mathcal{W}_{1k}(\tau)$ can be decomposed as $\left(\mathcal{W}_{1k}u\right)(t) = \mathcal{G}_0\,u(t) + \mathcal{G}_1\left(u(t)\right)$. This algorithm is the simplest *NPGMV* solution shown in Fig. 5.

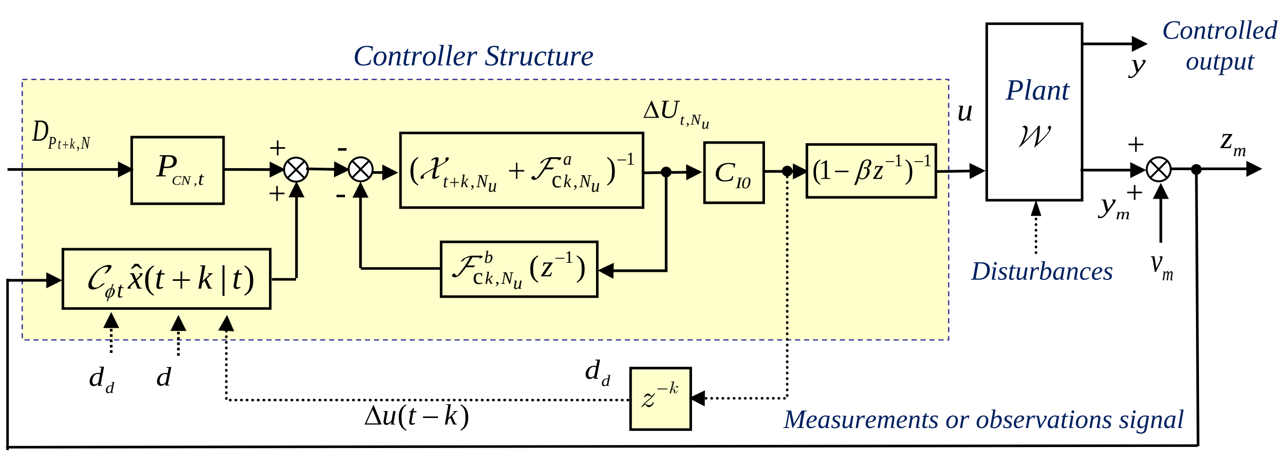

Fig. 5: **Simplified NPGMV Controller Structure for Predicted State Feedback**

8. Multivariable Control of a Two-Link Robotic Manipulator

One of the application areas for nonlinear predictive control is in industrial robotics, where the reference trajectory for the robot manipulator is defined in advance (welding or paint spraying robots). Consider for example a planar manipulator with two rigid links. The objective is to control the vector of joint angular positions q with the vector of torques τ applied at the manipulator joints, so that they follow a desired reference trajectory q_d. This problem was analysed in [18], and it was shown that a multi-loop *PD* controller could be used to control the links to desired fixed positions.

System model: The dynamics of the system are highly nonlinear and may be described by the following continuous-time coupled differential equations:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 + d_1 & -h(\dot{q}_1+\dot{q}_2) \\ h\dot{q}_1 & d_2 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

This equation may be written in the following more concise differential equation matrix form:

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{95}$$

The $H(q)$ is termed the inertia matrix, $C(q,\dot{q})\dot{q}$ is a vector of Centripetal and Coriolis torques, and $g(q)$ is a vector of torque components due to gravity. The parameters d_1 and d_2 represent the system damping due to friction (in the “ideal” nominal case $d_1 = d_2 = 0$). Assume the manipulator is operating in the horizontal plane, so that $g(q) = 0$. The components of the matrix H are defined as:

$$H_{11} = a_1 + 2a_3 \cos q_2 + 2a_4 \sin q_2\,, \quad H_{12} = H_{21} = a_2 + a_3 \cos q_2 + a_4 \sin q_2\,, \quad H_{22} = a_2$$

The parameters $h = a_3 \sin q_2 - a_4 \cos q_2$ and $a_1 = I_1 + m_1 l_{c1}^2 + I_2 + m_2 l_{c2}^2 + m_2 l_1^2$, $a_2 = I_2 + m_2 l_{c2}^2$, $a_3 = m_2 l_1 l_{c2} \cos\delta_e$ and $a_4 = m_2 l_1 l_{c2} \sin\delta_e$. The following numerical values of parameters were used for the simulation trials m_1 = 1, I_1 = 0.12, l_1 = 1, l_{c1} = 0.5, m_2 = 2, I_2 = 0.25 , l_{c2} = 0.6, δ_e = 30^o (see [18])). The above system has the state-dependent equation form. This is clear by rewriting the previous equations, where the invertability of the matrix H is a physical property of the system, as:

$$\dot{x}_q = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & -H^{-1}(q)C(\dot{q}) \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ H^{-1}(q) \end{bmatrix}\tau$$

$$y = \ddot{q} = \begin{bmatrix} 0 & -H^{-1}(q)C(\dot{q}) \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + H^{-1}(q)\tau \tag{96}$$

