Uniqueness in the Freedericksz transition with weak anchoring

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Abstract

In this paper we consider a boundary value problem for a quasilinear pendulum equation with nonlinear boundary conditions that arises in a classical liquid crystals setup, the Freedericksz transition, which is the simplest opto-electronic switch, the result of competition between reorienting effects of an applied electric field and the anchoring to the bounding surfaces. A change of variables transforms the problem into the equation $x_{\tau\tau} = -f(x)$ for $\tau \in (-T, T)$, with boundary conditions $x_\tau = \pm\frac{\beta}{T} f(x)$ at $\tau = \mp T$, for a convex nonlinearity f. By analyzing an associated inviscid Burgers' equation, we prove uniqueness of monotone solutions in the original nonlinear boundary value problem.

This result has been for many years conjectured in the liquid crystals literature, e. g. in E. G. Virga, Variational Theories for Liquid Crystals,Chapman & Hall, London, 1994 and in I. W. Stewart, The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction, Taylor & Francis, London, 2003.

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1 Introduction

In the liquid crystalline phase of matter, molecular self-organisation produces orientational order where the rod or disc-like molecules preferentially align approximately parallel to each other [5]. The orientational order within the liquid crystal allows us to define an anisotropic axis, the axis of rotational symmetry. This anisotropic axis, the average molecular direction at that point in the material, is a macroscopic variable, called the director $\mathbf{n}$ (a unit vector), and may vary in space (and change with time) to create director distortion structures which increase the stored elastic energy of the system. The organic molecules that form the liquid crystal material may contain aromatic ring structures which allow a magnetisation to be induced when placed in an external magnetic field. The director can therefore be influenced by the application of such a magnetic field or, through electrostatic or steric interactions, may be influenced by the presence of a bounding surface.

The birefringence of a liquid crystal material and the ability to alter that birefringence through an applied field are factors that enable liquid crystals to be used in a display system. It is the versatility, portability and space saving aspects of these liquid crystal displays (LCDs) that mean that they are now ubiquitous in modern life.

The competing influences of the orienting effect at the bounding surfaces and the applied magnetic (or electric) field in the bulk of the liquid crystal which can produce a sharp transition between two alternative molecular configurations (the "on" and "off" states in a LCD). The *Freedericksz Transition* is a classic phenomenon in liquid crystal physics that demonstrates this competition between surface and bulk effects [6]. The facts that the vast majority of liquid crystal displays in the world today use a similar balance of surface and bulk effects, and that the Freedericksz transition is such a simple experimental system mean that it is still used and studied seventy years after it was discovered.

The Freedericksz cell consists of a layer of liquid crystal material sandwiched between two planar substrates (see Fig. 1). On the inner surfaces of the two substrates (the sides closest to the liquid crystal) polymer alignment layers have been deposited. These polymer alignment layers induce an orientational effect on the liquid crystal molecules close to the substrates by introducing an alignment direction along which the director would prefer to lie. With a low magnetic field the director is governed by alignment layers (see Fig. 1(a)) but as the magnetic field strength increases past a critical value a distorted state is energetically favoured and the director attempts to align with the magnetic field direction.

In terms of a theoretical model of the system it is usually assumed that the equilibrium director structure is determined by the minimisation of the total free energy of the liquid crystal. The free energy consists of contributions from the elasticity

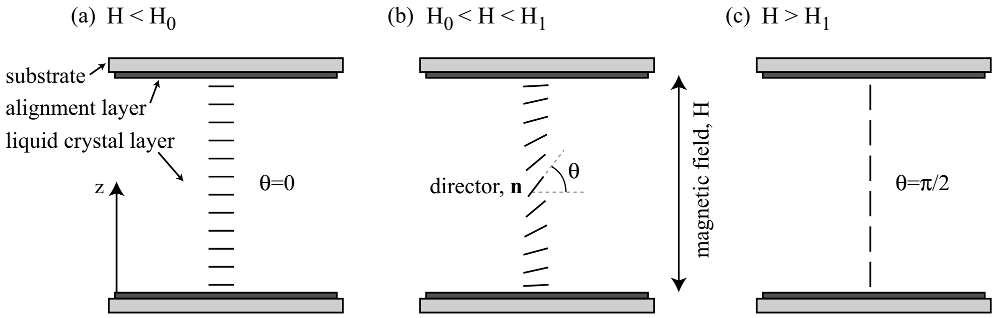

Fig. 1. The Freedericks cell: (a) With an applied magnetic field below the first critical value, $H < H_0$, the liquid crystal director is governed by the preferred alignment layer direction (the x direction). (b) With an applied field $H_0 < H < H_1$, the director attempts to reorient to align with the field direction. (c) With an field strength greater than the second critical value, $H > H_1$, the director reorients throughout the layer to align with the field direction.

of director distortions within the layer, the interaction between the director and the magnetic field and the interaction between the director and the surface alignment layers. This standard modelling approach is well documented and can be found in textbooks such as [5] and [15].

