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RAZUMIKHIN-TYPE THEOREMS ON EXPONENTIAL STABILITY OF NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. Recently, we initiated in [*Systems Control Lett.*, 26 (1995), pp. 245–251] the study of exponential stability of neutral stochastic functional differential equations, and in this paper, we shall further our study in this area. We should emphasize that the main technique employed in this paper is the well-known Razumikhin argument and is completely different from those used in our previous paper [*Systems Control Lett.*, 26 (1995), pp. 245–251]. The results obtained in [*Systems Control Lett.*, 26 (1995), pp. 245–251] can only be applied to a certain class of neutral stochastic functional differential equations excluding neutral stochastic differential delay equations, but the results obtained in this paper are more general, and they especially can be used to deal with neutral stochastic differential delay equations. Moreover, in [*Systems Control Lett.*, 26 (1995), pp. 245–251], we only studied the exponential stability in mean square, but in this paper, we shall also study the almost sure exponential stability. It should be pointed out that although the results established in this paper are applicable to more general neutral-type equation, for a particular type of equation discussed in [*Systems Control Lett.*, 26 (1995), pp. 245–251], the results there are sharper.

Key words. exponential stability, Razumikhin-type theorem, Brownian motion, Doob martingale inequality, Borel–Cantelli lemma

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1. Introduction.

Deterministic neutral functional differential equations and their stability have been studied by many authors, e.g., Haddock et al. [3], Hale and Lunel [4], and the references therein. Motivated by the chemical-engineering systems as well as the theory of aeroelasticity, Kolmanovskii and Nosov [8] introduced the neutral stochastic functional differential equations of the form

$$d[x(t) - G(x_t)] = f(t, x_t)dt + g(t, x_t)dw(t) \tag{1.1}$$

on $t \geq 0$ with initial data $x_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$. (For notation, please see section 2 below.) Kolmanovskii and Nosov [8] not only established the theory of existence and uniqueness of the solution to (1.1) but also investigated the stability and asymptotic stability of the equations (see also Kolmanovskii and Myshkis [7]). However, the exponential stability of such equations has not been studied until recently by the author in [11]. To be more precise, let us give the definition of exponential stability.

DEFINITION 1.1. *Denote by $x(t;\xi)$ the solution of equation* (1.1). *The trivial solution of equation* (1.1) *is said to be exponentially stable in mean square if there exists a pair of positive constants γ and M such that*

$$E|x(t;\xi)|^2 \leq Me^{-\gamma t} \sup_{-\tau \leq \theta \leq 0} E|\xi(\theta)|^2, \quad t \geq 0,$$

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or, equivalently,

$$\limsup_{t\to\infty} \frac{1}{t} \log E|x(t;\xi)|^2 \le -\gamma$$

for all $\xi \in L^2_{\mathcal{F}_0}([-\tau,0];R^n)$*. The trivial solution of equation* (1.1) *is said to be almost surely exponentially stable if there is a positive constant* $\bar{\gamma}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t;\xi)| \le -\bar{\gamma} \quad a.s.$$

for all $\xi \in L^2_{\mathcal{F}_0}([-\tau,0];R^n)$*.*

In this paper, we shall further our study in this area. We should emphasize that the main technique employed in this paper is the well-known Razumikhin argument (see Razumikhin [13], [14]). To explain this technique, applying Itô's formula to $e^{\lambda t}|x(t) - G(x_t)|^2$, one may see that to have exponential stability in mean square, it would require that

$$E\Big(2(\phi(0) - G(\phi))^T f(t,\phi) + \mathrm{trace}[g^T(t,\phi)g(t,\phi)]\Big) \le -\lambda E|\phi(0) - G(\phi)|^2 \tag{1.2}$$

for all $t \ge 0$ and *all* $\phi \in L^2_{\mathcal{F}_t}([-\tau,0];R^n)$. As a result, one would be forced to impose very severe restrictions on the functions $f(t,\phi)$ and $g(t,\phi)$. However, by the Razumikhin argument, one needs to require that (1.2) hold only for those $\phi \in L^2_{\mathcal{F}_t}([-\tau,0];R^n)$ satisfying

$$E|\phi(\theta)|^2 < qE|\phi(0) - G(\phi)|^2, \quad -\tau \le \theta \le 0,$$

but not necessarily for all ϕ, where $q > 1$ is a constant. Hence the restrictions on the functions $f(t,\phi)$ and $g(t,\phi)$ can be weakened considerably. This is the basic idea exploited in this paper.

This main technique of this paper is completely different from those used in our previous paper [11]. The results obtained in [11] can be applied only to a certain class of neutral stochastic functional differential equations excluding neutral stochastic differential delay equations, but the results obtained in this paper are much more general, and they especially can be used to deal with neutral stochastic differential delay equations. Moreover, in [11], we only studied the exponential stability in mean square, but in this paper, we shall also study the almost sure exponential stability. It should be pointed out that although the results established in this paper are applicable to more general neutral-type equations, for a particular class of equations discussed in [11], the results there are sharper. (Please see section 5 below for details.) Of course, this is not surprising because the results obtained by applying a particular technique to a particular equation are generally sharper than those obtained by using a general technique which is applicable to more general equations.

