NGMV Control of Delayed Piecewise Affine Systems

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Abstract: *A Nonlinear Generalized Minimum Variance* (*NGMV*) control algorithm is introduced for the control of piecewise affine (*PWA*) systems. Under some conditions, discrete-time *PWA* systems can be transferred into an equivalent state-dependent nonlinear system form. The equivalent state-dependent systems maintain the hybrid nature of the original *PWA* systems and includes both the discrete and continuous signals in one general description. In a more general way, the process is assumed to include common delays in input or output channels of magnitude k. Then the *NGMV* control strategy [1] can be applied. The *NGMV* controller is related to a well-known and accepted solution for time delay systems (*Smith Predictor*) but has the advantage that it may stabilize open-loop unstable processes [2].

1. Introduction

Control algorithms developed for piecewise affine systems are often designed using optimal control or *Model Predictive Control* (*MPC*) techniques. The first hybrid *MPC* algorithm, developed for mixed logical dynamical systems (equivalent to piecewise affine systems under certain mild conditions), was presented in [3]. Unfortunately, this algorithm has the drawback that it has a high on-line computational demand. This is mainly caused by the mixed integer quadratic programming problem (*NP* hard) that has to be solved on-line, at each discrete-time instant.

The delayed discrete-time *PWA* model is more general than most *PWA* models. It includes disturbances and what might be significant time-delays. Many existing control strategies are not so effective for this type of problem. State dependent systems are easier to understand and design than hybrid systems including both state and input constraints and involving explicit switching conditions. This motivates the development of an equivalent state-dependent framework of *PWA* systems that absorb these features in the system model.

State dependent systems can arise when parametric uncertainty is present in a model [4], or when the actual *Nonlinear* (*NL*) system can be approximated by a state-dependent system and an *LTI* model may be a very poor approximation. The other advantages of this model are:

1. A state-dependent model needs less supervision by logical constructs than controllers developed with traditional techniques for hybrid systems.

2. System time-delays and disturbances are more naturally modeled in the plant than some other hybrid control system models (eg. *MLD*).

3. They are easy to extend to systems with other types of nonlinearity or uncertainties.

After obtaining the hybrid system in the state-dependent form, the so called *Nonlinear Generalized Minimum Variance* (*NGMV*) controller, which is very simple to compute and implement, can be applied. In the following this transformation process and the properties of the resulting *NGMV* control law are explored. The focus is on implementation and design issues.

The rest of the paper is organized as follows. In section 2, the definitions of *PWA* systems and state-dependent systems are presented and the method by which a *PWA* system is transformed into a state-dependent system, which includes the hybrid characteristics, is discussed. In section 3, the general system models, that include three different types of subsystems, are described. The *NGMV* control law for this system is considered in section 4. Finally, conclusions are drawn in section 5.

2. Discrete-Time PWA Systems and State-dependent Systems

2.1 Piecewise Affine Systems

In this work, we focus on delayed discrete-time *PWA* systems, whose state-space representation is:

$$x(t+1) = A_i x(t) + B_i u(t-k) + D_i d(t) \tag{1a}$$

$$y(t) = C_i x(t) + E_i u(t-k) \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output and $d \in \mathbb{R}^n$ is the disturbance and k denotes the magnitude of the common delay elements. Each affine subsystem $(A_i, B_i, C_i, D_i, E_i)$, $i = 1, \ldots, s$ is defined on a cell $\Omega_i \subset \mathbb{R}^n \times \mathbb{R}^m$ that is a polyhedron. Moreover, in order to simplify the exposition, we assume that our cells are polyhedral sets defined by matrices G_{ix} , h_{ix} G_{iu} and h_{iu} as follows:

$$\Omega_i = \{\begin{bmatrix} x \\ u \end{bmatrix} \mid G_{ix} x(t) \le h_{ix} \wedge G_{iu} u(t) \le h_{iu}\} \tag{2}$$

The cells satisfy $\Omega_i \cap \Omega_j = \phi, \ \forall i \ne j$, their union defines the admissible set of states and inputs $\Omega = \cup_{i=1}^{s} \Omega_i$. Note that the delayed discrete-time *PWA* system defined in (1) can also include measurement noise $v(t)$ e.g. rewrite (1b) as $y(t) = C_i x(t) + E_i u(t-k) + v(t)$. However, in this section, we use system (1) to define the delayed discrete-time *PWA* systems. The model with output measurement noise $v(t)$ will be defined in a general format in section 3.