The concept of *weak anchoring* was theoretically introduced by Rapini and Papoular [11] who suggested a simple energy for the surface alignment/director interaction. In their model, the director is not rigidly anchored to lie in one direction at the surfaces but allowed to deviate from the preferred direction if, for instance, an alternative influence such as a magnetic field forces the director in a different direction. However, this deviation from the preferred alignment direction would produce an increase in energy and this would have to be balanced with the competing influence of the magnetic field. In this situation there are then two critical magnetic field strengths. The first $H = H_0$ is the field strength below which there is no distortion and the director aligns, throughout the layer, with the preferred alignment direction (see Fig. 1(a)). Above the second critical field strength $H = H_1$ the anchoring of the director at the surface breaks and the director fully aligns with the magnetic field (see Fig. 1(c)). Between these two critical field strengths the director attempts to align with the magnetic field direction in the bulk of the layer but close to the surfaces it attempts to align with alignment direction (see Fig. 1(b)).

The Freedericksz transition with weak anchoring has been considered by a number of authors in various geometries and the critical magnetic field strength for a transition from an undistorted state to a distorted state ($H = H_0$) was calculated by Nehring et al. [9]. In [17], the author proves that there is a (pitchfork) bifurcation from the constant solution $\theta = 0$ at $H = H_0$, a bifurcation from the constant solution $\theta = \pi/2$ at H_1 and conjectures the uniqueness of the nontrivial solution in (H_0, H_1). In this paper we prove that this conjecture is indeed correct.

The free energy of the system is discussed in, for instance, [15]. Our starting point is the free energy functional

$$F(\theta) = \int_{-d}^{d} \left[k(\theta)\theta_z^2 + h(\theta)\right] \, dz + \tau_0\omega[\sin^2(\theta(-d)) + \sin^2(\theta(d))], \tag{1.1}$$

where $\theta(z)$ is the angle between the director and the x-axis, z is the spatial coordinate perpendicular to the liquid crystal layer (see Fig. 1) and d is the thickness of the liquid crystal layer.

In eq. (1.1) we have used the shorthand notation,

$$k(\theta) = k_1 \cos^2\theta + k_3 \sin^2\theta \quad \text{and} \quad h(\theta) = -\frac{1}{2}\chi_a H^2 \sin^2\theta, \tag{1.2}$$

where $k_1,\ k_3$ are elastic constants, χ_a is the magnetic susceptibility of the liquid crystal material, H is the magnitude of the magnetic field strength, τ_0 is the anchoring strength and ω determines whether the alignment layer prefers for the director to be parallel or perpendicular to the alignment direction. The constants $d, \tau_0, \omega, k_1, k_3, \chi_a, H$ are all positive.

A simple computation shows that the Euler–Lagrange equation corresponding to this free energy functional is

$$2\sqrt{k(\theta)}\left(\theta_z\sqrt{k(\theta)}\right)_z - h'(\theta) = 0, \quad \text{for } z \in (-d, d) \tag{1.3}$$

with the (natural) boundary conditions

$$\begin{aligned} 2k(\theta)\theta' &= \tau_o\omega\sin 2\theta, && \text{at } z = -d, \\ 2k(\theta)\theta' &= -\tau_o\omega\sin 2\theta, && \text{at } z = d. \end{aligned} \tag{1.4}$$

In order to determine the uniqueness of a nontrivial solution to the Euler–Lagrange equations, and associated boundary conditions, we need to prove the following theorem, conjectured, for example, in [17]:

Theorem 1.1 *For any given values of the constants χ_a, k_1, k_3, τ_0, ω there is an interval of values of H, (H_0, H_1), such that for $H \le H_0$ and for $H \ge H_1$ the only solutions of (1.3) with the boundary conditions (1.4) and taking values in $[0, \pi/2]$ are the constant solutions $\theta = 0$ and $\theta = \pi/2$, while for all $H \in (H_0, H_1)$ there exists in addition a unique non-constant solution of these equations taking values in $(0, \pi/2)$.*