In this paper, the theory of existence and uniqueness of the solutions will first be introduced very briefly in section 2. The main results of this paper will be shown in sections 3 and 4, where several useful criteria will be established on the exponential stability in mean square as well as the almost sure exponential stability for the trivial solution of equation (1.1). In section 5, we shall compare our new results with the previous ones obtained in [11]. To show the power of the Razumikhin argument, the general results established in sections 3 and 4 will be applied to deal with the exponential stability of neutral stochastic differential delay equations in section 6 and of linear neutral stochastic functional differential equations in section 7.

2. Neutral stochastic functional differential equations. Throughout the paper, unless otherwise specified, we let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $||\varphi|| = \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its norm $||A||$ is defined by $||A|| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $||\varphi||$). Moreover, let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ (i.e., $\mathcal{F}_t = \sigma\{w(s) : 0 \le s \le t\}$). For each $t \ge 0$, denote by $L^2_{\mathcal{F}_t}([-\tau, 0]; R^n)$ the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; R^n)$-valued random variables $\phi = \{\phi(\theta) : -\tau \le \theta \le 0\}$ such that $\sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2 < \infty$. Also, define $L^2_{\mathcal{F}_\infty}([-\tau, 0]; R^n) = \bigcup_{t\ge 0} L^2_{\mathcal{F}_t}([-\tau, 0]; R^n)$. Obviously, $C([-\tau, 0]; R^n) \subset L^2_{\mathcal{F}_\infty}([-\tau, 0]; R^n)$.

Consider the n-dimensional neutral stochastic functional differential equation

$$d[x(t) - G(x_t)] = f(t, x_t)dt + g(t, x_t)dw(t) \tag{2.1}$$

on $t \ge 0$ with initial data $x_0 = \xi$. Here

$$G : C([-\tau, 0]; R^n) \to R^n, \qquad f : R_+ \times C([-\tau, 0]; R^n) \to R^n,$$
$$g : R_+ \times C([-\tau, 0]; R^n) \to R^{n\times m}$$

are all continuous functionals. Moreover, $x_t = \{x(t + \theta) : -\tau \le \theta \le 0\}$, which is regarded as a $C([-\tau, 0]; R^n)$-valued stochastic process, and $\xi = \{\xi(\theta) : -\tau \le \theta \le 0\} \in L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$. An $\mathcal{F}_t$-adapted process $x(t), -\tau \le t < \infty$ (let $\mathcal{F}_t = \mathcal{F}_0$ for $-\tau \le t \le 0$), is said to be a solution of equation (2.1) if it satisfies the initial condition and, moreover, for every $t \ge 0$,

$$x(t) - G(x_t) = \xi(0) - G(\xi) + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)dw(s). \tag{2.1$'$}$$

To ensure the existence and uniqueness of the solution, one of the key hypotheses is the following:

(H) There is a constant $\kappa \in (0, 1)$ such that

$$E|G(\phi_1) - G(\phi_2)|^2 \le \kappa \sup_{-\tau\le\theta\le 0} E|\phi_1(\theta) - \phi_2(\theta)|^2$$

for all $\phi_1, \phi_2 \in L^2_{\mathcal{F}_\infty}([-\tau, 0]; R^n)$.

In addition, we need further hypotheses on f and g. For example, f and g are uniformly Lipschitz continuous, or they are locally Lipschitz continuous and satisfy the linear-growth condition. Under these hypotheses, Kolmanovskii and Nosov [8] showed that there is a unique continuous solution to equation (2.1), and any moment, especially the second moment, of the solution is finite. Since the existence and uniqueness of the solution are not the main topic of this paper, we shall not discuss them in detail. All we need to do in this paper is assume that a unique solution exists and is continuous and that its second moment is finite. The solution will be denoted by $x(t; \xi)$.

3. Exponential stability in mean square. In this section, we will investigate the exponential stability in mean square for the solution of equation (2.1). For the general theory on stochastic stability, we refer the reader to Arnold [1], Friedman [2], Has'minskii [5], Mao [9, 10], or Mohammed [12]. For the stability purpose of this paper, we always assume that $G(0) = 0$, $f(t,0) \equiv 0$, and $g(t,0) \equiv 0$. Therefore, equation (2.1) admits a trivial solution $x(t;0) \equiv 0$. The following Razumikhin-type theorem gives a sufficient condition for the exponential stability in mean square of this trivial solution.

THEOREM 3.1. *Assume that there is a constant $\kappa \in (0,1)$ such that*

$$E|G(\phi)|^2 \le \kappa \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2, \quad \phi \in L^2_{\mathcal{F}_\infty}([-\tau,0];R^n). \tag{3.1}$$

Let $q > (1-\sqrt{\kappa})^{-2}$. Assume furthermore that there is a $\lambda > 0$ such that

$$E\Big(2(\phi(0)-G(\phi))^T f(t,\phi) + trace[g^T(t,\phi)g(t,\phi)]\Big) \le -\lambda E|\phi(0)-G(\phi)|^2 \tag{3.2}$$

for all $t \ge 0$ and those $\phi \in L^2_{\mathcal{F}_t}([-\tau,0];R^n)$ satisfying

$$E|\phi(\theta)|^2 < qE|\phi(0)-G(\phi)|^2, \quad -\tau \le \theta \le 0.$$

Then for all $\xi \in L^2_{\mathcal{F}_0}([-\tau,0];R^n)$,

$$E|x(t;\xi)|^2 \le q(1+\sqrt{\kappa})^2 e^{-\bar{\gamma}t} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2, \quad t \ge 0, \tag{3.3}$$

where

$$\bar{\gamma} = \min\left\{\lambda,\ \frac{1}{\tau}\log\left[\frac{q}{(1+\sqrt{q\kappa})^2}\right]\right\} > 0. \tag{3.4}$$

In other words, the trivial solution of equation (2.1) is exponentially stable in mean square.