Although *PWA* systems have been studied in many papers, the system disturbances ($d(t)$ in system (1)) are not included in many of them, for example [5]. For those existing works focused on piecewise affine systems with disturbances [6] [7], disturbances can only belong to a small bounded set. Unlike these models, the disturbances in this work are assumed to be zero-mean, independent, Gaussian white noise. The model also includes an explicit common delay *k* on the input channel, which is not always included in the existing literature, where time-delays are often modeled using additional state variables, (see for example [8]).

A *PWA* system (1) is called well-posed [9], if $x(t+1), y(t)$ are uniquely defined functions of $x(t),\ u(t-k),\ d(t)$. For a well-posed *PWA* system, the sets Ω_i have mutually disjoint interiors, and are often defined as the partition of a convex polyhedral set. i.e. $\Omega_i \cap \Omega_j = \phi,\ \forall\, i \neq j$. Note that the well-posedness requirement of a *PWA* system is contradicted with the definition in equation (2), where Ω_i and Ω_j can have overlapping boundaries from the definition "$\leq$ ". To ensure the well-posedness, some of the inequalities in equation (2) have to be written in the form $G_{ix} x(t) < h_{ix}$ and $G_{iu} u(t) < h_{iu}$.

2.2 State dependent systems

A *state dependent system* involves state equation matrices that are time-varying since they depend upon the states and also upon control inputs, or even some other external parameters or command signal. Such an equation has the form:

$$x(t+1) = \mathcal{A}(x(t),u(t))x(t) + \mathcal{B}(x(t),u(t))u(t-k) + \mathcal{D}(x(t),u(t))d(t) \tag{3a}$$

$$y(t) = \mathcal{C}(x(t),u(t))x(t) + \mathcal{E}(x(t),u(t))u(t-k) \tag{3b}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$,$\mathcal{E}$ depend on $x(t)$ and $u(t)$. The actual dependence is on past values of $u(t)$ since different states will be affected by different delayed values of $u(t)$ (greater than a delay k). The state-dependent form involves a system model that is applicable to a wide range of nonlinear dynamical systems. It can express evolutions of continuous (linear) variables through linear dynamic equations, of discrete (nonlinear) variables through propositional logic statements, and also represent the mutual interaction between the two. State-dependent systems are therefore capable of modeling a broad class of systems, including *PWA* systems.

Proposition 1: Every well-posed *PWA* system (1) can be written as a state-dependent system (3). That is, for any feasible polyhedral partition of state plus input set $\Omega = \bigcup_{i=1}^{s} \Omega_{\mathrm{i}}$ and its corresponding parameters $(A_i, B_i, C_i, D_i, E_i)$, i = 1, . . . , s, of system (1), there exists a combination of $(\mathcal{A}(x,u),\ \mathcal{B}(x,u),\ \mathcal{C}(x,u),\ \mathcal{D}(x,u), \mathcal{E}(x,u))$ of system (3), such that all trajectories $x(t), u(t), y(t)$ of the *PWA* system (1) also satisfy the state-dependent model (3).

Proof: Consider the *PWA* system (1), to rephrase the condition (2) in logic form, we introduce an auxiliary logic variable $\delta_i(t) \in \{0,1\}$, where $\delta_i(t) = \begin{cases} 1 & if\ \ G_{ix}x(t) \le h_{ix} \wedge G_{iu}u(t) \le h_{iu} \\ 0 & otherwise \end{cases}$.

The well-posed system (1) with the partition (2), can then be written in the form:

$$x(t+1) = \sum_{i=1}^{s} \delta_i(t)[A_i x(t) + B_i u(t-k) + D_i d(t)] \tag{4a}$$

$$y(t) = \sum_{i=1}^{s} \delta_i(t)[C_i x(t) + E_i u(t-k)] \tag{4b}$$

The value of the logic variable $\delta_i(t) \in \{0,1\}$ in system (4) depends on the state and input variables $x(t)$ and $u(t)$. Define the *less than or equal* ($\le$) function LE(x, m) as: $LE(x,m) = \begin{cases} 1 & \text{if } \ x \le m \\ 0 & \text{otherwise} \end{cases}$

where constant $m \in \mathbb{R}^n$. Then,

$$\delta_i(t) = \prod_j LE(G_{ix}^j x(t), h_{ix}^j) \prod_l LE(G_{iu}^l u(t), h_{iu}^l) \tag{5}$$

where *j* and *l* denote the *j-th* row and the *l-th* row, respectively. By substituting (5) in (4a) and (4b) obtain:

$$x(t+1)=[\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)A_i]x(t)+[\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)B_i]u(t)+$$
$$[\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)D_i]d(t) \quad (6a)$$

$$y(t)=[\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)C_i]x(t)+[\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)E_i]u(t) \quad (6b)$$