We rewrite (1.3)-(1.4) in a more convenient form. Let $\Theta := \int_0^{\pi/2} \sqrt{k(s)}ds$ and set $\sqrt{k(\theta)}\theta_z = \frac{2\Theta}{\pi}x_z$ by defining $x = \frac{\pi}{2\Theta}\int_0^\theta \sqrt{k(s)}ds$. This function is monotone and we denote its inverse by $G(x)$. Then (1.3) becomes

$$x_{zz} - \frac{\pi}{2\Theta}\frac{h'(G(x))}{2\sqrt{k(G(x))}} = 0 \quad \text{for} \quad z \in (-d,\, d); \tag{1.5}$$

and the boundary conditions (1.4) become

$$x_z = \tau_0\omega\frac{\sin(2G(x)}{2\sqrt{k(G(x))}}\frac{\pi}{2\Theta} \text{ at } z = -d, \quad x_z = -\tau_0\omega\frac{\sin(2G(x))}{2\sqrt{k(G(x))}}\frac{\pi}{2\Theta}, \text{ at } z = d. \tag{1.6}$$

Note that the rest points in terms of x are the same as in terms of θ. We shall need the following lemma.

Lemma 1.2 *Let* $f(x) := \dfrac{\sin(2G(x))}{\sqrt{k(G(x))}}$*. Then, for* $x \in (0, \pi/2)$*,* $f''(x) < 0$*.*

Proof. If $k_1 = k_3$, then $f(x) = \dfrac{1}{\sqrt{k_1}}\sin 2x$ and the result follows immediately. So let us assume that $k_1 \neq k_3$. Then, by computing derivatives we obtain,

$$f'(G^{-1}(\theta))\frac{\pi}{2\Theta}\sqrt{k(\theta)} = \frac{d}{d\theta}\frac{\sin 2\theta}{\sqrt{k(\theta)}} = \frac{2}{k_1 - k_3}\left(k(\theta)^2 - k_1k_3\right)k(\theta)^{-3/2}.$$

Thus,

$$f'(G^{-1}(\theta)) = \frac{2\Theta}{\pi}\frac{2}{k_1 - k_3}\left(1 - \frac{k_1k_3}{k(\theta)^2}\right).$$

Therefore

$$f''(G^{-1}(\theta))\sqrt{k(\theta)} = \frac{2\Theta}{\pi}\frac{2}{k_1 - k_3}\frac{d}{d\theta}\left(1 - \frac{k_1k_3}{k(\theta)^2}\right) = -\left(\frac{2\Theta}{\pi}\right)^2\frac{4k_1k_3}{k(\theta)^3}\sin 2\theta,$$

and the result follows. □

Using the function $f(x) := \dfrac{\sin(2G(x))}{\sqrt{k(G(x))}}$ defined in Lemma 1.2, letting

$$T := \frac{1}{2}Hd\sqrt{\frac{\pi\chi_a}{2\Theta}}, \qquad \beta := \frac{\tau_0\omega\pi d}{4\Theta},$$

and defining the change of variables $z \mapsto \tau := \dfrac{1}{2}H\sqrt{\dfrac{\pi\chi_a}{2\Theta}}\,z$, the system (1.5)–(1.6) becomes

$$x_{\tau\tau} = -f(x), \qquad \text{for } \tau \in (-T, T) \tag{1.7}$$

$$x_\tau = \frac{\beta}{T} f(x), \qquad \text{at } \tau = -T \tag{1.8}$$

$$x_\tau = -\frac{\beta}{T} f(x), \qquad \text{at } \tau = T \tag{1.9}$$

We will show that the reason for the uniqueness in theorem 1.1 is two-fold: we need both the concavity of the function $f(x)$ and the special form of equation (1.7) with the boundary conditions (1.8)-(1.9), both of which involve the same function $f(x)$. We need also to use information derived from the fact that our system has a first integral.