In order to prove this theorem, let us present two useful lemmas.

LEMMA 3.2. *Let (3.1) hold for some $\kappa \in (0,1)$. Then*

$$E|\phi(0)-G(\phi)|^2 \le (1+\sqrt{k})^2 \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2$$

for all $\phi \in L^2_{\mathcal{F}_\infty}([-\tau,0];R^n)$.

Proof. For any $\varepsilon > 0$,

$$\begin{aligned} E|\phi(0)-G(\phi)|^2 &\le E|\phi(0)|^2 + 2E\big(|\phi(0)||G(\phi)|\big) + E|G(\phi)|^2 \\ &\le (1+\varepsilon)E|\phi(0)|^2 + (1+\varepsilon^{-1})E|G(\phi)|^2 \\ &\le \big[1+\varepsilon+\kappa(1+\varepsilon^{-1})\big] \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2. \end{aligned}$$

Therefore, the desired result follows by taking $\varepsilon = \sqrt{\kappa}$. The proof is complete. ∎

LEMMA 3.3. *Let (3.1) hold for some $\kappa \in (0,1)$. Let $\rho \ge 0$ and $0 < \gamma < \tau^{-1}\log(1/\kappa)$. Let $x(t)$ be a solution of equation (2.1). If*

$$e^{\gamma t}E|x(t)-G(x_t)|^2 \le (1+\sqrt{\kappa})^2 \sup_{-\tau\le\theta\le 0} E|x(\theta)|^2 \tag{3.5}$$

for all $0 \le t \le \rho$, then

$$e^{\gamma t}E|x(t)|^2 \le \frac{(1+\sqrt{\kappa})^2}{(1-\sqrt{\kappa e^{\gamma\tau}})^2} \sup_{-\tau\le\theta\le 0} E|x(\theta)|^2. \tag{3.6}$$

Proof. Let $\kappa e^{\gamma\tau} < \varepsilon < 1$. For $0 \le t \le \rho$, note that

$$E|x(t) - G(x_t)|^2 \ge E|x(t)|^2 - 2E\big(|x(t)||G(x_t)|\big) + E|G(x_t)|^2$$
$$\ge (1-\varepsilon)E|x(t)|^2 - (\varepsilon^{-1} - 1)E|G(x_t)|^2.$$

Hence

$$E|x(t)|^2 \le \frac{1}{1-\varepsilon}E|x(t) - G(x_t)|^2 + \frac{\kappa}{\varepsilon}\sup_{-\tau\le\theta\le 0} E|x(t+\theta)|^2.$$

By condition (3.5), we then derive that for all $0 \le t \le \rho$,

$$e^{\gamma t}E|x(t)|^2 \le \frac{1}{1-\varepsilon}\sup_{0\le t\le\rho}\Big[e^{\gamma t}E|x(t) - G(x_t)|^2\Big] + \frac{\kappa}{\varepsilon}\sup_{0\le t\le\rho}\Big[e^{\gamma t}\sup_{-\tau\le\theta\le 0}E|x(t+\theta)|^2\Big]$$
$$\le \frac{(1+\sqrt{\kappa})^2}{1-\varepsilon}\sup_{-\tau\le\theta\le 0}E|x(\theta)|^2 + \frac{\kappa e^{\gamma\tau}}{\varepsilon}\sup_{-\tau\le t\le\rho}\Big[e^{\gamma t}E|x(t)|^2\Big].$$

However, this holds for all $-\tau \le t \le 0$ as well. Therefore,

$$\sup_{-\tau\le t\le\rho}\Big[e^{\gamma t}E|x(t)|^2\Big] \le \frac{(1+\sqrt{\kappa})^2}{1-\varepsilon}\sup_{-\tau\le\theta\le 0}E|x(\theta)|^2 + \frac{\kappa e^{\gamma\tau}}{\varepsilon}\sup_{-\tau\le t\le\rho}\Big[e^{\gamma t}E|x(t)|^2\Big].$$

Since $1 > \kappa e^{\gamma\tau}/\varepsilon$, we see that

$$\sup_{-\tau\le t\le\rho}\Big[e^{\gamma t}E|x(t)|^2\Big] \le \frac{\varepsilon(1+\sqrt{\kappa})^2}{(1-\varepsilon)(\varepsilon - \kappa e^{\gamma\tau})}\sup_{-\tau\le\theta\le 0}E|x(\theta)|^2.$$

The required assertion (3.6) follows by taking $\varepsilon = \sqrt{\kappa e^{\gamma\tau}}$. The proof is complete. ◻

We can now begin to prove Theorem 3.1.

