UNCONTROLLED SPACECRAFT FORMATIONS ON TWO-DIMENSIONAL INVARIANT TORI

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Abstract: *The libration point regions provide best requirements for multi-spacecraft missions demanding relative spacecraft configuration with large distances among satellites. In this paper, quasi-periodic trajectories are studied and their dynamical properties and the wide parameter range of existing natural trajectories are emphasised. The compound of all quasi-periodic trajectories belonging to a certain invariant torus define a natural region building a surface structure. This enables the computation of quasi-periodic orbits by a numerical iterative mapping scheme determining the required frequencies and a parametric torus function in angular phase space. The scheme is applied to the Earth-Moon libration points* L_1 *and* L_2*, and quasi-periodic orbits are computed. The above mention surface structure, which is maintained during dynamical evolution, enables the study of geometric dependencies between spacecraft, e.g. distance evaluations. Formation flying capabilities are pointed out by determining initial states for spacecraft formations on quasi-periodic trajectories and studying how spacecraft naturally evolve on their trajectories maintaining relative distances. The trajectory design of uncontrolled spacecraft formations can either be supported by utilising a torus function representing the surface structure of a torus or by finding a suitable analytical representation of quasi-periodic trajectories. The limits of both are highlighted in this work. For some orbital parameters an analytical model deliver sufficient results, for others the introduced parametric function provide tools to utilise quasi-periodic orbits in future studies, such as transfer scenarios. An in-plane coordinate frame is introduced visualising the formation geometry and evaluating the performance of formations.*

1. Introduction

New mission concepts using large spacecraft formations emphasise the study of multiple spacecraft placed at large relative distances. Projects within the ESA Cosmic Vision are now in the study phase with the aim to trigger new scientific discoveries, such as the detection of asteroids and planets. Naturally existing trajectories meeting certain formation conditions without continuous station-keeping are promising high performance for experiments in fundamental physics, such as telescope assemblies or spectroscopy configurations. The libration point regions provide the appropriate framework. The Earth-Moon libration point regions provide a low-acceleration environment and the variety of different orbits that can be combined. Those orbits lie in the centre manifold existing within the vicinity of each libration points and the motion is classified into periodic and quasi-periodic orbits. In particular, quasi-periodic trajectories, which evolve on invariant tori, provide a set of trajectories that satisfy geometric formation constraints. This unique dynamics yield better candidates for relative spacecraft configurations supporting large distances among satellites compared to Earth orbits or orbits about other central bodies.

Much of the available research in formation flying near the libration points focus on the exploitation of periodic orbits for trajectory and formation design. Beside periodic, there are also quasi-periodic orbits around the libration points providing a second class of bounded motion. Kolemen and Olikara studied those and proposed numerical procedures for their generation and continuation [5, 7]. The algorithm uses a Fourier representation to describe an invariant curve representing the intersection of an invariant torus with a

Poincaré section. A more general introduction to invariant objects, in particular two-dimensional tori and their trajectories is given by Schilder [8]. He studies invariant tori for a generic dynamical system laying the fundamentals and with some modification, results can be applied to the dynamics used in this study. Barden investigates formation flying near libration points in the circular-restricted three-body problem (CR3BP) with a focus on the determination of the natural behaviour at the centre manifold [1]. A *string of pearls* was proposed to demonstrate that quasi-periodic trajectories evolving on an invariant torus are useful for formation flying. Hértier explores quasi-periodic Lissajous trajectories near a given reference orbit in the vicinity the Sun-Earth libration point L_2 for the placement of large formations of spacecraft [3, 2]. They derive natural regions where the geometry changes of formations are minimal. Further analyses considered natural and non-natural arcs for formation applications [4]. Most of the formation flying missions have been considering spacecraft at a relatively small distance from the reference orbit. However, observatory and interferometry missions in space have been the motivation for the analysis of large formations, in particular on quasi-periodic trajectories on invariant tori.

The primary goal of this study is to characterise the natural motion of spacecraft on two-dimensional invariant tori and to indicate properties of the motion that are beneficial for formation flying missions. A two-dimensional invariant torus can be described as a set of orbits that start on a surface and stay on that surface during the dynamical evolution. With the aim of identifying these orbits, a numerical approach was developed to parametrise invariant tori and determine their dynamical properties such as the frequency base enabling to introduce phasing constraints. Formations, created by multiple spacecraft, are placed on the surface structure of a torus and their performance is examined, and formation snapshots are introduced to explain and study their shape and orientation in space. The variation of the formation's geometry depends on the selection of the initial states on the torus and therefore on the distribution of the spacecraft on the surface of the torus, which is defined by two phase angles. The appropriate orientation and phasing conditions are derived from subspaces introduced by linearisation. A sophisticated solution is derived from the parametrization of the torus. Quasi-periodic orbits in the vicinity of Lagrange point L_1 in the Earth-Moon system were studied in detail. For some orbital parameters an analytical model delivers sufficient results, for others the introduced parametric function provide tools to utilise quasi-periodic orbits in future studies, such as transfer scenarios. This analysis details and expands the understanding of the natural dynamics and points out the advantages of quasi-periodic orbit and their application to the trajectory design of formation flying spacecraft missions.