We now briefly describe the content of the paper. The basic idea of our approach is to perform a phase plane analysis of the first order system arising from (1.7)

$$\begin{cases} x_\tau = y \\ y_\tau = -f(x). \end{cases} \tag{1.10}$$

We seek a solution of (1.10) starting on the graph of the function $y = \frac{\beta}{T} f(x)$, ending on the graph of the function $y = -\frac{\beta}{T} f(x)$, and taking exactly $2T$ units of "time" τ to complete the trajectory. Due to the symmetry of the problem with respect to reflection on the x-axis, it is enough to obtain a solution starting at a point with $y = \frac{\beta}{T} f(x)$ and taking exactly T units of "time" τ to reach the x-axis (Fig. 2).

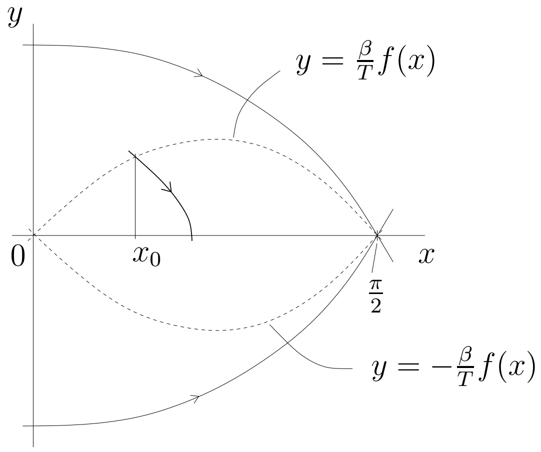

Fig. 2. The phase plane approach to (1.7), (1.8), (1.9). The case plotted is for sufficiently small β/T.

It is more convenient to work with the new reversed "time" $t = -\tau$ and, starting on the x-axis at $t = 0$, try to reach the final (in the new time) curve $\Lambda_T := \left\{(x, \frac{\beta}{T} f(x)) : x \in [0, \pi/2]\right\}$ at time $t = T$. For this we need to define the *isochronic* *set,* that is, the subset $\mathcal{H}_T$ of the phase plane which consists of those points that are attainable in T units of time, starting in the x-axis, and applying the flow generated by

$$\begin{cases} x' = -y \\ y' = f(x), \end{cases}$$

where $'$ denotes the derivative with respect to t. The main result will be proved once we conclude that the intersection between the final curve Λ_T and the isochronic set $\mathcal{H}_T$ is either empty or a singleton, and the set of values of the magnetic fields H for which the last case occurs is a bounded interval.

Clearly, for very large T the isochronic set $\mathcal{H}_T$ can be a quite complicated curve. However, we are only interested in solutions for which x is always in $[0, \pi/2]$. In these cases, the corresponding portion of $\mathcal{H}_T$, which we simply call the *isochrone,* is shown, by phase plane techniques, to be the graph of a function $x \mapsto h(x, T)$, $x \in [0, \pi/2]$. It is a remarkable fact that h satisfies a non-homogeneous Burgers' equation from which the necessary t evolution of (the relevant portion of) $\mathcal{H}_t$ can be obtained.

Phase plane analysis and results about isochrones will be the focus of Section 2. In Section 3 we seek to characterize the intersections between the $\mathcal{H}_T$ and Λ_T. This will be done by analyzing the monotonicity properties of the function $z(x) := h(x, T)/f(x)$ by the study of the evolution of a quantity related to $z'(x)$ along the characteristics of the Burgers' equation. This will complete the proof of Theorem 1.1.

2 Phase plane analysis and properties of the isochrone

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(0) = f(\pi/2) = 0$, positive and concave in $(0, \pi/2)$, with its unique maximum in this interval be located at $x = a$. Consider the following boundary value problem

$$\begin{cases} x' = -y \\ y' = f(x), \end{cases} \tag{2.1}$$

$$(x, y)(0) = (x_0, 0), \tag{2.2}$$

$$(x, y)(T) \in \Lambda_T. \tag{2.3}$$

We are interested in solutions (x, y) to (2.1)–(2.3) that lie in $[0, \pi/2] \times \mathbb{R}$. Since system (2.1) has the first integral

$$W(x, y) = \frac{1}{2}y^2 + F(x), \tag{2.4}$$

where $F(x) = \int_0^x f(s)ds$, the orbits in the phase plane of our system are thus level sets of the energy W, and we immediately conclude that the region of the phase plane of interest to our present study is the closed bounded set Ω whose boundary is made up of segments of the coordinate positive semi-axis and of the non-constant orbit $\gamma_{\pi/2}$ of (2.1) whose α-limit set is the equilibrium $\{(\pi/2, 0)\}$ (see Fig. 3).