Hence, the *PWA* system (1) is transformed into a NL state-dependent system (6) having the form of (3):

$$\mathcal{A}(x,u)=\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)A_i;\ \mathcal{B}(x,u)=\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)B_i;$$

$$\mathcal{C}(x,u)=\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)C_i;\ \mathcal{D}(x,u)=\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)D_i;$$

and $\mathcal{E}(x,u)=\sum_{i=1}^{s}\prod_j LE(G_{ix}^j x(t),h_{ix}^j)\prod_l LE(G_{iu}^l u(t),h_{iu}^l)E_i$ □

***Remarks*:** Note that the well-posedness of the original *PWA* system implies that $\delta_i(t)$ and $LE(\bullet,\bullet)$ are $\{0,1\}-valued$, and $\sum_{i=1}^{s}\delta_i=1$. In general, the feasible state plus input set Ω of (2) is non-convex, i.e. there must be some inequalities take the ‘<’ form. Nevertheless, the ‘<’ function can also be defined like the ‘≤’ function as $LT(x,m)=\begin{cases}1 & \text{if } x<m\\ 0 & \text{otherwise}\end{cases}$.

3. System Model Description

3.1 System Plants

In order to derive the control algorithm for state-dependent systems, we use the general system description in [1]. The plant is nonlinear and may include two nonlinear subsystems. Considering the input signals are normally bounded for *PWA* systems, the first *NL* subsystem is defined as a saturation type nonlinearity in this paper. The second is a so called *state-dependent NL equation form*. However, the reference and disturbance signals are assumed to have linear model representations. The system is shown in Fig.1, including the *nonlinear* plant model and the linear reference/disturbance models. The zero-mean white measurement noise is denoted $\{v(t)\}$ and it has a covariance matrix R_f. There is no loss of generality in assuming that the zero-mean, white noise signals $\{\omega(t)\}$ and $\{\xi_2(t)\}$, that feed the reference and disturbance models, have identity covariance matrices.

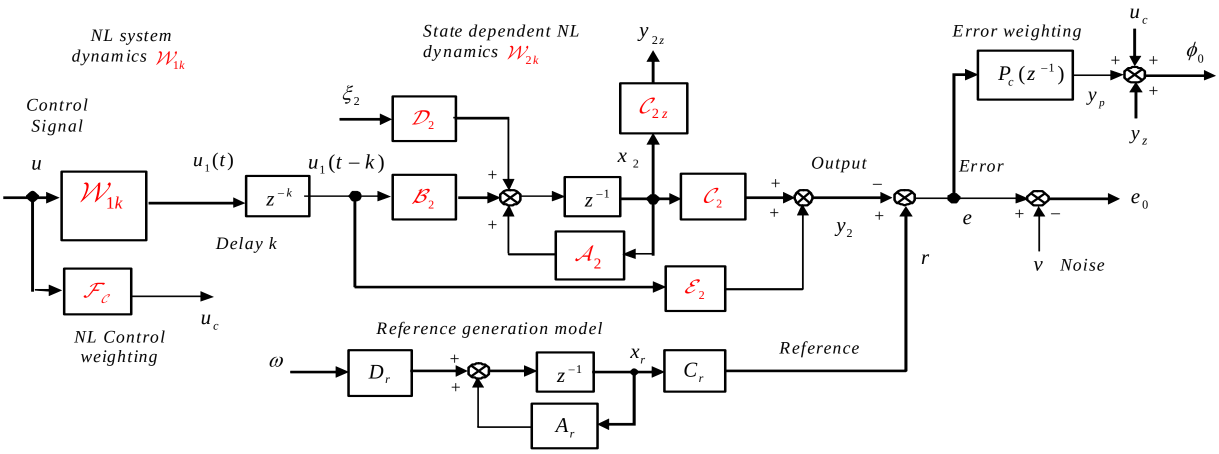

Fig 1: **The System Description**

Reference model: $$x_r(t+1) = A_r x_r(t) + D_r\omega(t), \quad x_r(t) \in R^{n_r} \tag{7}$$

$$r(t) = C_r x_r(t) \quad \text{and} \quad W_r(z^{-1}) = C_r(zI - A_r)^{-1} D_r \tag{8}$$