2. Dynamical Representation

The dynamical reference model used is the circular restricted three-body problem. It assumes that the Earth travels around the Sun in a circular orbit, whereas the spacecraft is modelled as a massless particle moving under the gravitational forces of the Earth (primary body) and the Moon (secondary body). In the CR3BP, the motion of the spacecraft is described in terms of rotating coordinates relative to the barycentre of the system primaries. In this frame, the rotating x-axis is directed from the primary to the secondary body. The second order differential equation of motion is introduced in the Euler-Lagrange form as

$$\begin{aligned}\ddot{x} &= 2\dot{x} + x - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu)\\ \ddot{y} &= -2\dot{x} + y - \frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y\\ \ddot{z} &= -\frac{1-\mu}{r_1^3}z - \frac{\mu}{r_2^3}z\end{aligned} \tag{1}$$

where $r_1^2 = (x-\mu)^2 + y^2 + z^2$ represents the distance from the spacecraft to the larger primary, and $r_2^2 = (x+1-\mu)^2 + y^2 + z^2$ to the larger primary. Contrarily to the Newtonian approach introduced previously, "the energy and momentum of the system are not conserved separately and a general analytical solution is not possible". A conserved quantity in the restricted-three body problem is introduced as an integral of motion, the Jacobian constant J which is directly related to the conserved energy given by

$$E = -\frac{C}{2} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \left\{ \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} - \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2} \right\} \tag{2}$$

Later the Jacobi constant is used to classify periodic and quasi-periodic trajectories. The Jacobi integral is used for the definition of forbidden regions in the rotating reference frame under the restrictions of the restricted problem formulation. The Jacobi constant value has no unit but an energy equivalent may be given, an example is $C = -0.4401$ is equal to a total energy of $E = -1.4233 kJ/kg$. Eq. 1 defines the motion of the spacecraft in a rotating coordinate system that is normalised, therefore dimensionless, such that the gravitational parameter G is equal one. The non-dimensional orbital period is normalised to 2π by the factor t^*, which is the inverse of the mean motion of the primaries. The factor for distance quantities is the characteristic length l^*, which is the distance between the primaries. For weights the characteristic mass m^* is the total mass of the system. The notion for units throughout this work is as follows: *length unit LU*, *time unit TU*, and the *mass unit MU*.

$$l^* = d_{p \to s} \qquad t^* = \left(\frac{l^{*3}}{Gm^*} \right)^{-1/2} \qquad m^* = m_{Sun} + m_{Earth/Moon} \tag{3}$$

The CR3BP possess five equilibrium points named ($L_1 - L_5$), which are also called Lagrange or libration points. L_1 and L_2 are the closest points to the secondary body. In the framework of the circular restricted three-body problem the linear six-dimensional phase space around the co-linear libration points can mathematically described by a centre-centre-saddle structure. This structure enables the classification of different periodic orbits that exist within the four-dimensional centre manifold. The value for the mass ratio is $\mu = 3.0406 \cdot 10^{-6}$. The subspaces are spanned by eigenvectors of the Monodromy matrix, which is a state transition matrix (STM) for a periodic orbit, mapping the initial state vector to the final state vector after one period. An independent basis is established building subspace, spanning the plane of the imaginary eigenvectors, other eigenvectors correspond to the stable and unstable direction. The periodic subspaces build the centre manifold, which is associated with families of periodic orbits, whereas many of these periodic orbits also possess centre components that correspond to quasi-periodic motions.

3. Quasi-Periodic Motion

In the following the analytic background of invariant tori is given. Certain periodic orbits enable the access to quasi-periodic motion. The equations of motion Eq. 1 in the CR3BP can be considered generically as an autonomous ordinary differential equation, assuming that those equations possess a quasi-periodic orbit as a solution that reside on an invariant tori about the corresponding periodic orbit. Starting from the behaviour of equilibrium points and periodic orbits studied in the previous chapter, another fundamental solutions are quasi-periodic orbits or in general p-dimensional invariant tori. In other words the closure of the quasi-periodic orbit is an invariant torus. It is often preferable to regard the torus directly as an invariant object, independent of a particular trajectory on its surface, see Fig. 2a [6]. The propagation of multiple trajectories densely fill and describe the surface of the invariant torus. Fig. 2b represents the surface to illustrate the concept. The results are invariant curves representing cross sections of the torus, they are visualized in Fig. 2b. The cross defines the centre of the curves, whereas the red markers highlight the zero direction from where the angle θ_1 is measured. The objective is to gain a better understanding of the motion of a spacecraft on a torus. This knowledge of the natural flow is very useful for trajectory and formation design. Quasi-periodic solutions of a non-linear system are described as the motion on a p-dimensional torus that is associated with p different internal frequencies. All trajectories of this flow are quasi-periodic functions of time and their properties strongly depend on arithmetical properties of the frequency base. In case of a 2-dimensional torus (2-torus) the frequency base has two entries, here defined as $\omega = \{\omega_1, \omega_2\}$.

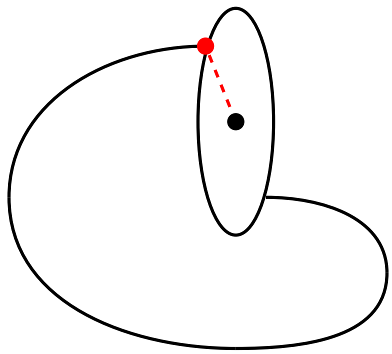


Figure 1: Visualization of the motion on a torus with a two-dimensional frequency base. The black circle represents a cross section of the torus.