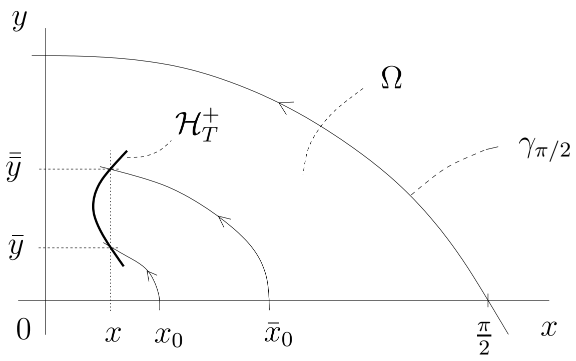

Fig. 3. The situation when $\mathcal{H}_T^+$ is not a graph of a function $x \mapsto h(x, T)$.

Denoting by φ^t the flow generated by (2.1), let

$$\mathcal{H}_T^+ := \Big\{(x, y) = \varphi^T(x_0, 0) : x_0 \in [0, \pi/2], \text{ and } \varphi^t(x_0, 0) \in \Omega, \forall\, t \in [0, T]\Big\}$$

be the subset of $\mathcal{H}_T$ corresponding to solutions that never leave Ω before time $t = T$. We start by showing the following:

Lemma 2.1 *There exists a smooth function $h : (0, \pi/2) \times [0, +\infty) \to \mathbb{R}$ such that, for each fixed $T \geq 0$, $\mathcal{H}_T^+$ is the graph of a map $x \mapsto h(x, T)$.*

Proof. For each $(t, x_0) \in \mathbb{R} \times (0, \pi/2)$, let us write $(x(t, x_0), y(t, x_0)) := \varphi^t(x_0, 0)$. By standard ODE theory, this solution exists for all real t, is unique and its dependence with respect to (t, x_0) is smooth. For each $T \geq 0$, define I_T as the subset of $(0, \pi/2)$ of all initial conditions x_0 for which $\forall_{t\in[0,T]},\ x(t, x_0) \in (0, \pi/2)$. Suppose that we have proved that,

$$\forall_{x_0 \in I_T}, \quad \frac{\partial x}{\partial x_0}(T, x_0) > 0. \tag{2.5}$$

Then, the map $x_0 \mapsto x(T, x_0)$ that takes I_T onto $(0, \pi/2)$ has a smooth inverse $x \mapsto x_0(T, x)$. In this situation, we can define a smooth function $h : (0, \pi/2) \times [0, +\infty) \to \mathbb{R}$ by

$$h(x, T) := y(T, x_0(T, x))\,.$$

Suppose that $\bar{y} = h(\bar{x}, T)$. Then, by definition, $\varphi^T(x_0(T, \bar{x}), 0) = (\bar{x}, \bar{y})$, thus proving that $(\bar{x}, \bar{y}) \in \mathcal{H}_T^+$. Conversely, let $(\bar{x}, \bar{y}) \in \mathcal{H}_T^+$. Then, $(\bar{x}, \bar{y}) = \varphi^T(\bar{x}_0, 0)$ for some $\bar{x}_0 \in I_T$. Therefore, by definition, $\bar{x}_0 = x_0(T, \bar{x})$ and $\bar{y} = y(T, x_0(T, \overline{x}))$, thus proving that $\bar{y} = h(\bar{x}, T)$ and our claim is proved.

Therefore, it remains to prove (2.5). The direct tackling of this question runs into problems due to the fact that the linear variational equation solved by $\frac{\partial x}{\partial x_0}$ involves differences whose signs are hard to handle. To overcome this difficulty, let us introduce the new variable $\sigma(t, x_0) := \frac{x(t,x_0)}{x_0}$, for each $t \in [0, T]$ and $x_0 \in I_T$. Note that $\sigma(t, x_0) \in [0, 1]$. Then, by (2.1)-(2.2),

$$\frac{\partial \sigma(t, x_0)}{\partial t} = -\frac{y(t, x_0)}{x_0}, \qquad \sigma(0, x_0) = 1.$$