Proof of Theorem 3.1. First, note that $q/(1+\sqrt{q\kappa})^2 > 1$ since $q > (1-\sqrt{k})^{-2}$ and hence $\bar\gamma > 0$. Now fix any $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$ and simply write $x(t;\xi) = x(t)$. Without any loss of generality, we may assume that $\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 > 0$. Let $\gamma \in (0, \bar\gamma)$ arbitrarily. It is easy to show that

$$0 < \gamma < \min\left\{\lambda,\ \frac{1}{\tau}\log\Big(\frac{1}{\kappa}\Big)\right\} \quad \text{and} \quad q > \frac{e^{\gamma\tau}}{(1-\sqrt{\kappa e^{\gamma\tau}})^2}. \tag{3.7}$$

We now claim that

$$e^{\gamma t}E|x(t) - G(x_t)|^2 \le (1+\sqrt{\kappa})^2\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 \quad \text{for all } t \ge 0. \tag{3.8}$$

If so, an application of Lemma 3.3 to (3.8) yields that

$$e^{\gamma t}E|x(t)|^2 \le \frac{(1+\sqrt{\kappa})^2}{(1-\sqrt{\kappa e^{\gamma\tau}})^2}\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 \le q(1+\sqrt{\kappa})^2\sup_{-\tau\le\theta\le 0}E|x(\theta)|^2$$

for all $t \ge 0$, where we have used (3.7), and the desired result (3.3) follows by letting $\gamma \to \bar\gamma$. The remainder of the proof is to show (3.8) by contradiction. Suppose (3.8) is not true. Then in view of Lemma 3.2, there is a $\rho \ge 0$ such that

$$e^{\gamma t}E|x(t) - G(x_t)|^2 \le e^{\gamma\rho}E|x(\rho) - G(x_\rho)|^2 = (1+\sqrt{\kappa})^2\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 \tag{3.9}$$

for all $0 \le t \le \rho$ and, moreover, there is a sequence of $\{t_k\}_{k\ge 1}$ such that $t_k \downarrow \rho$ and

$$e^{\gamma t_k} E|x(t_k) - G(x_{t_k})|^2 > e^{\gamma\rho} E|x(\rho) - G(x_\rho)|^2. \tag{3.10}$$

Applying Lemma 3.3, we derive from (3.9) that

$$\begin{aligned} e^{\gamma t} E|x(t)|^2 &\le \frac{(1+\sqrt{\kappa})^2}{(1-\sqrt{\kappa e^{\gamma\tau}})^2} \sup_{-\tau\le\theta\le 0} E|x(\theta)|^2 \\ &= \frac{e^{\gamma\rho}}{(1-\sqrt{\kappa e^{\gamma\tau}})^2} E|x(\rho) - G(x_\rho)|^2 \end{aligned}$$

for all $-\tau \le t \le \rho$. Particularly,

$$E|x(\rho+\theta)|^2 \le \frac{e^{\gamma\tau}}{(1-\sqrt{\kappa e^{\gamma\tau}})^2} E|x(\rho) - G(x_\rho)|^2 < qE|x(\rho) - G(x_\rho)|^2 \tag{3.11}$$

for all $-\tau \le \theta \le 0$, where (3.7) has been used once again. By assumption (3.2), we then have

$$E\Big(2(x(\rho) - G(x_\rho))^T f(\rho, x_\rho) + \text{trace}[g^T(\rho, x_\rho) g(\rho, x_\rho)]\Big) \le -\lambda E|x(\rho) - G(x_\rho)|^2.$$

Recalling $\gamma < \lambda$, we see by the continuity of the solution and the functionals G, f, and g (this is the standing hypothesis in this paper) that for all sufficiently small $h > 0$,

$$E\Big(2(x(t) - G(x_t))^T f(t, x_t) + \text{trace}[g^T(t, x_t) g(t, x_t)]\Big) \le -\gamma E|x(t) - G(x_t)|^2$$

if $\rho \le t \le \rho + h$. Now by Itô's formula, for all sufficiently small $h > 0$,

$$\begin{aligned} e^{\gamma(\rho+h)} & E|x(\rho+h) - G(x_{\rho+h})|^2 - e^{\gamma\rho} E|x(\rho) - G(x_\rho)|^2 \\ &= \int_\rho^{\rho+h} e^{\gamma t} \Big[\gamma E|x(t) - G(x_t)|^2 \\ &\quad + E\Big(2(x(t) - G(x_t))^T f(t, x_t) + \text{trace}[g^T(t, x_t) g(t, x_t)]\Big)\Big] dt \\ &\le 0; \end{aligned} \tag{3.12}$$

however, this contradicts (3.10), so (3.8) must hold. The proof is now complete. □

4. Almost sure exponential stability. In this section, we discuss the almost sure exponential stability for the neutral stochastic functional differential equations. It will be shown that under the linear-growth condition, the exponential stability in mean square implies the almost sure exponential stability.

THEOREM 4.1. *Let* (3.1) *hold for some* $\kappa \in (0,1)$. *Assume that there exists a positive constant* $K > 0$ *such that*

$$E\big(|f(t,\phi)|^2 + \text{trace}\big[g^T(t,\phi) g(t,\phi)\big]\big) \le K \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2 \tag{4.1}$$

for all $t \ge 0$ *and* $\phi \in L^2_{\mathcal{F}_\infty}([-\tau, 0]; R^n)$. *Assume also that the trivial solution of equation* (2.1) *is exponentially stable in mean square, that is, there exists a pair of positive constants* γ *and* M *such that*

$$E|x(t;\xi)|^2 \le M e^{-\gamma t} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2, \quad t \ge 0, \tag{4.2}$$

for all $\xi \in L^2_{\mathcal{F}_0}([-\tau,0];R^n)$*. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t;\xi)| \le -\frac{\bar\gamma}{2} \quad a.s., \tag{4.3}$$

where $\bar\gamma = \min\{\gamma, \tau^{-1}\log(1/\kappa)\}$*, that is, the trivial solution of equation* (2.1) *is also almost surely exponentially stable. In particular, if* (3.1)*,* (3.2)*, and* (4.1) *hold, then the trivial solution of equation* (2.1) *is almost surely exponentially stable.*

To prove the theorem, we need to present another lemma which is very useful in the study of the almost sure exponential stability of neutral stochastic functional differential equations.