Error weighting: $$x_p(t+1) = A_p x_p(t) + B_p\left(r(t) - y_2(t)\right), \qquad x_p(t) \in R^{n_p} \tag{9}$$

$$y_p(t) = C_p x_p(t) + E_p\left(r(t) - y_2(t)\right) \tag{10}$$

Nonlinear Plant: $$(\mathcal{W}u)(t) = z^{-k}(\mathcal{W}_k u)(t) \tag{11}$$

where k denotes the magnitude of the common delay elements in the output signal paths. The total forward path plant model: $$(\mathcal{W}u)(t) = z^{-k}(\mathcal{W}_{2k}\mathcal{W}_{1k}u)(t) \tag{12}$$

and $u_1(t) = \mathcal{W}_{1k}u(t)$. Although the first nonlinear subsystem $\mathcal{W}_{1k}$ can be a general nonlinear system [1], a saturation characteristic is defined here to ensure the input signal $u_1(t)$ is bounded i.e. $u_1(t) \in [u_{\min}, u_{\max}]$, where $u_{\min}$ and $u_{\max}$ are the lower and upper bounds. The second nonlinear subsystem $\mathcal{W}_{2k}$ is represented by the state-dependent model equations, shown in Fig.1.

Total Linear Sub-System State Equation Model: Combining the linear reference and error weighting model, obtain the augmented state equation for the total linear sub-system as:

$$x(t+1) = Ax(t) + Bu_0(t-k) + D\omega(t) \tag{13}$$

where $x(t) = [x_r^T, x_p^T]^T$ and $u_0(t-k) = y_2(t)$. Then:

$$x(t+1) = \begin{bmatrix} A_r & 0 \\ B_p C_r & A_p \end{bmatrix}\begin{bmatrix} x_r \\ x_p \end{bmatrix} + \begin{bmatrix} 0 \\ -B_p \end{bmatrix} u_0(t-k) + \begin{bmatrix} D_r \\ 0 \end{bmatrix}\omega(t) \tag{14}$$

The *resolvent operator* may now be defined as: $\Phi(z^{-1}) = (zI - A)^{-1}$ so that,

$$x(t) = \Phi(z^{-1})Bu_0(t-k) + \Phi(z^{-1})D\omega(t)$$

Note that he signal $\{\phi_0(t)\}$ is to be minimized in a variance sense as discussed in Section 4.

3.2 State Prediction Equations

The *Kalman filter* is needed to estimate the states of the combined linear model. These results are well known [10] and will be omitted here. For a time-invariant system define:

$$T_0(k, z^{-1}) = (I - A^k z^{-k})\Phi(z^{-1}) = z^{-1}\left(I + z^{-1}A + z^{-2}A^2 + \ \dots\ + z^{-k+1}A^{k-1}\right) \qquad (15)$$

which denotes a transfer operator with *finite pulse response*. It is then easy to show that a *k*-step ahead state prediction can be expressed as:

$$\hat{x}(t+k\,|\,t) = A^k\hat{x}(t\,|\,t) + T_0(k,z^{-1})By_2(t+k) = A^k\hat{x}(t\,|\,t) + A^{k-1}By_2(t) + A^{k-2}By_2(t+1) + \dots\dots + ABy_2(t+k-2) + By_2(t+k-1) \quad (16)$$

Now consider the second nonlinear system model in the so called *linear state dependent (LSD)* state-space form. [11]. It is the system defined in (3) with k steps common delay:

$$x_2(t+1) = \mathcal{A}_2(x_2, u_1)x_2(t) + \mathcal{B}_2(x_2, u_1)u_1(t-k) + \mathcal{D}_2(x_2, u_1)\xi_2(t) \qquad (17a)$$

$$y_2(t) = \mathcal{C}_2(x_2, u_1)x_2(t) + \mathcal{E}_2 u_1(t-k) \qquad (17b)$$

where x_2(t) is a vector of *sub-system* states, $u_1(t)$ is a vector of the *LSD* sub-system inputs, $y_2(t)$ is a vector of sub-system output signals and $\xi_2(t)$ is a vector of disturbance signals. The total combined vector of linear and state-dependent system model states is defined as $\mathcal{X}(t) = \begin{bmatrix} x(t)^T & x_2(t)^T \end{bmatrix}^T$ and to simplify notation in (17), write $\mathcal{A}_2(t) = \mathcal{A}_2(x_2(t), u_1(t))$ and similarly for $\mathcal{B}_2$, $\mathcal{C}_2$, $\mathcal{D}_2$ and $\mathcal{E}_2$.

