Stability of Hybrid Stochastic Retarded Systems

Lirong Huang, *Student Member, IEEE*, Xuerong Mao, and Feiqi Deng

***Abstract*—In the past few years, hybrid stochastic retarded systems (also known as stochastic retarded systems with Markovian switching), including hybrid stochastic delay systems, have been intensively studied. Among the key results, Mao *et al.* proposed the Razumikhin-type theorem on exponential stability of stochastic functional differential equations with Markovian switching and its application to hybrid stochastic delay interval systems. However, the importance of general asymptotic stability has not been considered. This paper is to study Razumikhin-type theorems on general p-th moment asymptotic stability of hybrid stochastic retarded systems. The proposed theorems apply to complex systems including some cases when the existing results cannot be used.**

***Index Terms*—Asymptotic stability, Markov chain, Razumikhin-type theorems, retarded systems, stochastic systems.**

I. Introduction

HYBRID systems are employed to model many practical systems where abrupt changes in system structure and parameters may occur (see, e.g., [4] and [8]). An area of particular interest has been the analysis of stability of hybrid systems (see, e.g., [1], [10], [18], and [19]). Recently, hybrid stochastic retarded systems (HSRSs), including hybrid stochastic delay systems (HSDSs), driven by continuous-time Markovian chains have been widely used since stochastic modeling plays an important role in many branches of science and engineering. Consequently, the stability analysis of HSRSs and HSDSs has been studied by many works, see, e.g., [12]–[16], [22]. Mao *et al.* [13] established a number of exponential stability criteria for stochastic differential delay equations with Markovian switching that apply for systems with constant delay and obtained exponential and asymptotic stability criteria for stochastic differential delay equations with Markovian switching [14], which are useful for systems with sufficient small constant delay. Mao [15] studied the exponential stability of linear stochastic delay interval systems with Markovian switching while Yue *et al.* [22] considered that of a class of stochastic systems with time delay, nonlinearity, and Markovian switching. These delay-dependent results use linear matrix inequality (LMI) techniques with Lyapunov functionals and require the time delay to be a constant or a differentiable function that varies slowly, or say, the derivative of which is bounded by a constant number less than one. To remove the restriction in [15] and allow the time delay to be a bounded variable only, Mao *et al.* [12], [16] proposed and employed the Razumikhin-type theorem on exponential stability.

The Razumikhin method is developed to cope with the difficulty arisen from the large, quickly varying, and nondifferentiable time delays. However, the importance of general asymptotic stability has not been considered. In many cases, the exponential stability of the equilibrium of the system is not necessary and to stabilize the system exponentially fast is economically, and sometimes practically, unfeasible. In fact, the criteria for exponential stability of HSRSs implicitly require the diffusion operator associated with the underlying HSRSs of the Lyapunov function along a solution of the system to be negative and have the same order as that of the function itself at some instants, which is not satisfied for many nonlinear systems. In these cases, the existing results (see [12]–[16], and [22]) cannot be applied. For example, consider the following scalar stochastic delay system is driven by a right-continuous Markov chain $r(t)$ that is independent of the one-dimensional (1-D) standard Brownian motion $B(t)$ and takes values in $S=\{1,2\}$ with generator

$$\Gamma=(\gamma_{ij})_{2\times 2}=\begin{pmatrix}-\gamma_{11} & \gamma_{12}\\ \gamma_{21} & -\gamma_{22}\end{pmatrix}$$
$$\gamma_{11}=\gamma_{12}>0,\quad \gamma_{21}=\gamma_{22}>0.$$

This HSDS is described as the following stochastic delay equation with Markovian switching:

$$\mathrm{d}x(t)=-\left[\frac{1}{2}x(t)+\zeta(x(t),r(t))\right]\mathrm{d}t +\sigma(x(t),x(t-h(t)),r(t))\mathrm{d}B(t) \quad (1)$$

on $t\geq 0$, where $h:R_+\to[-\tau,0)$ are Borel measurable and the nonlinear term $\zeta(x(t),r(t))$ and the diffusion term $\sigma(x(t),x(t-h(t)),i)$ are given as follows:

$$\zeta(x(t),i)=\begin{cases}\frac{1}{6}x^3(t), & i=1\\ \frac{1}{10}x(t)\sqrt{|x(t)|}, & i=2\end{cases}$$
$$\sigma(x(t),x(t-h(t)),i)=\begin{cases}\frac{\sqrt{2}}{4}x^2(t)+\frac{\sqrt{2}}{2}x(t-h(t)), & i=1\\ x(t-h(t)), & i=2\end{cases}$$

for all $t\geq 0$. We encounter a problem when we attempt to apply the existing results to analyze the stability of the solution to (1). To see this problem, let us set $V(x(t),t,r(t))=x^2(t)$ and calculate

$$\mathcal{L}V(x_t,t,i)\leq\begin{cases}-x^2(t)-\frac{1}{12}x^4(t)+x^2(t-h(t)), & i=1\\ -x^2(t)-\frac{1}{5}x^2(t)\sqrt{|x(t)|}+x^2(t-h(t)), & i=2\end{cases} \quad (2)$$

on $t\geq 0$, where operator $\mathcal{L}$ is defined in (4) or (25) (see, e.g., [12]). The higher order (higher than the order of $V(x)$) of polynomial $-2x(t)\zeta(x(t),r(t))$, time delay, and $\lambda_1=\lambda_2=1$

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L. Huang and X. Mao are with the Department of Statistics and Modeling Science, University of Strathclyde, Glasgow G1 1XH, U.K. (e-mail: lirong@stams.strath.ac.uk).

F. Deng is with the College of Automation Science and Engineering, South China University of Technology, Guangzhou 510640, China.

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(cf. Theorem 4.2, [12]) all appear on the right-hand side of inequality (2) and this prevents the exiting results from being used. However, the solution to (1) may be asymptotically stable in mean-square sense though, due to $\lambda_1 = \lambda_2 = 1$, it might be not exponentially stable (see [9] and [20]). This paper is to study the general asymptotic stability of HSRSs with Razumikhin-type arguments, which is a generalization of the result on exponential stability obtained in [12].