The parallel flow on an invariant torus with the frequency base ω is non-resonant or quasi-periodic, if the basic frequencies are rationally independent (no non-trivial linear combination with integers is equal to zero). In this case each trajectory densely covers the torus' surface. In the other case, when the frequencies are rationally dependent (integer k exist to solve equation $k_1\omega_1 + k_2\omega_2 = 0$) the torus collapses. A torus is the product of two imposed motions, which are directly linked to the frequencies, and can be described by a particle that is longitudinally moving about the torus structure with the frequency ω_1, while rotating with the frequency ω_2. The motion is visually described in Fig. 1. The time needed for one rotation is

$$T_i = \frac{2\pi}{\omega_i} \tag{4}$$

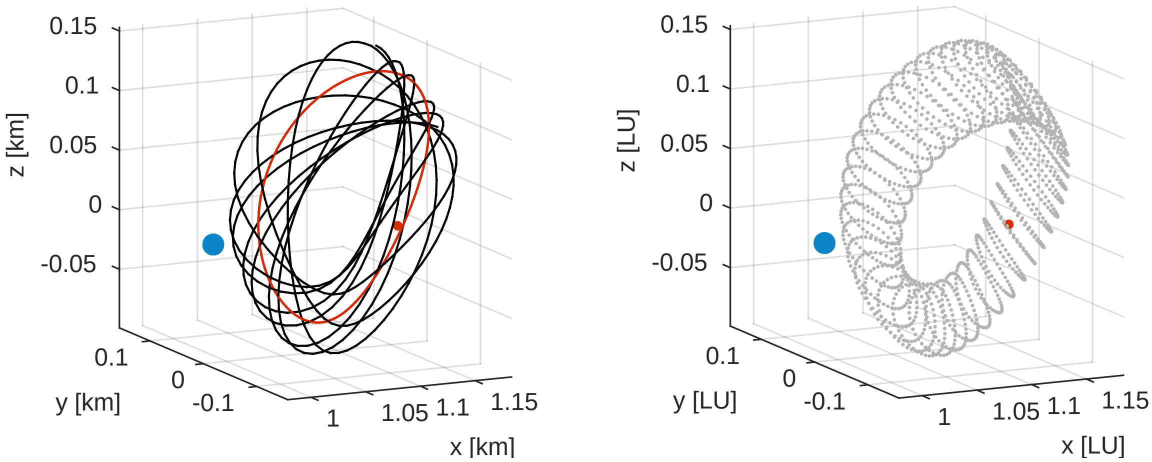
The rotation number of a torus is defined as

$$\rho = 2\pi\frac{\omega_1}{\omega_2} \tag{5}$$

which uniquely defines a torus with a given energy. In words the rotation number ρ represents the average movement in ω_2-direction when one revolution is done in the ω_1-direction.

3.1. Obtaining Quasi-Periodic Solutions

Quasi-periodic orbits exist in two parameter families. Mathematical they are often classified by the orbital energy E and the rotational parameter ρ. A unique definition is also possible by the frequencies ω_1 and ω_2. A detailed description of invariant tori and their existence is found in literature. Quasi-periodic trajectories existing in the dynamical regime of the three-body problem are difficult to characterise, as there is no set of orbital elements analogous to the orbital elements known in the two-body problem. Furthermore,



(a) (b)

Figure 2: (a) Periodic versus quasi-periodic orbit. (b) Torus function describing the entity of all quasi-periodic trajectories. The Moon is plotted in blue and the second libration point L_2 appears in red.

utilising the parametric representation the location of a spacecraft on the orbit is uniquely defined by two phase angles α and β. The angles are similar to the true anomaly known from the keplerian motion, they increase in the direction of motion and vary between 0 and 2π. The study of quasi-periodic orbits, e.g. the transfer design, benefits from a parametric description of trajectories and their associated manifolds, which simplifies the description of departure and arrival conditions along an orbit. The introduction of a parametric representation of quasi-periodic orbits enable the identification of a single orbit by its Jacobian constant C and the two frequencies ω_1 and ω_2. To fully specify a the location of a spacecraft on a quasi-periodic orbit, four parameters are required. In addition to the orbit identifier, which are two the Jacobian value, and a some second parameter e.g. the rotation number, two phase angles must be specified. "In short, specifying the amplitudes establishes the overall dimensions, while the phase angles establish a given point on the orbit." The selection of the origin of the phase angles is arbitrary. Here, the angle is measured clockwise from the x-axis, and the second angle results from the fact that the origin is the construction point for quasi-periodic orbits.

A transformation into angular torus coordinates (α, β) enables the introduction of a parametric representation of quasi-periodic trajectories. The two parameters, which are frequencies are defined as $\omega = \{\omega_1, \omega_2\}$. Any quasi-periodic orbit $x(t)$ can be written as

$$u\begin{pmatrix} \alpha + \omega_1 t \\ \beta + \omega_2 t \end{pmatrix} = \Phi_t\left(u\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right). \tag{6}$$