By using the non-negativity of $y(t, x_0)$ and the invariance of (2.4) for each fixed x_0, we obtain after computing derivatives,

$$\begin{aligned} y\frac{\partial}{\partial t}\frac{\partial \sigma}{\partial x_0} &= -y\frac{\partial}{\partial x_0}\frac{\sqrt{2F(x_0) - 2F(\sigma(t, x_0)x_0)}}{x_0} \\ &= \frac{G(x_0) - G(\sigma x_0)}{x_0^2} + f(\sigma x_0)\frac{\partial \sigma}{\partial x_0}, \end{aligned} \tag{2.6}$$

where, $G(u) := 2F(u) - uf(u)$. Since $G''(u) = -uf''(u) > 0$ for $u \in (0, \pi/2)$ by Lemma 1.2 and $G'(0) = 0$, it follows that $G'(u) > 0$, for all $u \in (0, \pi/2)$ and therefore, for $t \in (0, T]$,

$$\frac{G(x_0) - G(\sigma x_0)}{x_0^2} > 0. \tag{2.7}$$

For fixed x_0, let us write $u(t) := \frac{\partial \sigma}{\partial x_0}(t, x_0)$. Then, obviously, $u(0) = 0$. The differential equation (2.6) together with this initial condition defines a Cauchy problem which is singular at $t = 0$, since $y(0, x_0) = 0$. However, smooth dependence of $(x(t, x_0), y(t, x_0))$ with respect to (t, x_0) allows us to compute also $u'(0) = -\frac{\partial y}{\partial x_0}(0, x_0) = 0$, and, therefore, $u''(0) = \lim_{t\downarrow 0} \frac{u'(t)}{t} = \lim_{t\downarrow 0} \frac{2u(t)}{t^2}$. But since

$$\lim_{t\downarrow 0} \frac{f(\sigma(t)x_0)u(t)}{ty(t)} = f(x_0) \lim_{t\downarrow 0} \frac{t}{y(t)} \lim_{t\downarrow 0} \frac{u(t)}{t^2} = \lim_{t\downarrow 0} \frac{u(t)}{t^2} = \frac{u''(0)}{2},$$

by dividing both terms of the right-hand side of (2.6) by $ty(t)$ and taking limits as $t \downarrow 0$, we obtain

$$u''(0) = \frac{2}{x_0^2} \lim_{t\downarrow 0} \frac{G(x_0) - G(\sigma(t)x_0)}{ty(t)} = \frac{G'(x_0)}{x_0^2} > 0.$$

Therefore, there is $\tau \in (0, T]$ such that if $0 < t < \tau$ then $u(t) > 0$. Suppose now that there is some $t_1 \in [\tau, T]$ such that $u(t) > 0$, for all $t \in (0, t_1)$ and $u(t_1) = 0$. Then, again by (2.6), $u'(t_1) > 0$ which is absurd. We have proved that, for all $t \in (0, T]$, $u(t) > 0$.

But since $\partial x/\partial x_0 = x_0 u + \sigma$, this also proves that $\partial x(t, x_0)/\partial x_0 > 0$, for all $(t, x_0) \in (0, T] \times I_T$, and the proof is complete. □

The following two lemmas allow us to reduce our bifurcation problem to the study of a Cauchy problem defined by a well-known first order PDE and use characteristics to derive equations which will be crucial in the uniqueness result of the next section.

Lemma 2.2 *Let $(x,t) \mapsto h(x,t)$ be the function whose graph, for fixed t is the t-isochrone $\mathcal{H}_t^+$. Then, h satisfies the non-homogeneous inviscid Burgers' equation*

$$h_t - hh_x = f(x). \tag{2.8}$$

Proof. By definition of the isochrone, $y(t) - h(x(t),t) = 0$ holds for all values of t for which the orbit $(x(t),y(t))$ is in Ω. Applying the chain rule to this identit and using (2.1), we have

$$f(x) = y' = \frac{d}{dt}h(x,t) = h_t + h_x x' = h_t - hh_x,$$

as we wanted to prove. □

Lemma 2.3 *Let h be a solution of the non-homogeneous Burgers' equation (2.8). Then, along characteristics of (2.8) the following hold true:*

$$\frac{d}{dt}h = f(x) \tag{2.9}$$
$$\frac{d}{dt}h_t = h_x h_t \tag{2.10}$$
$$\frac{d}{dt}h_x = h_x^2 + f'(x) \tag{2.11}$$

Proof. By Lemma 2.1, h is smooth, and by Lemma 2.2, it satisfies equation (2.8). By the method of characteristics applied to (2.8), we immediately obtain the characteristic equations

$$\frac{d}{dt}x = -h \quad \frac{d}{dt}h = f(x).$$

Differentiating (2.8) with respect to t and to x, and using the characteristic equation for x we obtain equations (2.10) and (2.11), respectively. □

Remark 1 *It is interesting to observe the following: differentiating (2.8) twice with respect to x and using the characteristic equation for x we obtain*

$$\frac{d}{dt}h_{xx} = 3h_x h_{xx} + f''(x).$$

Integrating this equation and using the concavity of f proved in Lemma 1.2 immediately gives the concavity of the isochrone.