NOTATION

Throughout the paper, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. If x, y are real numbers, then $x \vee y$ denotes the maximum of x and y, and $x \wedge y$ stands for the minimum of x and y. Let $|\cdot|$ denote the Euclidean norm in R^n. Let $\tau \geq 0$ and $C([-\tau, 0]; R^n)$ denote the family of all continuous R^n-valued functions φ on $[-\tau, 0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. For $p > 0$ and $t \geq 0$, denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; R^n)$-valued random processes $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0} E|\phi(\theta)|^p < \infty$.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j : r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is known that almost all sample paths of $r(t)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $R_+ := [0, \infty)$.

Let us consider an n-dimensional HSRS

$$\mathrm{d}x(t) = f(x_t, t, r(t))\mathrm{d}t + g(x_t, t, r(t))\mathrm{d}B(t) \quad (3)$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -r \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. Moreover

$$f : C([-\tau, 0]; R^n) \times R_+ \times S \to R^n$$
$$g : C([-\tau, 0]; R^n) \times R_+ \times S \to R^{n\times m}$$

are measurable functions with $f(0, t, i) \equiv 0$ and $g(0, t, i) \equiv 0$ for all $t \geq 0$. Thus, (3) admits a trivial solution $x(t; 0) \equiv 0$. Here, $x_t = \{x(t + \theta) : -r \leq \theta \leq 0\}$ is regarded as a $C([-r, 0]; R^n)$-valued stochastic process. We assume that f and g are sufficiently smooth so that (3) only has continuous solutions on $t \geq 0$, any version of which is denoted by $x(t; x_0)$ or $x(t; \xi)$ in this paper. For example, f and g satisfy the local Lipschitz condition and the linear growth condition, see [12]–[16] and references therein.

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator associated with system (3), $\mathcal{L}$, from $C([-\tau, 0]; R^n) \times R_+ \times S$ to R by

$$\begin{aligned} \mathcal{L}V(x_t, t, i) = & V_t(x, t, i) + V_x(x, t, i)f(x_t, t, i) \\ & + \frac{1}{2} trace\left[g^T(x_t, t, i)V_{xx}(x, t, i)g(x_t, t, i)\right] \\ & + \sum_{j=1}^{N} \gamma_{ij}V(x, t, j) \end{aligned} \quad (4)$$

where

$$V_t(x, t, i) = \frac{\partial V(x, t, i)}{\partial t}$$
$$V_x(x, t, i) = \left(\frac{\partial V(x, t, i)}{\partial x_1}, \ldots, \frac{\partial V(x, t, i)}{\partial x_n}\right)$$
$$V_{xx}(x, t, i) = \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

In this paper, we let $\mathcal{K}$ denote the class of continuous strictly increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathcal{K}_\infty$ denote the class of functions $\mu \in \mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. Functions in $\mathcal{K}$ and $\mathcal{K}_\infty$ are called class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions, respectively. If $\mu \in \mathcal{K}$, its inverse function is denoted by μ^{-1} with domain $[0, \mu(\infty))$. We also denote by $\mu \in V\mathcal{K}$ and $\mu \in C\mathcal{K}$ if $\mu \in \mathcal{K}$ and μ is convex and concave, respectively.

The purpose of this paper is to further develop the Razumikhin-type theorems on stability of HSRSs initiated by [12]. Let us begin with the following definitions (see, e.g., [2], [6], [7], and [11])

1) Definition 2.1: The trivial solution of (3) or, simply, (3) is said to be:

1) stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $k > 0$, there exists $\delta = \delta(\varepsilon, k) > 0$ such that
$$P\{|x(t; \xi)| < k, \text{ for all } t \geq 0\} \geq 1 - \varepsilon$$
whenever $\|\xi\| < \delta$;
2) stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0, 1)$, there exists $\delta_0 = \delta_0(\varepsilon)$ such that
$$P\{\lim_{t\to\infty} x(t; \xi) = 0\} \geq 1 - \varepsilon$$
whenever $\|\xi\| < \delta_0$;
3) globally stochastically asymptotically stable if it is stochastically stable and, moreover, for all $\xi \in C^b_{\mathcal{F}_0}([-r, 0]; R^n)$,
$$P\{\lim_{t\to\infty} x(t; \xi) = 0\} = 1.$$

2) Definition 2.2: The trivial solution of (3) or, simply, (3) is said to be:

1) pth ($p > 0$) moment stable if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$E|x(t;\xi)|^p \le \varepsilon, \quad \forall\, t \ge 0$$

whenever $\|\xi\|^p < \delta_0$;

2) pth moment asymptotically stable if it is p-th moment stable and, moreover, for every $\varepsilon > 0$, there exist $\delta_0 = \delta_0(\varepsilon)$ and $T = T(\varepsilon)$ such that

$$E|x(t;\xi)|^p \le \varepsilon, \quad \forall\, t \ge T$$

whenever $\|\xi\|^p < \delta_0$;

3) globally pth moment asymptotically stable if it is pth moment stable and, moreover, for all $\xi \in C^b_{\mathcal{F}_0}([-r,0];R^n)$,

$$\lim_{t\to\infty} E|x(t;\xi)|^p = 0.$$

According to the above definitions, it is easy to verify that (global) pth moment (asymptotic) stability implies (global) stochastic (asymptotic) stability.