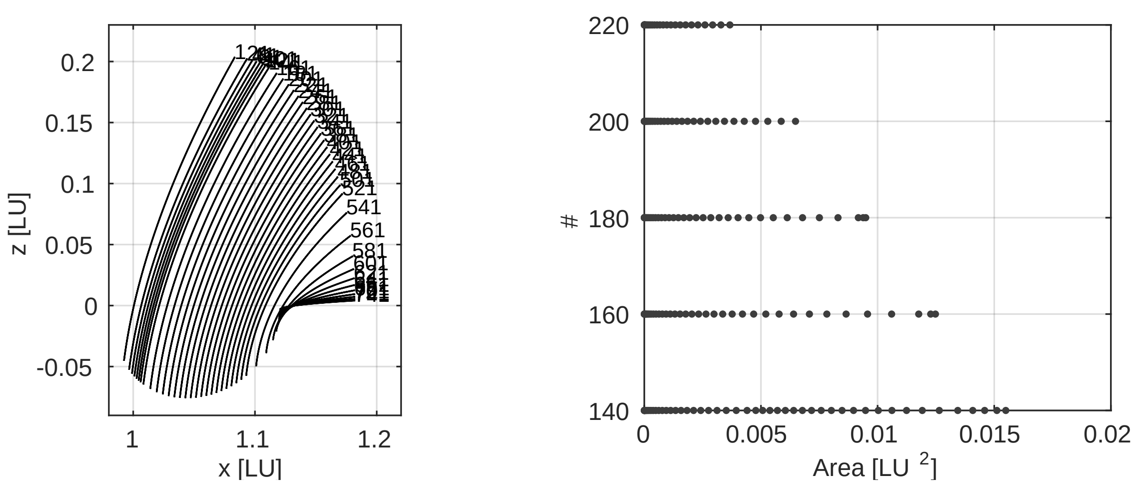
which need to be solved. A convenient way to obtain quasi-periodic solutions is to use a numerical approach which is based on a Poincaré section and the approximation of invariant curve by Fourier coefficients. Either Poincaré maps or stroboscopic maps find application to solve Eq. 6 depending on the method. For a description of this method, see [5]. The method uses a reduction of the original system, where periodic orbits are represented by points, while quasi-periodic orbits are represented by closed curves. The method reduces the calculation of quasi-periodic orbits to a search for periodic orbits that return to a closed curve on a section plane. Those curves are modelled by truncated Fourier coefficients in position and velocity space.

3.2. Families of Quasi-Periodic Trajectories

Sample families are computed for tori in the vicinity of Lagrange points L_1 and L_2 in the Earth-Moon system. Periodic orbits that initialise the quasi-periodic orbit are vertical lyapunov, Northern and Southern halo orbit and families of the 1-period and 3-period distant retrograde orbits. The extend and boundaries are assessed for the mentioned families. The results presented here do not present a complete study of all existing families, e.g. missing are planar distant retrograde orbits. The capability of the computational scheme is illustrated by the following. Furthermore, the notion of types of quasi-periodic orbits studied in this work is introduced along with properties of the two-parameter continuation process. The four-dimensional centre manifold around L_1 is occupied by quasi-periodic orbits of two different families, the Lissajous family around vertical Lyapunov orbits, and the two-dimensional invariant tori around halo orbits. In other words, periodic orbits in the libration point regions are surrounded by a variety of quasi-periodic orbits. The focus in this paper is set on quasi-periodic orbits that have an underlying halo orbit. The numerical methods outlined in the previous section allow to compute individual trajectories and the corresponding invariant tori. One well-known type of periodic trajectory is the set of halo orbit families that are symmetric across the xz rotating plane. These orbits serve as periodic trajectories for the calculation of families of invariant tori. The northern halo orbit family near the L_1 Sun-Earth libration point appears in Fig. 3. The circle indicates the position of the Earth, and the cross the libration point L_1. By applying this method to a wide range of periodic halo orbits of the Northern L_1 family, the family of invariant tori are computed. Tab. 1 shows the orbital period, and size parameters for a small, medium and large halo orbit within the family, those orbits are highlighted in Fig. 3.

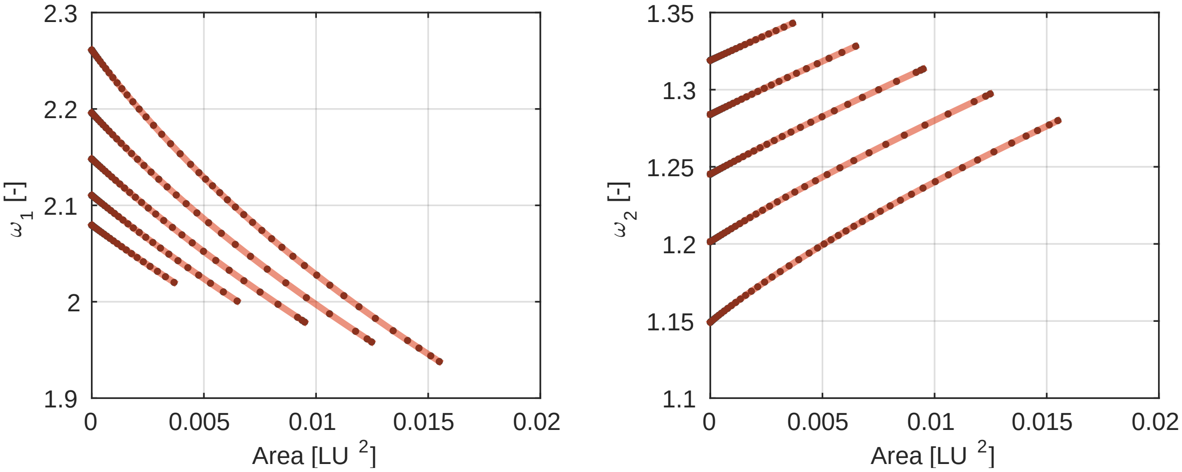
The three types of periodic orbits near the L_2 Earth/Moon Lagrange point are plotted in Fig. 3 over the Jacobian constant in a range of $C = \{3.038 \text{ to } 3.149\}$. The arrows indicate the order of the orbits by an increasing C. The circle indicates the position of the Earth, the cross Lagrange point L_2. The corresponding frequencies vary for the halo orbits between $\omega_{hn} = \{2.1183 \text{ to } 2.1183\}$, for the vertical Lyapunov $\omega_{vl} = \{1.6017 \text{ to } 1.7657\}$, and for the horizontal ones $\omega_{hl} = \{1.5676 \text{ to } 1.8331\}$ $(2\pi / t^*)$.