3 Application to the weak Freedericksz transition

Lemma 3.1 *For every $T > 0$, the intersection of $\mathcal{H}_T^+$ and Λ_T is either empty or a single point.*

Proof. Now we prove that for each $T > 0$ and $\alpha > 0$, the intersection between the graphs of $h(\cdot, T)$ and αf, in the interval $(0, \pi/2)$ is either the empty set or a single point. Define, for each $x \in (0, \pi/2)$

$$z(x) := \frac{h(x,T)}{f(x)} \,.$$

For the value of T that we have fixed and for each particular x^*, $z(x^*)$ gives the unique value of α for which the above curves intersect at $x = x^*$. This function is well defined since $f(x) > 0$, for $x \in (0, \pi/2)$. Now, if our claim were false, $z(x)$ would not be monotone. We now prove that this is impossible and, in fact, that z is strictly decreasing. By taking derivatives, at a particular $\bar{x} \in (0, \pi/2)$, we have

$$z'(\bar{x}) = \frac{h_x(\bar{x},T)f(\bar{x}) - h(\bar{x},T)f'(\bar{x})}{f(\bar{x})^2} \,.$$

Consider the projected characteristic $t \mapsto x(t)$ such that $x(T) = \bar{x}$. Let us define, for $t \in [0, T]$,

$$H(t) := h_x(x(t),t)f(x(t)) - h(x(t),t)f'(x(t)) \,.$$

Obviously, $H(0) = 0$, since $h(x, 0) \equiv 0$. Then, by taking into account the evolution of h and h_x along the characteristics of (2.8), given by equations (2.9) and (2.11), we conclude that

$$\begin{aligned}\frac{d}{dt}H &= h_x\frac{d}{dt}f + f\frac{d}{dt}h_x - h\frac{d}{dt}f' - f'\frac{d}{dt}h \\ &= -h_x h f' + f(f' + h_x^2) + h^2 f'' - f'f \\ &= h_x(h_x f - hf') + h^2 f''\end{aligned}$$

Then H satisfies the following linear Cauchy problem:

$$\frac{d}{dt}H - h_x(x(t),t)H = h^2 f'' \,, \qquad H(0) = 0 \,.$$

Keeping the notation h, h_x for their evaluations at $(x(t), t)$, by the variation of constants formula, we conclude that

$$H(T) = \int_0^T e^{-\int_t^T h_x} h^2 f''\, dt < 0 \,.$$

But this proves that

$$h_x(\bar{x},T)f(\bar{x}) - h(\bar{x},T)f'(\bar{x}) < 0,$$

and therefore, $z'(\bar{x}) < 0$, concluding our proof. □

Proof (of Theorem 1.1). Let χ_a, k_1, k_3, τ_0 and ω be fixed positive constants. Then $\beta = \frac{\tau_0 \omega \pi d}{4\Theta}$ is fixed and $T \propto H$. For all values of H, the constant functions $\theta = 0$ and $\theta = \pi/2$ are solutions to (1.3)-(1.4). By Lemma 3.1 we have that, for each value of H, the problem (1.3)-(1.4) has at most one further (nonconstant) solution taking values in $[0, \pi/2]$.

It remains to be proved that the set of values of H for which such a nonconstant solution exists is a bounded interval. This can be achieved by applying the relevant results in [17]. However, since the argument is simple and rather brief once the behaviour of h has been studied, we choose, for completeness' sake, to include it here.