II. Asymptotic Stability of HSRSs

As the main results of this paper, we present the Razumikhin-type theorems on general stability of HSRSs (3) as follows. ■

1) Theorem 3.1: Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in C\mathcal{K}_\infty$, and $w : R^n \times R_+ \times S \to R_+$ be a nonnegative continuous function with $w(x,t,i) > 0$ if $E|x(t)|^p > 0$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that

$$u(|x|^p) \le V(x,t,i) \le v(|x|^p), \quad \forall\,(x,t) \in R^n \times [-\tau,\infty) \tag{5}$$

and, moreover, for all $1 \le i \le N$

$$E\mathcal{L}V(\phi,t,i) \le -w(\phi(0),t,i) \tag{6}$$

for all $t \ge 0$ and those $\phi \in L^p_{\mathcal{F}_t}([-\tau,0];R^n)$ satisfying

$$\min_{1\le k\le N} EV(\phi(\theta),t+\theta,k) < q(\max_{1\le k\le N} EV(\phi(0),t,k),i) \tag{7}$$

on $-\tau \le \theta \le 0$, where $q : R \times S \to R$ is a continuous nondecreasing function with respect to $s \in R$ for all $s \ge 0$ and $1 \le i \le N$. Moreover, $q(s,i) > s$ for all $s > 0$ and $1 \le i \le N$. Then, the trivial solution of HSRS (3) is globally pth moment asymptotically stable.

Proof: Fix the initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$ and extend $r(t)$ to $[-\tau,0)$ by setting $r(t) = r(0)$ for all $t \in [-\tau,0)$. Noting that $x(t;\xi)$ is continuous and $r(t)$ is right continuous for all $t \ge 0$, we see that $EV(x(t),t,r(t))$ is right continuous on $t \ge -\tau$. Define

$$U(t) = \sup_{-\tau\le\theta\le 0} EV(x(t+\theta),t+\theta,r(t+\theta)) \quad \forall\, t \ge 0. \tag{8}$$

We claim that

$$D_+U(t) = \limsup_{h\to 0+} \frac{U(t+h)-U(t)}{h} \le 0 \quad \forall\, t \ge 0. \tag{9}$$

To show inequality (9), for each $t \ge 0$ (fix t for the moment), we define

$$\bar\theta = \max\{\theta \in [-\tau,0] \mid EV(x(t+\theta),t+\theta,r(t+\theta)) = U(t)\} \tag{10}$$

Obviously, $\bar\theta$ is either less than 0 or equal to 0.

If $\bar\theta < 0$, then

$$\begin{aligned} EV(x(t+\theta),t+\theta,r(t+\theta)) &< EV(x(t+\bar\theta),t+\bar\theta,r(t+\bar\theta)) \\ &= U(t) \qquad \forall\,\theta \in (\bar\theta,0]. \end{aligned} \tag{11}$$

It follows from the right continuity of $EV(x(t),t,r(t))$ that for every sufficiently small $h > 0$

$$EV(x(t+h),t+h,r(t+h)) \le U(t)$$

hence

$$U(t+h) \le U(t) \text{ and } D_+U(t) \le 0.$$

If $\bar\theta = 0$, then

$$\begin{aligned} EV(x(t+\theta),t+\theta,r(t+\theta)) &\le EV(x(t),t,r(t)) \\ &= U(t) \qquad \forall\,\theta \in [-\tau,0]. \end{aligned} \tag{12}$$

Note that either $EV(x(t),t,r(t)) = 0$ or $EV(x(t),t,r(t)) > 0$. In the former case, i.e., $EV(x(t),t,r(t)) = 0$, inequalities (12) and (5) yield that $x(t+\theta) = 0$ a.s. for all $-\tau \le \theta \le 0$. Recalling that $f(0,t,i) = 0$ and $g(0,t,i) = 0$, we see $x(t+h) = 0$ for all $h > 0$, hence $U(t+h) = 0$ and $D_+U(t) = 0$. In the other case when $EV(x(t),t,r(t)) > 0$, the above inequality (12) implies

$$\begin{aligned} EV(x(t+\theta),t+\theta,r(t+\theta)) &\le EV(x(t),t,r(t)) \\ &< q(EV(x(t),t,r(t)),r(t)) \\ &\qquad \forall\,\theta \in [-\tau,0]. \end{aligned} \tag{13}$$

Consequently, inequality (7) holds, that is,

$$\min_{1\le k\le N} EV(x(t+\theta),t+\theta,k) < q(\max_{1\le k\le N} EV(x(t),t,k),r(t))$$

on all $-\tau \le \theta \le 0$. Moreover, by condition (5) and Jensen's inequality, $EV(x(t),t,r(t)) > 0$ yields $E|x(t)|^p > 0$. Thus, by condition (6), we have

$$E\mathcal{L}V(x_t,t,i) < 0 \tag{14}$$

for all $1 \le i \le N$. By the right continuity of the processes concerned, we see that, for all $h > 0$ sufficiently small, we have

$$E\mathcal{L}V(x_s,s,i) \le 0 \quad \forall\, t \le s \le t+h, 1 \le i \le N.$$

By a formula derived from generalized Itô's lemma (see [17] and [12]) and Fubini's theorem, we observe

$$\begin{aligned} &EV(x(t+h), t+h, r(t+h)) \\ &\quad = EV(x(t), t, r(t)) + \int_t^{t+h} E\mathcal{L}V(x_s, s, r(s))\mathrm{d}s \\ &\quad \le EV(x(t), t, r(t)). \end{aligned} \tag{15}$$

Hence, we have

$$U(t+h) = U(t) = EV(x(t), t, r(t)) \quad D_+U(t) = 0.$$

Inequality (9) has been proved. It follows immediately that

$$U(t) \le U(0), \quad \forall\, t \ge 0. \tag{16}$$

Together with the definition of $U(t)$, condition (5) and Jensen's inequality, the above inequality (16) yields

$$E|x(t)|^p \le u^{-1}(v(\|\xi\|^p)), \quad \forall\, t \ge 0. \tag{17}$$

So, for any $\epsilon > 0$, we can find $\delta(\epsilon) = v^{-1}(u(\epsilon))$ such that

$$E|x(t)|^p \le \epsilon, \quad \forall\, t \ge 0$$

whenever $\|\xi\|^p < \delta(\epsilon)$. The pth moment stability is proved.