The continuation parameter for the family is the area that is confined by the invariant curve. This parameter indicates the maximal size of existing tori. Later, the maximal mean distance for spacecraft formations placed



(a) (b)

Figure 3: (a) Northern halo family around L_1 in synodic reference frame. (b) Maximal size of tori around halo orbits.



(a) (b)

Figure 4: Relation between ω_1 and (a) ω_2 and (b) ω_1 plotted as a function of the continuation parameter of the torus family A for several periodic orbits (numbered).

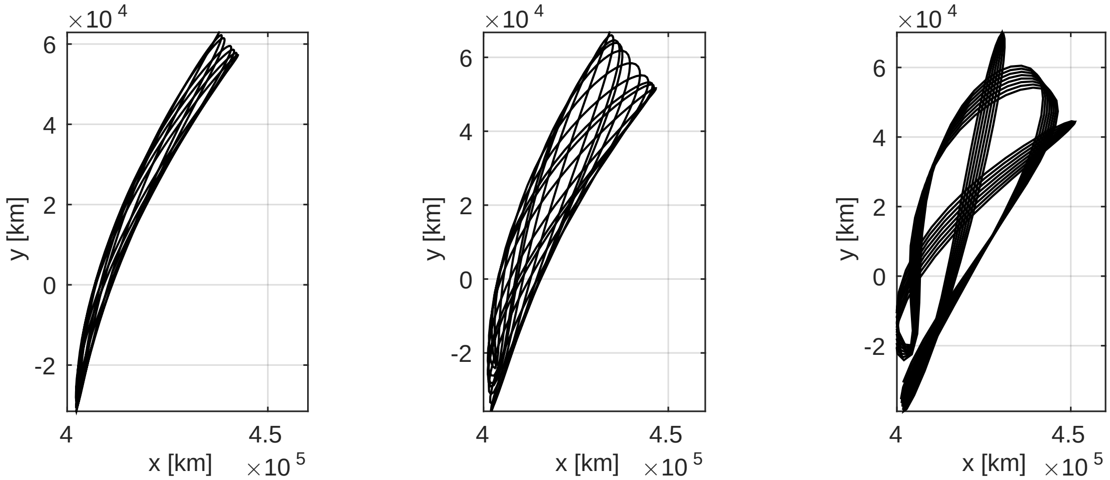
on the torus can be derived. The size is plotted as a function of the orbit number within the halo family in Fig. 3b. The area under the curve in Fig. 3a show possible geometries. The existence of tori strongly depends on the periodic orbit. For orbit number 120, the continuation procedure fails, due to a resonance.

The stability parameters, the energy level and the frequencies are not presented in this plot. For this purpose another plot is introduced combining those parameters. The Jacobian constant and the two frequencies uniquely describe an orbit of the quasi-periodic family. In Fig. 4a and Fig. 4b contains this information. The existence is restricted to right by the orbital period of the generating periodic orbit and the corresponding ω_2 as the argument of the complex eigenvalue of the monodromy matrix. The maximal extend is given by the same pair of frequencies of the horizontal lyapunov orbits.

The arithmetical properties of the frequency base defines if the torus is densely fill by the quasi-periodic trajectory or if a resonance within the motion exists. The torus is not filled by the trajectory and the Poincaré section plane is not described by a curve, but a few discrete points. Three invariant tori from the family around the periodic orbit number 80 out of the halo orbits are shown in Fig. 5. The size, frequency base and

Halo orbit	orb #	T_{period} [adim]	A_x [adim]	A_z [adim]
small	1	3.058	0.00145	0.00144
medium	86	2.945	0.00288	0.00692
large	166	2.193	0.00306	0.00749
Torus	A	ω_1	ω_2	ρ
small	5.858e-07	2.120	1.554	8.574
medium	2.535e-06	2.126	1.559	8.568
large	9.243e-06	2.087	1.538	8.529

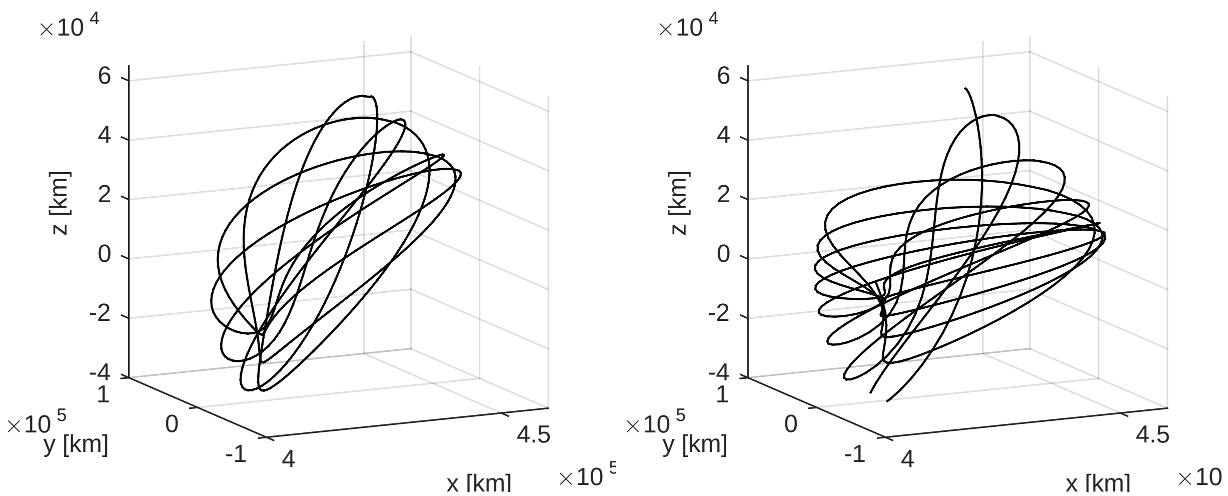
Table 1: Properties of three halo orbits within the L_1 family (top). Frequencies and size of tori shown in Fig. 5 (bottom).