Prediction model: The *k-steps* prediction of the state and output signals can similarly be defined:

$$\mathcal{T}_1(k,z^{-1}) = \mathcal{A}_2(t+k-1)\mathcal{A}_2(t+k-2)\dots\mathcal{A}_2(t+1)\mathcal{B}_2(t)z^{-k} + \dots + \mathcal{A}_2(t+k-1)\mathcal{B}_2(t+k-2)z^{-2} + \mathcal{B}_2(t+k-1)z^{-1} \quad (18)$$

Estimate:

$$\hat{x}_2(t+k\,|\,t) = \mathcal{A}_2(t+k-1)\mathcal{A}_2(t+k-2)\dots\mathcal{A}_2(t)\hat{x}_2(t\,|\,t) + \mathcal{T}_1(k,z^{-1})u_1(t) \qquad (19)$$

$$\hat{y}_2(t+k\,|\,t) = \mathcal{C}_2(t+k)\mathcal{A}_2(t+k-1)\mathcal{A}_2(t+k-2)\dots\mathcal{A}_2(t)\hat{x}_2(t\,|\,t) + \left(\mathcal{C}_2(t+k)\mathcal{T}_1(k,z^{-1}) + \mathcal{E}_2(t+k)\right)u_1(t) \quad (20)$$

4. Nonlinear Generalized Minimum Variance Control Law

4.1 NGMV Control Problem

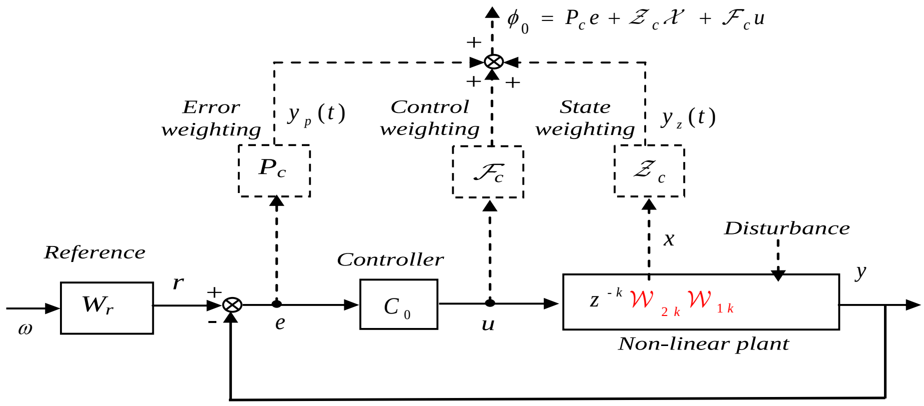

Fig. 2: Single Degree of Freedom Closed-Loop Control for the NL Hybrid Plant

The optimal *NGMV* control problem involves the minimization of the variance of the signal $\{\phi_0(t)\}$ shown in Fig. 2. The signal $\{\phi_0(t)\}$ is to be minimized in a variance sense, where:

$$\phi_0(t) = P_c e(t) + (\mathcal{Z}_c \mathcal{X})(t) + (\mathcal{F}_c u)(t) \tag{21}$$

and the cost index:

$$J = E\{\phi_0^T(t)\phi_0(t)\} = E\{trace\{\phi_0(t)\phi_0^T(t)\}\} \tag{22}$$

where $E\{\cdot\}$ denotes the unconditional expectation operator. The signal $\{\phi_0(t)\}$ involves an error signal dynamic cost-function weighting matrix $P_c(z^{-1})$, that is represented by a linear state-space sub-system with output $y_p(t) = P_c e(t)$. The criterion also includes the nonlinear state weighting term $y_z(t) = (\mathcal{Z}_c \mathcal{X})(t)$ and enables a limit to be introduced on all the linear and state-dependent subsystem states, that are to be penalized. That is, the weighting on a certain combination of states may be included in the criterion via the signal $y_z(t) = (\mathcal{Z}_c \mathcal{X})(t) = C_{1z} x(t) + \mathcal{C}_{2z} x_2(t)$, where $y_{1z}(t) = \mathcal{C}_{1z} x(t)$ and $y_{2z}(t) = \mathcal{C}_{2z} x_2(t)$. The operator $\mathcal{Z}_c$ can include dynamics and *NL* terms, and for simplicity can be augmented to the second linear state-dependent (*LSD*) sub-system states. The final term in the criterion is the *NL* dynamic control signal costing operator term $(\mathcal{F}_c u)(t)$. If the smallest delay in each output channel of the plant is of magnitude k-steps this implies the control at time t affects the output at least k steps later. For this reason the control signal costing can be defined to have the form:

$$(\mathcal{F}_c u)(t) = z^{-k}(\mathcal{F}_{ck} u)(t) \tag{23}$$

Typically this will be a linear operator but it may also be chosen to be *NL* to cancel the plant input nonlinearities in appropriate cases. The control weighting operator $\mathcal{F}_{ck}$ is assumed to be full rank and invertible. The choice of dynamic weightings is critical to the design and the weighting P_c is typically a low-pass filter and $\mathcal{F}_c$ is a high-pass filter.