LEMMA 4.2. *Assume that there exists a constant* $\kappa \in (0,1)$ *such that*

$$|G(\varphi)|^2 \le \kappa \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|^2, \quad \varphi \in C([-\tau,0];R^n). \tag{4.4}$$

Let $z : [-\tau,\infty) \to R^n$ *be a continuous function. Let* $0 < \gamma < \tau^{-1}\log(1/\kappa)$ *and* $H > 0$*. If*

$$|z(t) - G(z_t)|^2 \le He^{-\gamma t} \quad \text{for all } t \ge 0, \tag{4.5}$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \log|z(t)| \le -\frac{\gamma}{2}. \tag{4.6}$$

Proof. Choose any $\varepsilon \in (\kappa e^{\gamma\tau}, 1)$. In the same way as in the proof of Lemma 3.3, we can show that for any $T > 0$,

$$\sup_{0\le t\le T}\Big[e^{\gamma t}|z(t)|^2\Big] \le \frac{H}{1-\varepsilon} + \frac{\kappa e^{\gamma\tau}}{\varepsilon} \sup_{-\tau\le t\le T}\Big[e^{\gamma t}|z(t)|^2\Big].$$

It then follows that

$$\Big(1 - \frac{\kappa e^{\gamma\tau}}{\varepsilon}\Big) \sup_{0\le t\le T}\Big[e^{\gamma t}|z(t)|^2\Big] \le \frac{H}{1-\varepsilon} + \frac{\kappa e^{\gamma\tau}}{\varepsilon} \sup_{-\tau\le t\le 0} |z(t)|^2.$$

Consequently,

$$\limsup_{t\to\infty} \frac{1}{t} \log|z(t)| \le -\frac{\gamma}{2},$$

as required. The proof is complete. $\square$

Proof of Theorem 4.1. First, note that condition (4.1) implies condition (4.4) since $C[-\tau;0];R^n) \subset L^2_{\mathcal{F}_\infty}([-\tau,0];R^n)$. Now fix any initial data ξ and write the solution $x(t;\xi) = x(t)$ simply. By the well-known Doob martingale inequality (cf. Karatzas and Shreve [6]), the Hölder inequality, and condition (4.2), we can easily derive that for any integer $k \ge 1$,

(4.7)

$$\begin{aligned}
&E\Big(\sup_{0\le\theta\le\tau} |x(k\tau+\theta) - G(x_{k\tau+\theta})|^2\Big) \\
&\le 3E|x(k\tau) - G(x_{k\tau})|^2 + 3K(\tau+4)\int_{k\tau}^{(k+1)\tau}\Big(\sup_{-\tau\le\theta\le 0} E|x(s+\theta)|^2\Big)ds \\
&\le \Big(6M(1+\kappa)e^{-\bar\gamma(k\tau-\tau)} + 3K(\tau+4)M\int_{k\tau}^{(k+1)\tau} e^{-\bar\gamma(s-\tau)}ds\Big)\Big[\sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2\Big] \\
&\le Ce^{-\bar\gamma k\tau},
\end{aligned}$$

where $C = 3Me^{\bar{\gamma}\tau}\big[2(1+\kappa)+K(\tau+4)\big]\sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2$. Let $\varepsilon \in (0, \bar{\gamma})$ be arbitrary. It then follows from (4.7) that

$$P\left(\omega : \sup_{0\le\theta\le\tau} |x(k\tau+\theta) - G(x_{k\tau+\theta})|^2 > e^{-(\bar{\gamma}-\varepsilon)k\tau}\right) \le Ce^{-\varepsilon k\tau}.$$

In view of the well-known Borel–Cantelli lemma, we see that for almost all $\omega \in \Omega$,

$$\sup_{0\le\theta\le\tau} |x(k\tau+\theta) - G(x_{k\tau+\theta})|^2 \le e^{-(\bar{\gamma}-\varepsilon)k\tau} \tag{4.8}$$

holds for all but finitely many k. Hence for all $\omega \in \Omega$ excluding a P-null set, there exists a $k_o(\omega)$ for which (4.8) holds whenever $k \ge k_o$. In other words, for almost all $\omega \in \Omega$,

$$|x(t) - G(x_t)|^2 \le e^{-(\bar{\gamma}-\varepsilon)(t-\tau)} \quad \text{if } t \ge k_o\tau.$$

However, $|x(t) - G(x_t)|^2$ is finite on $[0,\ k_o\tau]$. Therefore, for almost all $\omega \in \Omega$, there exists a finite number $H = H(\omega)$ such that

$$|x(t) - G(x_t)|^2 \le He^{-(\bar{\gamma}-\varepsilon)t} \quad \text{for all } t \ge 0.$$

An application of Lemma 4.2 now yields

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \le -\frac{\bar{\gamma}-\varepsilon}{2} \quad \text{a.s.},$$

and the desired result (4.3) follows by letting $\varepsilon \to 0$. The proof is complete. ◻

5. Comparison with existing results. Recently, in [11], we studied the exponential stability in mean square for a class of neutral stochastic functional differential equations using a completely different technique from the one in this paper. The aim of this section is to compare our previous results in [11] with our new results in this paper. The equation studied in [11] is of the form

$$d[x(t) - G(x_t)] = [f_1(t, x(t)) + f_2(t, x_t)]dt + g(t, x_t)dw(t) \tag{5.1}$$

on $t \ge 0$ with initial data $x_0 = \xi$, where $f_1 : R_+ \times R^n \to R^n,\ f_2 : R_+ \times C([-\tau, 0]; R^n) \to R^n$, and G and g are the same as before. Let us first state a useful result.