(a) (b) (c)

Figure 5: Trajectories in the synodic reference frame emanating around three different sized tori.

cross section for all three tori are put together in the bottom part of Tab. 1. The frequency ω_1 and the fraction $\frac{\omega_1}{\omega_2}$ are plotted for the family of tori in Fig. 4. For $\frac{\omega_1}{\omega_2} = 1.5$ a resonant and near resonant are highlighted and the corresponding trajectories on the tori are plotted in Fig. 6.



(a) (b)

Figure 6: Trajectories near resonances of the two frequencies. (a) Near resonance, (b) torus collapse to a triple-period orbit.

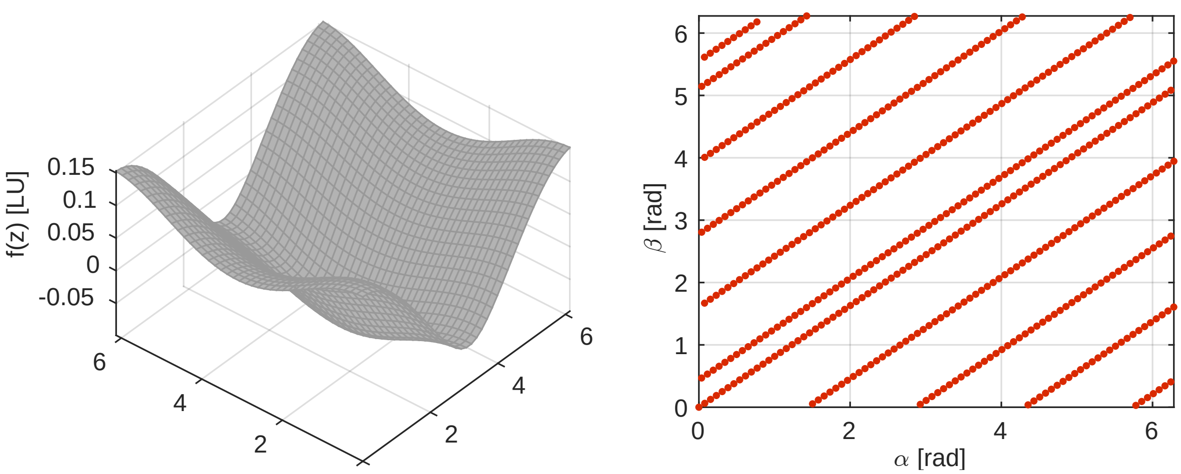
4. Parametrization of a Quasi-Periodic Trajectories

Several tools are available to study the properties of fixed points or periodic orbits, such as Poincaré maps. The study of quasi-periodic motions in particular on an invariant torus is difficult and no numerical tools are available. One issue is to get a parametrization of the torus for further studies. In order to find a parametrization for a torus, a point on the surface of the torus is described by two angular coordinates. A transformation from Cartesian coordinates to torus coordinates (θ_1, θ_2, u) is introduced. The value u for the corresponding values of θ_1 and θ_2 are expressed by trigonometric polynomials The aim of our approach is a numerical approximation of the parametrization of the original system starting from an already calculated torus. Any quasi-periodic solution $x(t)$ can be written as $x(t) = u(\omega t)$, where u is a torus function, whereas ω are the basic frequencies of a solution $x(t)$. This transformation maps the torus in an area $[0, 2\pi)^2$ in the parameter space. After the transformation in torus coordinates an equation to determine u is required, which is a solution of the invariance condition. An approach to obtain the parametrization in torus coordinates is presented in the following, together with tools derived from the parametrization.

The analytic approximation with real-valued Fourier coefficients provides a simple tool for mission design by providing the trajectory without interpolating data. The state functions u_x, u_y, $\dot{u}_x$, and $\dot{u}_y$ are smooth functions of τ and the amplitude A_x. They are written as

$$
\begin{aligned}
u(\alpha, \beta) &= \sum_{n=-N/2}^{N/2} \sum_{m=-M/2}^{M/2} c_{n,m} e^{-i\alpha_n n - i\beta_m m} \\
\dot{u}(\alpha, \beta) &= \frac{1}{\sqrt{NM}} + \sum_{n=-N/2}^{N/2} \sum_{m=-M/2}^{M/2} c_{n,m} e^{i\alpha_n n + i\beta_m m}
\end{aligned}
\tag{7}
$$

The parametrization of a torus is a tool to study the quasi-periodic motion from a dynamical system perspective. The parametrization provides the entire set of trajectories on the torus. Orbits on the neighbourhood of others



(a) (b)

Figure 7: (a) Torus function u plotted in two-dimensional angular phase space. (b) Characteristics represent an orbit on the torus surface.