4.2 Solution of the NGMV Optimal Control Problem

The solution of the optimal control problem may be obtained by expanding the expression for the inferred output $\{\phi_0(t)\}$ and by then introducing a prediction equation. Recall,

$$\phi_0(t) = P_c\, e(t) + (\mathcal{Z}_c\mathcal{X})(t) + (\mathcal{F}_c\, u)(t) = y_p(t) + y_z(t) + (\mathcal{F}_c\, u)(t) \tag{24}$$

The first error weighting term may be written in a more concise form, using (10), as:

$$y_p(t) = P_c\, e(t) = C_{1p}x(t) + E_{1p}y_2(t) \tag{25}$$

where $x(t) \in R^n$, $E_{1p} = -E_p$. Similarly, if the linear and nonlinear subsystem state weightings are denoted by C_{1z} and $\mathcal{C}_{2z}$, respectively, then the state weighting term:

$$y_z(t) = (\mathcal{Z}_c\mathcal{X})(t) = C_{1z}x(t) + \mathcal{C}_{2z}x_2(t) \tag{26}$$

Hence, the inferred output or signal to be minimized:

$$\phi_0(t) = \mathcal{C}_\phi\mathcal{X}(t) + E_\phi u_0(t-k) + (\mathcal{F}_c u)(t) \tag{27}$$

where $\mathcal{C}_\phi = \begin{bmatrix} C_{\phi 1} & \mathcal{C}_{\phi 2} \end{bmatrix} = \begin{bmatrix} (C_{1p} + C_{1z}) & \mathcal{C}_{2z} \end{bmatrix}$ and $E_\phi = E_{1p}$. Thence,

$$\phi_0(t) = \mathcal{C}_\phi\mathcal{X}(t) + E_\phi(\mathcal{W}_{2k}\mathcal{W}_{1k}\, u)(t-k) + (\mathcal{F}_c\, u)(t) \tag{28}$$

In the set of channels with explicit delay k the control signal affects the *outputs* $\phi_0(t)$ at least k - steps later and the control signal weighting is therefore defined to have the form $(\mathcal{F}_c u)(t) = z^{-k}(\mathcal{F}_{ck}\, u)(t)$. Substituting into (28) obtain:

$$\phi_0(t) = \mathcal{C}_\phi\mathcal{X}(t) + \left((E_\phi\, \mathcal{W}_{2k}\mathcal{W}_{1k} + \mathcal{F}_{ck})u\right)(t-k) \tag{29}$$

The prediction may be obtained in terms of (16) and (19) as:

$$\hat{\mathcal{X}}(t+k\,|\,t) = \begin{bmatrix} \hat{x}(t+k\,|\,t) \\ \hat{x}_2(t+k\,|\,t) \end{bmatrix} = \begin{bmatrix} A^k\hat{x}(t\,|\,t) + T_0(k,z^{-1})By_2(t+k) \\ \mathcal{A}_2(t+k-1)\mathcal{A}_2(t+k-2)\ldots\mathcal{A}_2(t)\hat{x}_2(t\,|\,t) + \mathcal{T}_1(k,z^{-1})u_1(t) \end{bmatrix}$$

$$= \begin{bmatrix} A^k & 0 \\ 0 & \mathcal{A}_2(t+k-1)\mathcal{A}_2(t+k-2)\ldots\mathcal{A}_2(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t|t) \\ \hat{x}_2(t|t) \end{bmatrix} + \begin{bmatrix} T_0(k,z^{-1})B\mathcal{W}_{2k} \\ \mathcal{T}_1(k,z^{-1}) \end{bmatrix} u_1(t)$$

which may be written more concisely, with an obvious definition of matrix terms, as:

$$\hat{\mathcal{X}}(t+k|t) = \mathcal{A}_\phi^k \hat{\mathcal{X}}(t|t) + \mathcal{T}_\phi(k,z^{-1})u_1(t) \tag{30}$$