Now, we proceed to show the convergence of $E|x(t)|^p \to 0$ as $t \to \infty$. Fix any initial data $\xi \in C^b_{\mathcal{F}_0}([-r, 0]; R^n)$. Let $\delta > 0$ and $\varepsilon > 0$ be such that $\|\xi\|^p < \delta$ and $U(0) < v(\delta) = u(\varepsilon)$. So, by inequalities (16) and (17), we have $EV(x(t), t, r(t)) < v(\delta)$ and $E|x(t)|^p < \varepsilon$ for all $t \ge 0$. Suppose $0 < \beta \le \varepsilon$ is arbitrary. We need to show there is a number $T = T(\beta, \delta)$ such that $E|x(t)|^p \le \beta$ for all $t \ge T$. This will be true by condition (5) and Jensen's inequality if we show that $EV(x(t), t, r(t)) \le u(\beta)$ for all $t \ge T$.

From the property of function $q(s, i)$, there is a positive real number $a > 0$ such that $q(s, i) - s > a$ for all $u(\beta) \le s \le v(\delta)$ and $1 \le i \le N$. Let J be the minimal nonnegative integer such that $u(\beta) + Ja \ge v(\delta)$, and $\gamma = \inf\{w(x(t), t, i) : \beta \le E|x(t)|^p \le \varepsilon, t \ge 0, 1 \le i \le N\}$. So $\gamma > 0$, since $w(x(t), t, i) \ge 0$ with $w(x(t), t, i) > 0$ if $E|x(t)|^p > 0$. Let $\tilde{\tau} = \tau \vee v(\delta)/\gamma$ and $T_j = j\tilde{\tau}$ with $j = 0, 1, \ldots, J$.

We claim that $EV(x(t), t, r(t)) \le u(\beta)$ for all $t \ge T_J$. First we show that $EV(x(t), t, r(t)) \le u(\beta) + (J-1)a$ for all $t \ge T_1$. Let $t_1 = \inf\{t \ge T_0 : EV(x(t), t, r(t)) \le u(\beta) + (J-1)a\}$. If $t_1 > T_1$, then, $\forall\, T_0 \le t \le T_1$, we have

$$\begin{aligned} &q(\max_{1\le k\le N} EV(x(t), t, k), r(t)) \\ &\quad \ge q(EV(x(t), t, r(t)), r(t)) \\ &\quad > EV(x(t), t, r(t)) + a \\ &\quad \ge u(\beta) + Ja \\ &\quad \ge v(\delta) \\ &\quad > EV(x(t+\theta), t+\theta, r(t+\theta)) \\ &\quad \ge \min_{1\le k\le N} EV(x(t+\theta), t+\theta, k) \quad \forall\, \theta \in [-\tau, 0]. \end{aligned}$$

This, by condition (6), implies

$$\begin{aligned} E\mathcal{L}V(x_t, t, r(t)) &\le -w(x(t), t, r(t)) \le -\gamma, \\ &\forall\, T_0 \le t \le T_1. \end{aligned}$$

Consequently, by formula (15), we see

$$\begin{aligned} EV(x(T_1), T_1, r(T_1)) &\le EV(x(T_0), T_0, r(T_0)) - \gamma(\bar{T}_1 - \bar{T}_0) \\ &< v(\delta) - \gamma\tilde{\tau} \\ &\le 0 \end{aligned}$$

which contradicts the positive property of $EV(x(t), t, r(t))$. So, $t_1 \le T_1$ and $E\mathcal{L}V(x(t_1), t_1, r(t_1)) \le -\gamma$. In fact, $\forall\, t_{11} \in \{t \ge T_0 : EV(x(t), t, r(t)) = u(\beta) + (J-1)a\}$, we have $E\mathcal{L}V(x_{t_{11}}, t_{11}, r(t_{11})) \le -\gamma$ because

$$\begin{aligned} &q(\max_{1\le k\le N} EV(x(t_{11}), t_{11}, k), r(t)) \\ &\quad \ge q(EV(x(t_{11}), t_{11}, r(t_{11})), r(t)) \\ &\quad > u(\beta) + Ja \\ &\quad \ge v(\delta) \\ &\quad > EV(x(t_{11}+\theta), t_{11}+\theta, r(t_{11}+\theta)) \\ &\quad \ge \min_{1\le k\le N} EV(x(t+\theta), t+\theta, k) \quad \forall\, \theta \in [-\tau, 0]. \end{aligned}$$

Thus, we have $EV(x(t), t, r(t)) \le u(\beta) + (J-1)a$ for all $t \ge T_1$.

Define $t_j = \inf\{t \ge T_{j-1} : EV(x(t), t, r(t)) \le u(\beta) + (J-j)a\}$ for $j = 2, 3, \ldots, J$. By the same type of reasoning as above, we have

$$EV(x(t), t, r(t)) \le u(\beta) + (J-j)a$$

for all $t \ge T_j$ and $j = 2, 3, \ldots, J$.

In particular, $EV(x(t), t, r(t)) \le u(\beta)$ for all $t \ge T_J$. This completes the proof. ■

2) Theorem 3.2: Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in C\mathcal{K}_\infty$, and $w : R^n \times R_+ \times S \to R_+$ be a nonnegative continuous function with $w(x, t, i) > 0$ if $E|x(t)|^p > 0$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that

$$u(|x|^p) \le V(x, t, i) \le v(|x|^p), \quad \forall\, (x, t) \in R^n \times [-\tau, \infty) \tag{18}$$

and, moreover, for all $1 \le i \le N$

$$E\mathcal{L}V(\phi, t, i) \le -w(\phi(0), t, i) \tag{19}$$

for all $t \ge 0$ and those $\phi \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ satisfying

$$\min_{1\le k\le N} EV(\phi(\theta), t+\theta, k) < \max_{1\le k\le N} E\bar{q}(V(\phi(0), t, k), i) \tag{20}$$

on $-\tau \le \theta \le 0$, where $\bar{q} : R \times S \to R$ is a continuous nondecreasing function with respect to $s \in R$ for all $s \ge 0$ and $1 \le i \le N$. Moreover, for all $1 \le i \le N$, $\bar{q}(s, i) > s$ for all $s > 0$ and $\bar{q}(s, i)/s > 1$ as $s \to \infty$. Then, the trivial solution of HSRS (3) is globally pth moment asymptotically stable.