THEOREM 5.1. *Let* (3.1) *hold. Assume that there are two positive constants* λ_1 *and* λ_2 *such that*

$$E\Big(2(\phi(0) - G(\phi\,))^T[f_1(t, \phi(0)) + f_2(t, \phi)] + trace[g^T(t, \phi)g(t, \phi)]\Big)$$
$$\le -\lambda_1 E|\phi(0)|^2 + \lambda_2 \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2 \tag{5.2}$$

for all $t \ge 0$ *and* $\phi \in L^2_{\mathcal{F}_\infty}([-\tau, 0]; R^n)$. *If*

$$0 < \kappa < \frac{1}{4} \quad \text{and} \quad \lambda_1 > \frac{\lambda_2}{(1-2\sqrt{\kappa})^2}, \tag{5.3}$$

then the trivial solution of equation (5.1) *is exponentially stable in mean square.*

Proof. By condition (5.3), we can choose q such that

$$\frac{1}{\kappa} > q > \frac{1}{(1-\sqrt{\kappa})^2} \quad \text{and} \quad \lambda_1 > \frac{\lambda_2 q}{(1-\sqrt{\kappa q})^2}. \tag{5.4}$$

By defining $f(t,\varphi) = f_1(t,\varphi(0)) + f_2(t,\varphi)$ for $t \ge 0$ and $\varphi \in C([-\tau,0];R^n)$, equation (5.1) can be written as equation (2.1), so all that we need to do is verify condition (3.2). To do so, let $t \ge 0$ and $\phi \in L^2_{\mathcal{F}_t}([-\tau,0];R^n)$, satisfying

$$E|\phi(\theta)|^2 < qE|\phi(0) - G(\phi)|^2, \quad -\tau \le \theta \le 0.$$

Note that for any $\varepsilon > 0$,

$$-E|\phi(0)|^2 \le -\frac{1}{1+\varepsilon} E|\phi(0) - G(\phi)|^2 + \frac{1}{\varepsilon} E|G(\phi)|^2. \tag{5.5}$$

It then follows from (5.2) and (5.5) that

$$\begin{aligned} &E\Big(2(\phi(0) - G(\phi))^T[f_1(t,\phi(0)) + f_2(t,\phi)] + \mathrm{trace}[g^T(t,\phi)g(t,\phi)]\Big) \\ &\quad \le -\lambda_1 E|\phi(0)|^2 + \lambda_2 \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2 \\ &\quad \le -\Big[\lambda_1\Big(\frac{1}{1+\varepsilon} - \frac{\kappa q}{\varepsilon}\Big) - \lambda_2 q\Big] E|\phi(0) - G(\phi)|^2. \end{aligned} \tag{5.6}$$

In particular, choose $\varepsilon = \sqrt{\kappa q}/(1-\sqrt{\kappa q})$ and hence

$$\Big[\lambda_1\Big(\frac{1}{1+\varepsilon} - \frac{\kappa q}{\varepsilon}\Big) - \lambda_2 q\Big] = \lambda_1(1-\sqrt{\kappa q})^2 - \lambda_2 q > 0,$$

where we have used (5.4). In other words, condition (3.2) is satisfied and hence the conclusion follows from Theorem 3.1. The proof is complete. □

To compare this result with one in our previous paper [11], let us introduce another new notation $\mathcal{W}([-\tau,0];R_+)$, which is the family of all Borel-measurable bounded nonnegative functions $\eta(\theta)$ defined on $-\tau \le \theta \le 0$ such that $\int_{-\tau}^0 \eta(\theta)d\theta = 1$. In [11], conditions (3.1) and (5.2) were strengthened as follows: There is a constant $\kappa \in (0,1)$ and a function $\eta_1 \in \mathcal{W}([-\tau,0];R_+)$ such that

$$|G(\varphi)|^2 \le \kappa \int_{-\tau}^0 \eta_1(\theta)|\varphi(\theta)|^2 d\theta \quad \text{for all } \varphi \in C([-\tau,0];R^n); \tag{5.7}$$

moreover, there exists a function $\eta_2(.) \in \mathcal{W}([-\tau,0];R_+)$ and two positive constants λ_1 and λ_2 such that

$$\begin{aligned} &2(\varphi(0) - G(\varphi))^T[f_1(t,\varphi(0)) + f_2(t,\varphi)] + \mathrm{trace}[g^T(t,\varphi)g(t,\varphi)] \\ &\quad \le -\lambda_1|\varphi(0)|^2 + \lambda_2 \int_{-\tau}^0 \eta_2(\theta)|\varphi(\theta)|^2 d\theta \end{aligned} \tag{5.8}$$

for all $t \ge 0$ and $\varphi \in C([-\tau,0];R^n)$. These two conditions are indeed stronger than (3.1) and (5.2), respectively. For example, if (5.7) holds, then for any $\phi \in$

$L^2_{\mathcal{F}_\infty}([-\tau,0];R^n)$,

$$E|G(\phi)|^2 \le \kappa \int_{-\tau}^0 \eta_1(\theta) E|\phi(\theta)|^2 d\theta$$
$$\le \kappa \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2 \int_{-\tau}^0 \eta_1(\theta)d\theta = \kappa \sup_{-\tau\le\theta\le 0} E|\phi(\theta)|^2,$$

that is, (3.1) holds. However, what we gained in [11] with this price paid is the following sharper result.