Note that the predicted values of the state related terms in (29) therefore become:

$$\mathcal{C}_\phi \hat{\mathcal{X}}(t+k|t) = C_{\phi 1}A^k\hat{x}(t|t) + \mathcal{C}_{\phi 2}\mathcal{A}_2(t+k-1)\mathcal{A}_2(t+k-2)\ \ldots\ \mathcal{A}_2(t)\hat{x}_2(t|t) + (C_{\phi 1}T_0(k,z^{-1})B\mathcal{W}_{2k} + \mathcal{C}_{\phi 2}\mathcal{T}_1(k,z^{-1}))u_1(t) \tag{31}$$

Prediction Equation: The k steps ahead prediction of the signal $\phi_0(t)$, from (29) and (30),

$$\hat{\phi}_0(t+k|t) = \mathcal{C}_\phi \hat{\mathcal{X}}(t+k|t) + \left((E_\phi \mathcal{W}_{2k}\mathcal{W}_{1k} + \mathcal{F}_{ck})u\right)(t)$$

$$= \mathcal{C}_\phi \mathcal{A}_\phi^k \hat{\mathcal{X}}(t|t) + \mathcal{C}_\phi \mathcal{T}_\phi(k,z^{-1})(\mathcal{W}_{1k}u)(t) + \left((E_\phi \mathcal{W}_{2k}\mathcal{W}_{1k} + \mathcal{F}_{ck})u\right)(t)$$

$$= \mathcal{C}_\phi \mathcal{A}_\phi^k \hat{\mathcal{X}}(t|t) + \left(\left((\mathcal{C}_\phi \mathcal{T}_\phi(k,z^{-1}) + E_\phi \mathcal{W}_{2k})\mathcal{W}_{1k} + \mathcal{F}_{ck}\right)u\right)(t) \tag{32}$$

The cost-function involves the minimization of the *weighted error* and *control signals*, in a variance sense. The variance $J = E\{\phi_0(t+k)^T \phi_0(t+k)\}$ may be written in terms of the prediction $\hat{\phi}_0(t+k|t)$ and the prediction error: $\tilde{\phi}_0(t+k|t)$, using the orthogonality properties [12], as:

$$J = \mathrm{E}\{\phi_0(t+k|t)^T \hat{\phi}_0(t+k|t)\} + \mathrm{E}\{\tilde{\phi}_0(t+k|t)^T \tilde{\phi}_0(t+k|t)\} \tag{33}$$

The prediction error $\tilde{\phi}_0(t+k|t)$ does not depend upon control action and hence the cost is minimized by setting the predicted values of the signal $\phi_0(t)$, for k steps ahead, to zero.

Theorem 4.1: NGMV Controller for State Dependent and NL Systems

The *NGMV* optimal controller to minimize the variance of the weighted error, states and control signals may be computed from the following state and operator equations. The assumption is made that the nonlinear operator $\left(\mathrm{P_c}\,\mathcal{W}_k - \mathcal{N}_c - \mathcal{F}_{ck}\right)$ has a finite gain m_2 stable causal inverse, due to the choice of weighting operators $P_c, \mathcal{Z}_c$ and $\mathcal{F}_c$.

Optimal control signal: The optimal *NGMV control action* can be computed as:

$$\mathrm{u(t)} = -\left(\mathcal{F}_{ck} + \left(\mathcal{C}_\phi \mathcal{T}_\phi(\mathrm{k},\mathrm{z}^{-1}) + \mathrm{E}_\phi \mathcal{W}_{2k}\right)\mathcal{W}_{1k}\right)^{-1} \mathcal{C}_\phi \mathcal{A}_\phi^k \hat{\mathcal{X}}(\mathrm{t}|\mathrm{t}) \tag{34}$$

or $$u(t) = -\mathcal{F}_{ck}^{-1}\left(\mathcal{C}_\phi \mathcal{A}_\phi^k \hat{\mathcal{X}}(t\,|\,t) + \left(\mathcal{C}_\phi \mathcal{T}_\phi(k, z^{-1}) + E_\phi \mathcal{W}_{2k}\right)(\mathcal{W}_{1k} u)(t)\right) \tag{35}$$

where $E_\phi = -E_p$. The controller structure corresponding to Equation (35) is shown in Fig. 3