Proof: As above, the proof is composed of tow parts. The first part to show the pth moment stability of (3) is similar to that for Theorem 3.1. One only needs to note that from the property of $\bar{q}(s,i)$ holds

$$\begin{aligned} &E\bar{q}(V(x(t),t,r(t)),r(t)) \\ &\quad\geq \int_{0<V<\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}P \\ &\qquad + \int_{V\to\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}P \\ &\quad> \int_{0<V<\infty} V(x(t),t,r(t)))\mathrm{d}P \\ &\qquad + \int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}P \\ &\quad= EV(x(t),t,r(t)) \quad \forall\, t\geq 0. \end{aligned} \tag{21}$$

Inequalities (12) and (21) imply that condition (20) is satisfied. Moreover, $EV(x(t),t,r(t))>0$ implies $E|x(t)|^p>0$. Thus, by condition (19) and the property of $w(x,t,i)$, we are led to (14) in the case when $EV(x(t),t,r(t))>0$.

The other part to show the convergence of $E|x(t)|^p\to 0$ as $t\to\infty$ is slightly different and given as follows.

Numbers $\delta,\varepsilon,\gamma$, and $\tilde{\tau}$ are defined as above while the positive real number $\bar{a}=a_1\wedge a_2$, where $a_1>0$ and $a_2>0$ are such that, for all $1\leq i\leq N$, we have

$$\begin{aligned} &\bar{q}(s,i)-s>a_1 \quad \forall\, u(\beta)\leq s<\infty \\ &\frac{\bar{q}(s,i)-s}{s}>a_2 \text{ as } s\to\infty. \end{aligned}$$

Let us now consider the expectation of function $V(x(t),t,r(t))$ for any $t\geq 0$

$$\begin{aligned} EV(x(t),t,r(t)) = &\int_{V<u(\beta)} V(x(t),t,r(t))\mathrm{d}P \\ &+ \int_{u(\beta)\leq V<\infty} V(x(t),t,r(t))\mathrm{d}P \\ &+ \int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}P. \end{aligned}$$

Obviously, there is a positive number $0<\bar{p}<1$ such that

$$\alpha_1\vee\alpha_2\geq\bar{p} \tag{22}$$

for any $t\geq 0$ whenever $EV(x(t),t,r(t))\geq u(\beta)$, where

$$\begin{aligned} \alpha_1 &= P\left\{u(\beta)\leq V(x(t),t,r(t))<\infty\right\} \\ \alpha_2 &= \int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}P. \end{aligned}$$

Let $\bar{J}$ be the minimal nonnegative integer such that $u(\beta)+\bar{J}\bar{p}\bar{a}\geq v(\delta)$, and $\bar{T}_j=j\tilde{\tau}$ with $j=0,1,\ldots,\bar{J}$.

To prove that $EV(x(t),t,r(t))\leq u(\beta)$ for all $t\geq\bar{T}_{\bar{J}}$, we first show that $EV(x(t),t,r(t))\leq u(\beta)+(\bar{J}-1)\bar{p}\bar{a}$ for all $t\geq\bar{T}_1$. Let $\bar{t}_1=\inf\{t\geq\bar{T}_0 : EV(x(t),t,r(t))\leq u(\beta)+(\bar{J}-1)\bar{p}\bar{a}\}$. If $\bar{t}_1>\bar{T}_1$, then $\forall\,\bar{T}_0\leq t\leq\bar{T}_1$, we have

$$\begin{aligned} &\max_{1\leq k\leq N} E\bar{q}(V(x(t),t,k),r(t)) \\ &\quad\geq E\bar{q}(V(x(t),t,r(t)),r(t)) \\ &\quad= \int_{V<u(\beta)} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}P \\ &\qquad+ \int_{u(\beta)\leq V<\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}P \\ &\qquad+ \int_{V\to\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}P \\ &\quad> \int_{V<u(\beta)} V(x(t),t,r(t))\mathrm{d}P \\ &\qquad+ \int_{u(\beta)\leq V<\infty} \left[V(x(t),t,r(t))+\bar{a}\right]\mathrm{d}P \\ &\qquad+ (1+\bar{a})\int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}P \\ &\quad\geq EV(x(t),t,r(t))+\bar{p}\bar{a} \\ &\quad\geq u(\beta)+\bar{J}\bar{p}\bar{a} \\ &\quad\geq v(\delta) \\ &\quad> EV(x(t+\theta),t+\theta,r(t+\theta)) \\ &\quad\geq \min_{1\leq k\leq N} EV(x(t+\theta),t+\theta,k) \end{aligned} \tag{23}$$

for all $\theta\in[-\tau,0]$. This, by condition (19), implies

$$\begin{aligned} E\mathcal{L}V(x_t,t,r(t)) \leq -\,w(x(t),t,r(t)) \leq -\gamma, \\ \forall\,\bar{T}_0\leq t\leq\bar{T}_1. \end{aligned}$$

Consequently, we see

$$\begin{aligned} &EV(x(\bar{T}_1),\bar{T}_1,r(\bar{T}_1)) \\ &\quad\leq EV(x(\bar{T}_0),\bar{T}_0,r(\bar{T}_0))-\gamma(\bar{T}_1-\bar{T}_0) \\ &\quad< v(\delta)-\gamma\tilde{\tau} \\ &\quad\leq 0 \end{aligned}$$

which contradicts the positive property of $EV(x(t),t,r(t))$. Thus, $\bar{t}_1\leq\bar{T}_1$ and $E\mathcal{L}V(x(\bar{t}_1),\bar{t}_1,r(\bar{t}_1))\leq-\gamma$. Moreover, $\forall\,\bar{t}_{11}\in\{t\geq\bar{T}_0 : EV(x(t),t,r(t))=u(\beta)+(\bar{J}-1)\bar{p}\bar{a}\}$, we have $E\mathcal{L}V(x_{\bar{t}_{11}},\bar{t}_{11},r(\bar{t}_{11}))\leq-\gamma$ because inequality (20), or say, (23) holds on $t=\bar{t}_{11}$. Thus, we have $EV(x(t),t)\leq u(\beta)+(\bar{J}-1)\bar{p}\bar{a}$ for all $t\geq\bar{T}_1$.