can be easily computed. The parametrization reveals invariant curves at equidistant time steps on the torus suitable for formations, see Fig. 8. The blue circles are invariant curves, the zero directions for the angle θ_2 are indicated by red markers. A torus mesh is easily created from the parametrization enabling distance evaluations between points on the torus and the nearby space. The x-component of the torus function is plotted in angular phase space in Fig. 7a. A linear characteristic in angular phase space is the map of a quasi-periodic trajectories on the torus. Formation distance and phasing consideration can be extracted from the pattern of the characteristics in Fig. 7b. For the transfer design it is important to have a parametric description of quasi-periodic trajectories and their associated manifolds enabling the identification of a single orbit by its Jacobian constant C and two frequencies ω_1 and ω_2. Furthermore, utilising the parametric representation the location of a spacecraft on the orbit is uniquely defined by two phase angles α and β. The angles are similar to the true anomaly known from the keplerian motion. A sample function is plotted for a quasi-periodic solution with Jacobi constant $C = 3$ is plotted in Fig. 7a. Here, the parametric function is computed on a $50x50$ grid.

$$f(\alpha, \beta) = u(\alpha_0 + \omega_1 \cdot t, \beta_0 + \omega_2 \cdot t) \tag{8}$$

A trajectory generated by utilising the torus function u is suitable as input to a higher-order corrections or multiple shooting scheme to transition to an more precise ephemeris model. Since the torus function can be evaluated efficiently, quasi-periodic orbits from various starting points (α_0, β_0) can be generated.

5. Relative Spacecraft Configurations on Invariant Tori

Some research is only feasible when several spacecraft collaborate, e.g. astronomical apertures with a reflector and detector require a physical distance in between. Achieving spacecraft formation patterns on natural existing trajectories reduces the thrust auth/magnitude. The first step is the definition of suitable formation constraints. In an optimisation distance, angle, and other geometric constraints are imposed, and the best phasing of two and more spacecraft are studied. With a set of natural trajectories at hand along with their amplitudes and orbital periods pairs can be determined that fulfil certain requirements. In a first example

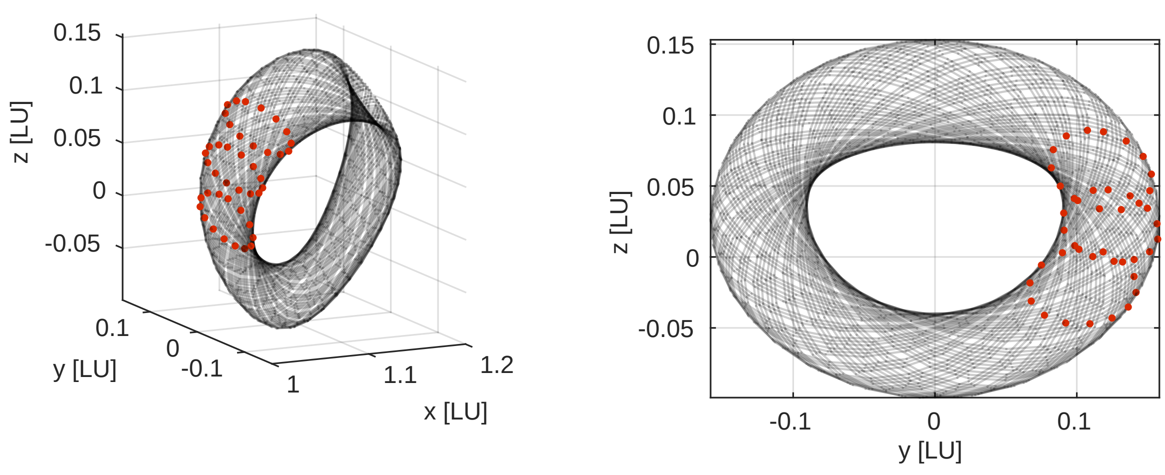


Figure 8: (a) Quasi-periodic trajectory shown in the synodic reference frame. Four snapshots of the formation in time (red dots). (b) As seen from Earth.

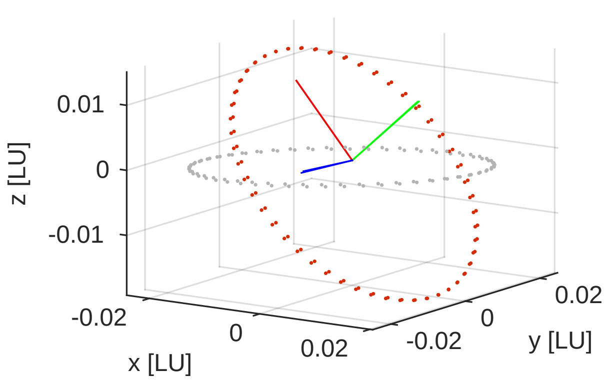
the variations of the euclidean distance between two spacecraft over time is minimized. The configuration design is supported by introduced performance indexes. equal angle separations, equal arc length separations, and equal time separations.

After gaining a fundamental understanding of the motion associated with invariant tori, therefore quasi-periodic trajectories, aspects of formation flying can be studied. The compound of all trajectories belonging to a certain invariant torus defines a natural region building a surface structure. During dynamical evolution trajectories remain on this surface enabling the study of geometric dependencies between spacecraft, e.g. distance evaluations. In this analysis the behaviour of spacecraft placed along quasi-periodic trajectories is explored and relative motion performance factors are evaluated. A quasi-periodic trajectory is shown in Fig. 8 (grey). The initial states for a spacecraft formation are defined at the intersection of a cutting plane with the torus. The evolution of the formation is indicated by red points at four moments in time.

Initially, the set of position vectors describe a planar curve, The motion in the yz-plane as seen from Earth is almost circular and bounded as time proceeds, see Fig. 8b. As the spacecraft proceed in time, they move longitudinally along the underlying halo orbit, and describe latitudinally a winding motion. These components are significant aspects of the natural motion and effect the evolution of the curve in time. The curve's shape contracts and expands, whereas the orientation of the plane in space changes as the trajectories are propagated forward in time. Phasing conditions are introduced and the evolution of the formation is studied in the following.