We start by noting that neither in the case $T \to 0$ nor if $T \to +\infty$ does a solution to $Th(x,T) = \beta f(x)$ exist in $(0, \pi/2)$. This is so by continuity and because the left hand-side is zero if $T = 0$ whereas it is pointwise convergent to infinity when $T \to +\infty$. Of course the right hand side is positive, bounded, and independent of T. So, there exists $T_0 < T_1$ such that, for either $T < T_0$ or $T > T_1$ it is true that $\mathcal{H}_T^+ \cap \Lambda_T = \varnothing$. Suppose the set of values of T for which this intersection is not empty is not an interval. Then, there would exist $T' < T < T''$ such that $\mathcal{H}_T^+ \cap \Lambda_T = \varnothing$ and the intersections for the values of T' and T'' are non-empty. But this is impossible by continuity, and by the properties of h and f studied previously: if the intersection is empty at $T > T'$, then, for each $x \in (0, \pi/2)$, $Th(x,T)$ must be above $\beta f(x)$, and thus for all $T'' > T$ is must remain above.

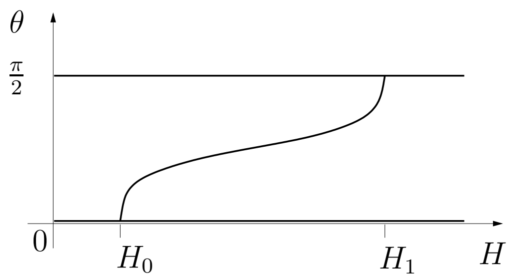

Fig. 4. Bifurcation diagram for the solutions of (1.2)-(1.4) with θ taking values in $\left[0, \frac{\pi}{2}\right]$ (Freedericksz transition with weak anchoring).

So, reverting to the original variables θ and H, we can draw the bifurcation diagram of Figure 4, and thus the statement of Theorem 1.1 has been proved. □

4 Remarks

One could prove the same result using the time-map methods of Schaaf [12] and working with elliptic integrals [10]. The method we have chosen here seems to us more transparent. The connection between isochrones in second order ODEs and the Burgers' equation certainly can be exploited in other contexts. For example, it seems to us possible, by using ray tracing methods, to recover the classical result of Smoller and Wasserman [13]. Note that in the application here the shock propagated away from the area of interest; this is not the case in the Smoller and Wasserman context and multivalued "solutions" of Burgers' equation have to be considered then.

The connection between the Burgers' equation with source, (2.8) satisfied by $h(x,t)$ and the Hamilton-Jacobi equation associated with the Hamiltonian (2.4) has been observed by J. Robbins (private communication) and by an anonymous referee. Indeed, if $S(x,t)$ satisfies the Hamilton-Jacobi equation

$$S_t = W(S_x, x)$$

with the initial condition $S(x,0) = \text{const}$, differentiating the Hamilton–Jacobi with respect to x shows that $h(x,t) = S_x(x,t)$. An (implicit) interpretation of S_x as an isochrone curve can be found in, for example, [7, Theorem 13.10], but to the best of our knowledge, the present paper is the first application of these ideas to multiplicity questions in boundary value problems.

It is worth commenting on the importance of uniqueness of solutions in such liquid crystal systems. The Freedericksz transition is often used as both a simple test experiment for new liquid crystals or alignment layers. It is also used extensively in the measurement of certain material parameters (k_1, k_3 and τ_0) and as such the comparison between experimental and theoretical results is crucial to the development of new liquid crystal materials. In particular the elastic constant k_3 is measured using information from the non-trivial solution. Because the exact form of this solution cannot be directly experimentally measured it is essential that we have confidence that the observed effect is the same as the theoretical solution it is being compared to. This result therefore gives complete reassurance that the experimentally measured solution can be compared with the theoretical model and that the measurements of k_3 are unambiguous.

Possible extensions of this result are numerous. There are three classical geometries for the Freedericksz transition, the splay (considered here), bend and twist transitions [5]. Relatively simple symmetry operations (in the case of the bend transition) or a simplification of our result here (twist transition) mean that this uniqueness result is true for all three cases. It would be interesting to extend this analysis to more complicated situations such as the transition of a twisted nematic cell with weak anchoring, where there are two couple equations and two types of weak anchoring to consider.

It should also be possible to extend the present analysis to more complicate forms of the surface energy [8,14,18,1–3]. The physically correct form of this energy term has been disputed for some time and it would be very interesting to see if this type of uniqueness proof could be applied to systems with other forms of surface energy. Again this is important in providing confidence in various experimental measurements. Other approximate surface energies, for particularly complicated substrates, have recently been proposed that make the system bistable [4,16]. An investigation of uniqueness in these systems would be extremely interesting as it could have applications to a number of bistable liquid crystal display technologies.

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