Define $\bar{t}_j=\inf\{t\geq\bar{T}_{j-1} : EV(x(t),t,r(t))\leq u(\beta)+(\bar{J}-j)\bar{p}\bar{a}\}$ for $j=2,3,\ldots,\bar{J}$. By the same type of reasoning, we have

$$EV(x(t),t,r(t))\leq u(\beta)+(\bar{J}-j)\bar{p}\bar{a}$$

for all $t\geq\bar{T}_j$ and $j=2,3,\ldots,\bar{J}$.

Therefore, $EV(x(t),t,r(t))\leq u(\beta)$ for all $t\geq\bar{T}_J$. The proof is complete. ■

III. Application

Hybrid stochastic delay systems (HSDSs) described with stochastic differential delay equations with Markovian switching

[redacted]

are an important class of HSRSs that are frequently used in engineering. As an illustrative example of applications of our new results, we consider the following HSDEs.

Let us consider the HSDSs of the form

$$\mathrm{d}x(t) = F(x(t), x(t-h(t)), t, r(t))\mathrm{d}t + G(x(t), x(t-h(t)), t, r(t))\mathrm{d}B(t) \quad (24)$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, where $h : R_+ \to [0, \tau]$ is Borel measurable while

$$F : R^n \times R^n \times R_+ \times S \to R^n$$

and

$$G : R^n \times R^n \times R_+ \times S \to R^{n\times m}$$

are measurable functions with $F(0, 0, t, i) \equiv 0$ and $g(0, 0, t, i) \equiv 0$ for all $t \geq 0$. Assume that (24) only has continuous solutions. This is a special case of (3) with

$$f(\phi, t, i) = F(\phi(0), \phi(-h(t)), t, i)$$
$$g(\phi, t, i) = G(\phi(0), \phi(-h(t)), t, i)$$

for $(\phi, t, i) \in C([-\tau, 0]; R^n) \times R_+ \times S$. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, for the special case of (24) the operator $\mathcal{L}$ defined in (4) becomes from $R^n \times R^n \times R_+ \times S$ to R as

$$\begin{aligned}\mathcal{L}V(x, y, t, i) = {} & V_t(x, t, i) + V_x(x, t, i)F(x, y, t, i) \\ & + \frac{1}{2}\text{trace} \\ & \times \left[G^T(x, y, t, i)V_{xx}(x, t, i)G(x, y, t, i)\right] \\ & + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j). \end{aligned} \quad (25)$$

To give our new result for the HSDSs (24), let us introduce one more notation that $L^p_{\mathcal{F}_t}([\Omega; R^n)$ are the collection of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; R^n)$-valued random variables X such that $E|X|^p < \infty$ and state the corresponding version of Theorem 3.2 for (24) as follows.

1) Theorem 4.1: Let $p > 0$, $c_2 \geq c_1 > 0$ and $w : R^n \times R_+ \times S \to R_+$ be a nonnegative continuous function with $w(X, t, i) > 0$ for $E|X|^p > 0$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that

$$c_1|X|^p \leq V(X, t, i) \leq c_2|X|^p, \quad \forall\, (x, t) \in R^n \times [-\tau, \infty) \quad (26)$$

and, moreover, for all $1 \leq i \leq N$, let

$$E\mathcal{L}V(X, Y, t, i) \leq -w(X, t, i) \quad (27)$$

for all $t \geq 0$ and those $X, Y \in L^p_{\mathcal{F}_t}(\Omega; R^n)$ satisfying

$$\min_{1\leq k\leq N} EV(Y, t-h(t), i) < \max_{1\leq k\leq N} E\bar{q}(V(X, t, k), i) \quad (28)$$

where $\bar{q} : R \times S \to R$ is a continuous nondecreasing function with respect to $s \in R$ for all $s \geq 0$ and $1 \leq i \leq N$. Moreover, for all $1 \leq i \leq N$, $\bar{q}(s, i) > s$ for all $s > 0$ and $\bar{q}(s, i)/s > 1$ as $s \to \infty$. Then, the trivial solution of HSDS (24) is globally pth moment asymptotically stable.

This is a corollary from Theorem 3.2 and will be used to establish the following useful result.

2) Theorem 4.2: Let $p > 0$, $c_2 \geq c_1 > 0$, $\lambda_{0i} \geq \lambda_{1i} \geq 0$ and $\lambda : R \times S \to R$ be a continuous nondecreasing function with respect to $s \in R$ for all $s \geq 0$ and $1 \leq i \leq N$. Moreover $\lambda(s, i)/s > 0$ for all $s > 0$ and $1 \leq i \leq N$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that inequality (26) is satisfied and, moreover, for all $X, Y \in R^n$, $t \geq 0$, and $1 \leq i \leq N$, assume

$$\begin{aligned}\mathcal{L}V(X, Y, t, i) \leq {} & -\lambda_{0i} \max_{1\leq k\leq N} V(X, t, k) \\ & + \lambda_{1i} \min_{1\leq k\leq N} V(Y, t-h(t), k) \\ & - \lambda(\max_{1\leq k\leq N} V(X, t, k), i). \end{aligned} \quad (29)$$

Then, the trivial solution of HSDS (24) is globally pth moment asymptotically stable.