THEOREM 5.2 (Mao [11]). *Let* (5.7) *and* (5.8) *hold. If* $\lambda_1 > \lambda_2$ *and* $\kappa \in (0,1)$, *then the trivial solution of equation* (5.1) *is exponentially stable in mean square.*

Obviously, $\lambda_1 > \lambda_2$ and $\kappa \in (0,1)$ are much sharper than (5.3). The disadvantage of Theorem 5.2 is that (5.7) and (5.8) are somehow too restricted. For instance, Theorem 5.2 is not applicable to the case of neutral stochastic differential delay equations. However, Theorem 5.1 can be applied to deal with the delay case easily. Let us now turn to this topic.

6. Neutral stochastic differential delay equations. As an application, let us apply the theory established in the previous sections to deal with the exponential stability of neutral stochastic differential delay equations of the form

$$d[x(t) - \bar{G}(x(t-\tau))] = \bar{f}(t, x(t), x(t-\tau))dt + \bar{g}(t, x(t), x(t-\tau))dw(t) \tag{6.1}$$

on $t \ge 0$ with initial data $x_0 = \xi$, where $\bar{G}: R^n \to R^n$, $\bar{f}: R_+ \times R^n \times R^n \to R^n$, and $\bar{g}: R_+ \times R^n \times R^n \to R^{n\times m}$. As before, assume that equation (6.1) has a unique global solution denoted by $x(t;\xi)$ and, moreover, $\bar{G}(0) = 0$, $\bar{f}(t,0,0) = 0$, and $\bar{g}(t,0,0) = 0$. We now employ Theorem 5.1 to establish one useful corollary.

COROLLARY 6.1. *Assume that there is a constant* $\kappa \in (0,1)$ *such that*

$$|\bar{G}(x)|^2 \le \kappa |x|^2, \quad x \in R^n. \tag{6.2}$$

Assume also that there are two positive constants λ_1 *and* λ_2 *such that*

$$2(x - \bar{G}(y))^T \bar{f}(t,x,y) + \mathrm{trace}[\bar{g}^T(t,x,y)\bar{g}(t,x,y)] \le -\lambda_1|x|^2 + \lambda_2|y|^2 \tag{6.3}$$

for all $t \ge 0$ *and* $x, y \in R^n$. *If*

$$0 < \kappa < \frac{1}{4} \quad \text{and} \quad \lambda_1 > \frac{\lambda_2}{(1-2\sqrt{\kappa})^2}, \tag{6.4}$$

then the trivial solution of equation (6.1) *is exponentially stable in mean square.*

This corollary follows directly from Theorem 5.1 since equation (6.1) can be written as equation (5.1) by defining

$$G(\varphi) = \bar{G}(\varphi(-\tau)), \qquad f_1(t,x) = \bar{f}(t,x,0),$$

$$f_2(t,\varphi) = -\bar{f}(t,\varphi(0),0) + \bar{f}(t,\varphi(0),\varphi(-\tau)), \qquad g(t,\varphi) = \bar{g}(t,\varphi(0),\varphi(-\tau))$$

for $t \ge 0$, $x \in R^n$ and $\varphi \in C([-\tau,0];R^n)$. Of course, we can directly apply Theorems 3.1 and 4.1 to obtain a more general result. For this purpose, let us introduce another new notation $L^2_{\mathcal{F}}(\Omega;R^n)$, which is the family of R^n-valued $\mathcal{F}$-measurable random variables X such that $E|X|^2 < \infty$.

THEOREM 6.2. *Let* (6.2) *hold with* $\kappa \in (0,1)$. *Let* $q > (1-\sqrt{\kappa})^{-2}$. *Assume that there is a constant* $\lambda > 0$ *such that*

$$E\Big(2(X-\bar{G}(Y))^T\ \bar{f}(t,X,Y) + \mathrm{trace}[\bar{g}^T(t,X,Y)\bar{g}(t,X,Y)]\Big) \\ \le -\lambda E|X-\bar{G}(Y)|^2 \tag{6.5}$$

for all $t \ge 0$ *and those* $X, Y \in L^2_{\mathcal{F}}(\Omega; R^n)$ *satisfying* $E|Y|^2 < qE|X-\bar{G}(Y)|^2$. *Then the trivial solution of equation* (6.1) *is exponentially stable in mean square. Furthermore, if there is a positive constant* K *such that*

$$|\bar{f}(t,x,y)|^2 + \mathrm{trace}[\bar{g}^T(t,x,y)\bar{g}(t,x,y)] \le K(|x|^2+|y|^2), \quad x,y \in R^n, \tag{6.6}$$

then the trivial solution of equation (6.1) *is also almost surely exponentially stable.*

This theorem follows directly from Theorems 3.1 and 4.1 since equation (6.1) can be written as equation (2.1) by defining

$$G(\varphi) = \bar{G}(\varphi(-\tau)), \qquad f(t,\varphi) = \bar{f}(t,\varphi(0),\varphi(-\tau)), \qquad g(t,\varphi) = \bar{g}(t,\varphi(0),\varphi(-\tau))$$

for $t \ge 0$ and $\varphi \in C([-\tau,0]; R^n)$.