8.1 Two Link Robot Arm State-Dependent Solution

It was noted above that the two-link robot arm equations are in fact in a natural state-dependent form. In this case the input sub-system can be replaced by the identity and all the non-linear model can be absorbed in the state-dependent output sub-system. The control costing term is linear in this case and hence the solution is given by equation (94) and the controller can be implemented as in Fig. 5. The performance of the unconstrained *NPGMV* controller is shown in Fig. 6 for a changing reference and stochastic disturbance inputs. The interaction is clearly evident leading to large torque changes. The results for a well-tuned *PID* controller (actually *PD* terms) are also shown in Fig. 6. Note that the *PID* controller did not include any rate limits on plant inputs, as in the original publication, but the predictive control solutions both included such limits (in the constrained case taken account of directly). The *PID* becomes unstable with such limits and the predictive control results are therefore impressive.

To reduce the amplitude of control signals the constrained solution can be applied, which means applying a quadratic-programming solution to minimise (83), using the same matrices involved in (94). The area where the largest changes arise is illustrated in the expanded time-scale shown in Fig. 7. Implementing the constrained solution using quadratic programming is relatively simple in this *NPGMV* case. It is not of course very meaningful to compare the actual values of the dynamically weighted *NPGMV* cost-function. This only serves as a mathematical means to obtain desired system properties and by definition the *optimal NPGMV* controller will always provide the lowest cost for the *NPGMV* cost-function. The Table 1 of variances below has therefore been computed for the individual plant inputs and outputs, to enable a comparison of the different controls. Clearly a dynamically weighted predictive controller does not minimise the variances of these signals (this would require a minimum variance controller). The cost-function is simply a mechanism for controller design, like frequency response shaping of the sensitivities. This is also a multivariable problem, and it is not therefore simply variances that are important. Clearly cost weighting gains can easily be modified to change the importance of limiting particular inputs and outputs. Since the plant rate limits were only applied to the predictive controls the results are good as mentioned.

Table 1: **Variances for PD and NPGMV Unconstrained and Constrained Controllers**

	RMS error (q_1)	RMS error (q_2)	St Dev (τ_1)	St Dev (τ_2)
PD	12.20	4.21	240.45	146.90
NPGMV Unconstrained	8.10	6.90	517.48	313.59
NPGMV Constrained	12.75	1.27	206.23	243.84