5.1. Phasing Multiple Spacecraft

The variation in the distance between the spacecraft and the orientation in space depends on the phasing/initial placement of the spacecraft on the torus' surface. It is important that a spacecraft keeps their relative distance within the formation and that the orientation is bounded. Depending on the initial set of state vectors, it is possible for the formation to evolve along the surface of the torus such that the relative positions of each spacecraft in the formation are unaltered and the relative distances are closely bounded. A linear approximation for this phasing problem is derived from the the Monodromy matrix and their periodic



(a)

Figure 9: (a) Visualisation of in-plane and out-of-plane parameter vectors and planes (dotted circles) describing the formation orientation at a certain time.

subspaces. Their eigenvalues and the associated eigenvectors indicate the linear stability of the halo orbit and characterize the nearby motion. Specially, the two-dimensional subspace spanned by the two complex eigenvectors define the plane for the initial states of the formation that stay bounded in the linear system. An exact solution is based on the torus parametrization and their invariant curves. The planar approximation of the invariant curves coincide with the subspaces described above. Phasing conditions are pointed out, and natural regions are identified for formations on the torus where the variations of the distance and orientation between spacecraft are minimal.

5.2. Analytical definition of motion on a torus

In the previous, a natural solution for multiple spacecraft moving in a relative configuration within this dynamical system on a torus structure is presented. No manoeuvres are necessary and all spacecraft proceed naturally and along their paths and their motion is constrained within limits. The type of natural motion is an option for formation flying. However, some missions may requiring a tight pre-specified formation forcing specified configurations. The knowledge of the natural motion on the torus, in particular the frequencies and the plane orientation from the parametrization, is used to propose a modified torus motion in the in-plane coordinate frame as following. This section describes the details find suitable parameters to describe the motion of the spacecraft by a set of analytical equations. The knowledge of the natural motion on the torus, in particular the frequencies and the plane orientation from the parametrization, is used to propose a modified torus motion in the in-plane coordinate frame as following

$$
\begin{aligned}
x &= r \cdot cos(\omega_2 t + \phi_0) \\
y &= r \cdot sin(\omega_2 t + \phi_0) \\
z &= 0
\end{aligned}
\tag{9}
$$

where ω_2 is the rotating frequency introduced for the torus motion, describing the winding motion component. The radius of the desired formation is defined by r, whereas the phasing ϕ varies between the spacecraft in

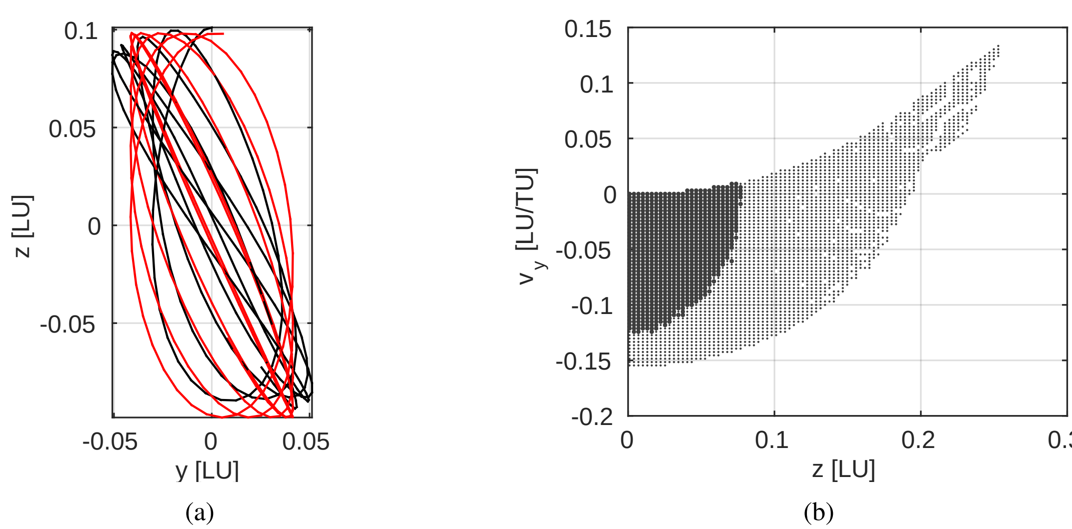


Figure 10: (a) Natural trajectory on torus versus modified motion described by Eq. 9. (b) Region where the description by analytical formulas of the quasi-periodic motion may be justified and produce accurate results.

the formation. Only the small highlighted region in Fig. 10b allows a approximation with the introduced equations in Eq. 9. For other quasi-periodic trajectories the previously introduced torus function u is suitable to approximate state vectors along the orbit.

6. Summary and Concluding Remarks

In the present study, natural quasi-periodic trajectories near libration point regions evolving on quasi-periodic invariant tori were studied. Quasi-periodic trajectories were computed using a numerical method, starting around periodic halo orbits in the CR3BP. A numerical method to obtain the torus frequency base is proposed. The parametrization in the two-dimensional angular phase space of the torus' surface is introduced which enables to further study the motion with tools known from the dynamical system theory. By introducing a certain phasing of spacecraft on the torus, trajectories that are particularly suitable for satellite formations emerge. The relative positions of each spacecraft in the formation are unaltered and the relative distances and orientation are closely bounded. The analysis of formations lead to the definition of an analytic torus motion. It is shown that only small amplitude quasi-periodic trajectories can be approximated by these analytical functions, for other tori the introduced parametrization is applicable. Future work might focus on implementing manoeuvres to find connecting orbits between different sized tori.

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