Proof: In condition (28), let

$$\bar{q}(s, i) = s + \frac{1}{2(1+\lambda_{1i})}\lambda(s, i). \quad (30)$$

For all $t \geq 0$ and $X, Y \in L^p_{\mathcal{F}_t}(\Omega; R^n)$ satisfying condition (28) with function (30), i.e.,

$$\begin{aligned}&\min_{1\leq i\leq N} EV(Y, t-h(t), i) \\ &\quad < \max_{1\leq i\leq N} EV(X, t, i) + \frac{1}{2(1+\lambda_{1i})} E\lambda(\max_{1\leq i\leq N} V(X, t, i))\end{aligned}$$

from inequality (29), Fatou's lemma, and condition (26), we have

$$\begin{aligned}E\mathcal{L}V(X, Y, t, i) \leq {} & -\lambda_{0i} \max_{1\leq k\leq N} EV(X, t, k) \\ & + \lambda_{1i} \min_{1\leq k\leq N} EV(Y, t-h(t), k) \\ & - E\lambda(\max_{1\leq k\leq N} V(X, t, k), i) \\ \leq {} & -(\lambda_{0i} - \lambda_{1i}) \max_{1\leq k\leq N} EV(X, t, k) \\ & - \frac{1}{2}E\lambda(\max_{1\leq k\leq N} V(X, t, k), i) \\ \leq {} & -\frac{1}{2}E\lambda(c_1|X|^p, i)\end{aligned}$$

for all $1 \leq i \leq N$. Since $\lambda(s, i)/s > 0$ for all $s > 0$ and $1 \leq i \leq N$, it is easy to verify that $E\lambda(c_1|X|^p, i) > 0$ if $E|X|^p > 0$. Let $w(X, t, i) = -1/2E\lambda(c_1|X|^p, i)$ in condition (27), then the conclusion follows from Theorem 4.1. ■

3) Remark 4.1: In many cases, this useful criterion may be applied with $\lambda(s, i) = \tilde{\lambda}_i s^{k_i}$, $k_i \geq 1$, and $\tilde{\lambda}_i > 0$ for $1 \leq i \leq N$. In a special case when $\lambda(s, i) = \tilde{\lambda}_0 s$ and $\tilde{\lambda}_0 > 0$ for all $1 \leq i \leq N$, the above result is exactly [12, Theorem 4.2]. However, our result works for the particular cases when $\lambda_{0i} - \lambda_{1i} = 0$ for some $1 \leq i \leq N$, to which the existing results (see [12]–[16] and [22]) do not apply.

Using the above skills, Theorem 4.2 can be developed to cope with systems with multiple delays of the form

$$\begin{aligned}\mathrm{d}x(t) = {} & F(x(t), x(t-h_1(t)), \ldots, x(t-h_L(t)), t, r(t))\mathrm{d}t \\ & + G(x(t), x(t-h_1(t)), \ldots, x(t-h_L(t)), t, r(t))\mathrm{d}B(t)\end{aligned} \quad (31)$$

on $t \geq 0$, where $h_l : R_+ \to [0, \tau]$ is Borel measurable, $l = 1, 2, \ldots, L$.

Let us state the following generalized result, which can be proven in the same way as in the proof of Theorem 4.2.

4) Theorem 4.3: Let $p > 0$, $c_2 \geq c_1 > 0$, and $\lambda_{0i} \geq 0, \lambda_{1i} \geq 0, \ldots \lambda_{Li} \geq 0$ such that $\lambda_{0i} \geq \sum_{l=1}^{L} \lambda_{li}$ for all $1 \leq i \leq N$. Let $\lambda : R \times S \to R$ be a continuous nondecreasing function with respect to $s \in R$ for all $s \geq 0$ and $1 \leq i \leq N$. Moreover $\lambda(s,i)/s > 0$ for all $s > 0$ and $1 \leq i \leq N$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that inequality (26) is satisfied and, moreover, for all $X, Y_1, \ldots, Y_L \in R^n$, $t \geq 0$ and $1 \leq i \leq N$,

$$\begin{aligned}\mathcal{L}V(X, Y_1, \ldots, Y_L, t, i) &\leq -\lambda_{0i} \max_{1\leq k\leq N} V(X, t, k) \\ &+ \lambda_{1i} \min_{1\leq k\leq N} V(Y_1, t - h_1(t), k) + \cdots \\ &+ \lambda_{Li} \min_{1\leq k\leq N} V(Y_L, t - h_L(t), k) \\ &- \lambda(\max_{1\leq k\leq N} V(X, t, k), i). \end{aligned} \tag{32}$$

Then, the trivial solution of HSDS (31) is globally pth moment asymptotically stable.

IV. Examples

1) Example 5.1: Let us now return to the scalar HSDS (1). For the previous calculation (2), let

$$\lambda_{01} = \lambda_{11} = 1, \quad \lambda(s,1) = \frac{1}{12}s^2,$$
$$\lambda_{02} = \lambda_{12} = 1, \quad \lambda(s,2) = \frac{1}{5}s^{5/4}$$

in condition (29). It immediately follows from Theorem 4.2 that the trivial solution of system (1) is mean-square asymptotically stable. Clearly, this is in fact an application of Theorem 3.2. Alternatively, we can use Theorem 3.1 and have the same conclusion. Let

$$q(s,i) = \begin{cases} s + \frac{1}{24}s^2, & i = 1 \\ s + \frac{1}{10}s^{5/4}, & i = 2 \end{cases}$$

in condition (7), then the previous calculation (2) yields

$$E\mathcal{L}V(x_t, t, i) \leq \begin{cases} -\frac{1}{24}Ex^4(t), & i = 1 \\ -\frac{1}{10}E\left[x^2(t)\sqrt{|x(t)|}\right], & i = 2 \end{cases}$$

when condition (7) is satisfied. Let

$$w(x,t,i) = \begin{cases} \frac{1}{24}(Ex^2(t))^2, & i = 1 \\ \frac{1}{10}(Ex^2(t))^{5/4}, & i = 2 \end{cases}$$

in inequality (6), then the inequality holds. According to Theorem 3.1, this implies that the trivial solution of system (1) is mean-square asymptotically stable.