7. Linear neutral stochastic functional differential equations. As another application, let us consider a linear neutral stochastic functional differential equation

$$d[x(t) - G(x_t)] = [-Ax(t) + B_0(x_t)]dt + \sum_{i=1}^m B_i(x_t)dw_i(t) \tag{7.1}$$

on $t \ge 0$ with initial data $x_0 = \xi$. Here A is an $n \times n$ constant matrix and

$$G(\varphi) = \int_{-\tau}^0 d\gamma(\theta)\varphi(\theta), \qquad B_i(\varphi) = \int_{-\tau}^0 d\beta_i(\theta)\varphi(\theta)$$

for $\varphi \in C([-\tau,0]; R^n)$, $0 \le i \le m$, where $\gamma(\theta) = (\gamma^{kl}(\theta))_{n\times n}$, $\beta_i(\theta) = (\beta_i^{kl}(\theta))_{n\times n}$ and all $\gamma^{kl}(\theta)$ and $\beta_i^{kl}(\theta)$ are functions of bounded variation on $-\tau \le \theta \le 0$. Let $V_{\gamma^{kl}}(\theta)$ denote the total variations of γ^{kl} on the interval $[-\tau,\theta]$ and let $V_\gamma(\theta) = ||V_{\gamma^{kl}}(\theta)||$. We can define $V_{\beta_i}(\theta)$ similarly. In particular, let

$$\hat{\gamma} = V_\gamma(0) \quad \text{and} \quad \hat{\beta}_i = V_{\beta_i}(0), \quad 0 \le i \le m.$$

Let us now impose the first assumption:

$$0 < \hat{\gamma} < \frac{1}{2}. \tag{7.2}$$

Then for any $\phi \in L^2_{\mathcal{F}_\infty}([-\tau,0]; R^n)$,

$$E|G(\varphi)|^2 \le \hat{\gamma}E\int_{-\tau}^0 dV_\gamma(\theta)|\varphi(\theta)|^2 \le \hat{\gamma}^2 \sup_{-\tau\le\theta\le 0} E|\varphi(\theta)|^2. \tag{7.3}$$

In other words, (3.1) is satisfied with $\kappa = \hat{\gamma}^2$. Moreover,

$$2E\big[|\phi(0)||G(\phi)|\big] \le \frac{\hat{\gamma}}{1-2\hat{\gamma}}E|\phi(0)|^2 + \frac{1-2\hat{\gamma}}{\hat{\gamma}}E|G(\phi)|^2 \\ \le \frac{\hat{\gamma}}{1-2\hat{\gamma}}E|\phi(0)|^2 + \hat{\gamma}(1-2\hat{\gamma})\sup_{-\tau\le\theta\le 0} E|\varphi(\theta)|^2. \tag{7.4}$$

Similarly, one can show that

$$2E\big[|\phi(0)||B_0(\phi)|\big] \le \frac{\hat\beta_0}{1-2\hat\gamma}E|\phi(0)|^2 + \hat\beta_0(1-2\hat\gamma)\sup_{-\tau\le\theta\le 0}E|\varphi(\theta)|^2, \tag{7.5}$$

$$2E\big[|G(\phi)||B_0(\phi)|\big] \le 2\hat\gamma\hat\beta_0 \sup_{-\tau\le\theta\le 0}E|\varphi(\theta)|^2, \tag{7.6}$$

and

$$\sum_{i=1}^m E|B_i(\phi)|^2 \le \left[\sum_{i=1}^m \hat\beta_i^2\right]\sup_{-\tau\le\theta\le 0}E|\varphi(\theta)|^2. \tag{7.7}$$

Let $\lambda_{\min}(A+A^T)$ denote the smallest eigenvalue of $A+A^T$. Using (7.4)–(7.7), we then see that

$$\begin{aligned} &E\left(2(\phi(0)-G(\phi))^T[-A\phi(0)+B_0(\phi)] + \sum_{i=1}^m E|B_i(\phi)|^2\right) \\ &\quad\le -\left[\lambda_{\min}(A+A^T) - \frac{\hat\gamma||A||+\hat\beta_0}{1-2\hat\gamma}\right]E|\phi(0)|^2 \\ &\qquad + \left[(\hat\gamma||A||+\hat\beta_0)(1-2\hat\gamma) + 2\hat\gamma\hat\beta_0 + \sum_{i=1}^m \hat\beta_i^2\right]\sup_{-\tau\le\theta\le 0}E|\phi(\theta)|^2. \end{aligned} \tag{7.8}$$

To close this paper, we apply Theorems 5.1 and 4.1 and conclude the following corollary.

COROLLARY 7.1. *Let* (7.2) *hold. If*

$$\lambda_{\min}(A+A^T) > \frac{2(\hat\gamma||A||+\hat\beta_0)}{1-2\hat\gamma} + \frac{1}{(1-2\hat\gamma)^2}\left[2\hat\gamma\hat\beta_0 + \sum_{i=1}^m \hat\beta_i^2\right],$$

then the trivial solution of equation (7.1) *is exponentially stable in mean square and is also almost surely exponentially stable.*

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