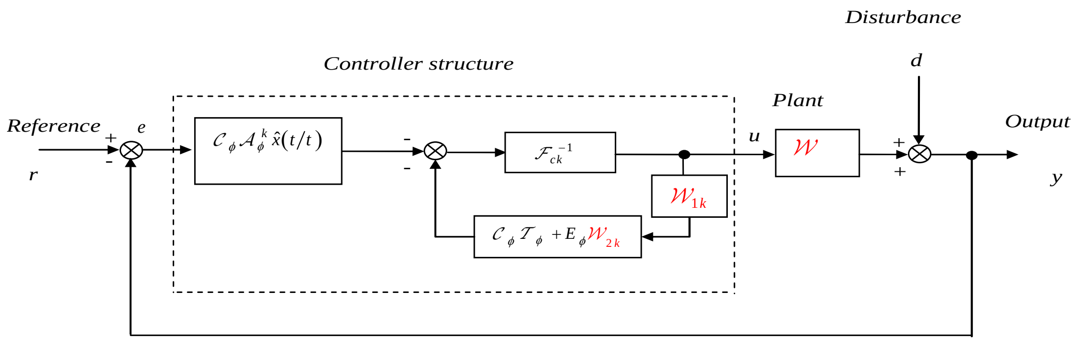

Fig. 3: **Control Signal Generation and Controller Modules**

Proof: For stability analysis the time functions can be considered to be contained in extensions of the *discrete Marcinkiewicz space* $m_2(R_+, R^n)$ [13] and [14]. This is the space of time sequences with time-averaged square summable signals, which have finite power. The aim of the nonlinear control design is then to ensure certain input-output maps are finite-gain m_2 stable and the cost-index is minimized.

Recalling from (19) that the estimate: $\hat{x}_2(t\,|\,t)$, depends only on $u_1(t) = (\mathcal{W}_{1k} u)(t)$, let

$$\hat{x}(t\,|\,t) = T_{f1}(z^{-1}) e_0(t) + T_{f2}(z^{-1}) u_0(t)$$

Combine linear and nonlinear terms in the state weighting operators as:

$$\left(\mathcal{N}_C\, u\right)(t) = \left(N_{C0}\, u_0\right)(t) + \left(\mathcal{N}_{C1}\, u_1\right)(t) \tag{36}$$

where $$\left(N_{C0}\, u_0\right)(t) = \left(C_{1z} \Phi(z^{-1}) B u_0(t)\right) \tag{37}$$

and $$\left(\mathcal{N}_{C1}\, u_1\right)(t) = \left(\mathcal{C}_{\phi 2} \mathcal{A}_2(t+k-1) \mathcal{A}_2(t+k-2) \ldots \mathcal{A}_2(t) \hat{x}_2(t\,|\,t) + \mathcal{C}_{\phi 2} \mathcal{T}_1(k, z^{-1}) u_1(t)\right) \tag{38}$$

and note from (19) that this implies:

$$\mathcal{C}_{\phi_2} \hat{x}_2(t+k\,|\,t) = \mathcal{C}_{\phi_2}\left(\mathcal{A}_2(t+k-1) \mathcal{A}_2(t+k-2) \ldots \mathcal{A}_2(t) \hat{x}_2(t\,|\,t) + \mathcal{T}_1(k, z^{-1}) u_1(t)\right) = \left(\mathcal{N}_{C1}\, u_1\right)(t).$$

Rearranging, the desired expressions for the *optimal control* and *plant output* signals become:

$$u(t) = \left(P_c \mathcal{W}_k - \mathcal{N}_C - \mathcal{F}_{ck}\right)^{-1} C_{\phi 1} A^k T_{f1}(z^{-1}) r(t) \tag{39}$$

$$\left(\mathcal{W} u\right)(t) = z^{-k} \mathcal{W}_k \left(P_c \mathcal{W}_k - \mathcal{N}_C - \mathcal{F}_{ck}\right)^{-1} C_{\phi 1} A^k T_{f1}(z^{-1}) r(t) \tag{40}$$

The assumption is made that the cost-weightings are chosen, so that the operator $\left(P_c \mathcal{W}_k - \mathcal{N}_C - \mathcal{F}_{ck}\right)^{-1}$ is also finite gain m_2 stable. Under this assumption the two systems in the expressions for the *control* and *output* signals (39) and (40) only involve finite gain stable systems. □

5. **Conclusions**

An *NGMV* controller for delayed *PWA* systems, whose switching sequence depends on the state and on the control input, has been proposed. These *PWA* systems can be translated into *NL* state-dependent systems by introducing some binary functions representing the conditions describing the crossing of the switching surfaces. The advantage of state-dependent systems over *PWA* systems is that state-dependent systems are much easier to design, as both the state and input constraints and the switching conditions, can all be included in the system model. The state-feedback *NGMV* design methodologies provide a possible relatively simple way to synthesize controllers for hybrid systems.

There are some hybrid systems where the switching conditions are more complicated and cannot be modeled as *PWA* systems. The state-dependent system may be extended to model these types of hybrid system in future. A discrete supervisor is needed for this extension and an *NGMV* controller is also needed for the continuous time control part.

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