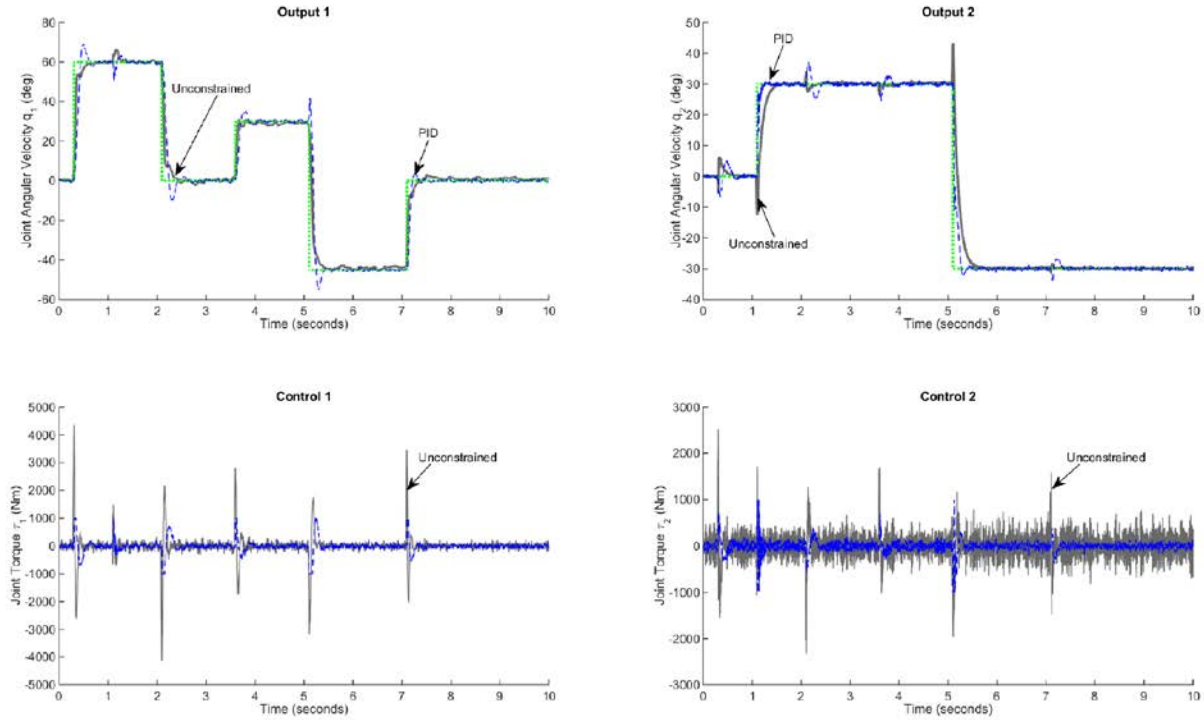

Fig. 6: **NPGMV and PID Control with Incremental Control Costing for Unconstrained Case, State-Dependent Model and Free Weighting Choice**

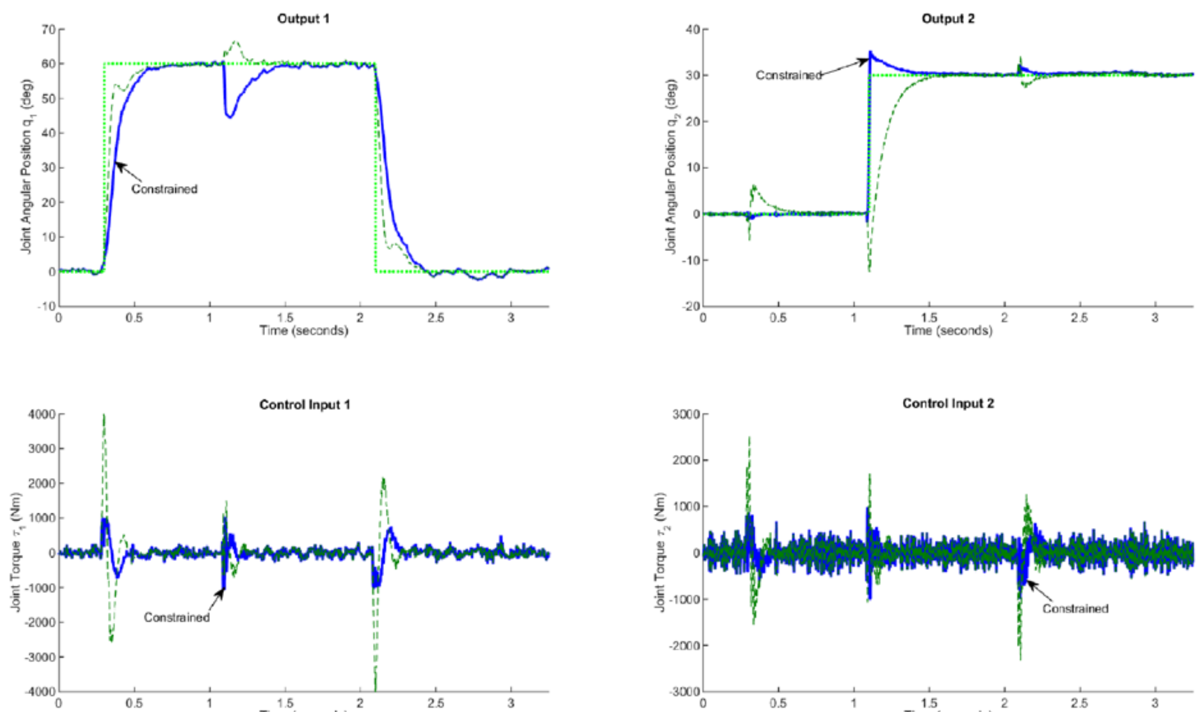

Fig. 7: **NPGMV Design for Incremental Control Action Cases and Free Error Weighting Choice For Constrained and Unconstrained Cases**

9. Concluding Remarks

The *NPGMV* control design problem for a state-dependent system involves a multi-step predictive control cost-function and future set-point information. The tracking results are more general than for *NGMV* designs because of the ability to distinguish between signals that are to be penalized and those which are measured. The use of either incremental control or control costing terms over a control horizon and control profile determined by the connection matrix, adds to the generality of the results. The simplified control structure has been shown to be particularly valuable for real applications, and avoids any algebraic-loop problem. The *NPGMV* control has the property that if the system is linear then the controller reduces to the *Generalised Predictive Controller* for state-dependent systems. The *NPGMV* controller offers greater flexibility compared with the *NGMV* and *NGPC* controllers, at the expense of some additional complexity in the implementation ([19], [20]).

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