2) Example 5.2: Let $r(t)$ be a right-continuous Markov chain taking values in $S = 1, 2$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and independent of the scalar Brownian motion $B(t)$. Let a_i, b_i, c_i, d_i be positive numbers with $b_i > c_i^2$ for $i = 1, 2$ and $h : R_+ \to [0, \tau]$ be Borel measurable. Consider the following HSDS:

$$\begin{aligned}\mathrm{d}x(t) = [\alpha(r(t))x(t) + \beta(x(t), r(t))]\mathrm{d}t \\ +\sigma(x(t), x(t - h(t)), r(t))\mathrm{d}B(t)\end{aligned} \tag{33}$$

on $t \geq 0$, where

$$\begin{aligned}\alpha(1) &= a_1, \quad \beta(x,1) = -b_1 x|x|, \\ |\sigma(x,y,1)| &\leq c_1\sqrt{|x^3|} + d_1|y|, \\ \alpha(2) &= -a_2, \quad \beta(x,2) = -b_2 x^3, \\ |\sigma(x,y,2)| &\leq c_2|x|^2 + d_2|y|.\end{aligned}$$

To examine the stability of system (33) in a mean-square sense, we construct a function $V : R \times R_+ \times S \to R_+$ by

$$V(x,t,i) = \begin{cases} \gamma_1 x^2, & i = 1 \\ \gamma_2 x^2, & i = 2 \end{cases}$$

where γ_1 and γ_2 are positive constants to be determined.

By calculation, we have

$$\begin{aligned}\mathcal{L}V(x(t), x(t-h(t)), t, 1) \leq &-[(1 - 2a_1)\gamma_1 - \gamma_2]x^2(t) \\ &+ 2d_1^2\gamma_1 x^2(t - h(t)) \\ &- 2(b_1 - c_1^2)\gamma_1|x^3(t)| \\ \mathcal{L}V(x(t), x(t-h(t)), t, 2) \leq &-[(1 + 2a_2)\gamma_2 - \gamma_1]x^2(t) \\ &+ 2d_2^2\gamma_2 x^2(t - h(t)) \\ &- 2(b_2 - c_2^2)\gamma_2 x^4(t).\end{aligned}$$

It is easy to show that there exist positive numbers γ_1 and γ_2 such that

$$\begin{aligned}(1 - 2a_1)\gamma_1 - \gamma_2 &\geq 2d_1^2\gamma_1 \\ (1 + 2a_2)\gamma_2 - \gamma_1 &\geq 2d_2^2\gamma_2\end{aligned}$$

when the following inequalities are satisfied:

$$\begin{cases} a_1 + d_1^2 < \frac{1}{2}, \\ a_2 > d_2^2, \\ \frac{1}{1 - 2(a_1 + d_1^2)} \leq 1 + 2(a_2 - d_2^2). \end{cases}$$

Since $b_i > c_i^2$ for $i = 1, 2$, by Theorem 4.2, we can conclude that system (33) is mean-square asymptotically stable if the above inequalities hold.

V. Conclusion

In this paper, the general pth moment asymptotic stability of HSRSs (3) is studied with Razumikhim-type arguments. Theorems on asymptotic stability are established. Their applications to HSDSs (24) and (31) are also proposed. The Razumikhin-type theorems work for many HSRSs including some complicated cases to which the existing results do not apply. In a special case of the above results when $w(x(t), t, r(t)) = \alpha(t)EV(x(t), t, r(t))$ for all $t \geq 0$ with $\alpha(t) > 0$, using the techniques similar to [16], Razumikhin-type theorems on generalized exponential stability of HSRSs (3) may be obtained. By Fatou's lemma, we note that conditions (7) and (20) are less conservative than that in the existing results (see [12] and [16]) and are convenient in application.

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Lirong Huang (S'08) received the B.E. and M.E. degrees from South China University of Technology, Guangzhou. He is currently working toward the Ph.D. degree at the University of Strathclyde, Glasgow, U.K.

His research is concerned with stability and stabilization of stochastic delay systems.

Xuerong Mao received the Ph.D. degree from Warwick University, Coventry, U.K., in 1989.

He was then SERC (Science and Engineering Research Council, U.K.) Post-Doctoral Research Fellow from 1989 to 1992. Moving to Scotland, he joined the Department of Statistics and Modelling Science, University of Strathclyde, Glasgow, as a lecturer in 1992, was promoted to Reader in 1995, and was made Professor in 1998, which post he still holds. He has authored five books and over 180 research papers. His main research interests lie in the field of stochastic analysis including stochastic stability, stabilization, control, numerical solutions.

Dr. Mao is a Fellow of the Royal Society of Edinburgh (FRSE). He is also a member of the editorial boards of several international journals, including the *Journal of Stochastic Analysis and Applications* and the *Journal of Dynamics of Continuous, Discrete and Impulsive Systems Series B.*

Feiqi Deng received the Ph.D. degree in control theory and control engineering from South China University of Technology, Guangzhou, in 1997.

From July 1997 to September 1999, he was an Associate Professor with the College of Automation Science and Engineering, South China University of Technology. Since October 1999, he has been a Professor and the Director of the Systems Engineering Institute of College of Automation Science and Engineering, South China University of Technology. His main research interests include stability, stabilization, and robust and variable structure control theory of complex systems, including time-delay systems, nonlinear systems and stochastic systems, and machine learning. In these areas, he has authored/coauthored more 400 journal papers and one book in English and Chinese. Since 2002, he has been serving as a vice chief editor of the *Journal of South China University of Technology* and editor of *Control Theory and Applications* as well as the *Journal of Systems Engineering and Electronics.*