

The Eternal Triangle

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Introduction

The story starts one lunchtime in August 1998. A group of bright young postgraduates are perusing the problems set in the 1998 International Mathematical Olympiad which I had brought back from Taiwan. The problems are tough. (You can see for yourself in my article in SMC Journal 28, pages 13-15.) I expected the students to run into trouble sooner or later but I had hoped that they would get a little further than the first problem. Among all these experts in things like fluid mechanics and numerical analysis, none knew what a cyclic quadrilateral was. (However, there were some novel suggestions!)

Although this does not quite represent the end of civilisation, it is nevertheless a poor show and epitomises the lack of basic geometrical knowledge among the vast majority of today's undergraduates in science and engineering. Forty years ago when I was a lad, life was rather different. Geometrical deductions were part of the curriculum for a large percentage of each cohort. This percentage was almost certainly too large, with many pupils finding the going very hard indeed. The 'solution' adopted was to do away with almost everything and introduce new approaches, with topics like transformation geometry and tessellations becoming flavour of the month.

One major casualty in this revolution was the idea of proof in mathematics. Geometry provided an excellent vehicle for developing powers of deductive reasoning, which are needed in all branches of mathematics. Undergraduates embarking on a first course in analysis are hit by a double whammy. The dreaded ϵ and δ are bad enough but, in addition to learning what is tantamount to a foreign language, students have to string the 'words' together to form sentences and logical arguments. A 'proof' produced nowadays by a typical student usually contains most of the right words but all in the wrong order.

In my view (and all views in this article are personal), the prospective scientists and

engineers are being short-changed by the way geometry is taught at present. At the other end of the spectrum are lots of people who will get through life perfectly happily without knowing that the altitudes of a triangle are concurrent. How to satisfy the needs of a disparate population is a challenge that needs to be faced.

Let's go back to the postgraduates eating their sandwiches in 1998. I decided there and then to give them a course of lectures on the geometry of the triangle. The response was remarkable. The topic was totally different from the usual material of such courses and they loved it! At one level they were exposed to sustained logical arguments. At another level they could appreciate the elegance of many of the results.

- Two non-parallel lines intersect at a point but we keep coming across sets of three concurrent lines (medians, altitudes, etc.)
- In general two circles either intersect at two distinct points or miss each other completely but we keep coming across circles which touch.
- Three non-collinear points are sufficient to determine a circle. What then is the likelihood of *nine* interesting points lying on one circle?

We have no right to expect any of these flukes and yet they are all nestling inside the humble triangle.

Come with me as we attempt to unravel the mysteries of what's going on. For some readers this will be a voyage of discovery, while for others it may be more of a trip Down Memory Lane. The material contains opportunities for the lowest attainers to draw pretty pictures as well as opportunities to teach high attainers how to construct rigorous proofs. For everyone, there is the chance to wallow in the sheer beauty of it all.

Congruent Triangles

Any triangle has three sides and three angles. However, we do not need all six bits of information to determine a triangle uniquely. For example, suppose we know that

$$AB = 2 \text{ cm}, \quad AC = 3 \text{ cm}, \quad \angle BAC = 45^\circ.$$

Draw a line segment of length 2 cm to represent AB . Form an angle of 45° and head off in this direction for a distance of 3 cm to arrive at C . The three vertices A , B and C are now known and the triangle is completely specified.

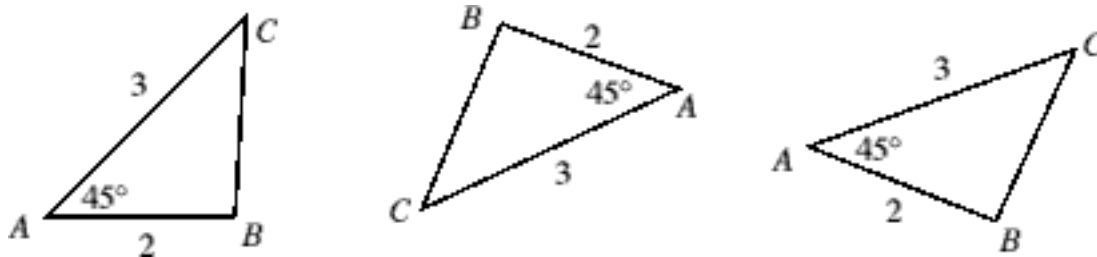


Figure 1: triangles given SAS

If the three triangles shown in Figure 1 were made of cardboard, any one could be placed exactly on top of either of the others. In this sense the triangles are regarded as equivalent or *congruent*. Knowing the lengths of the sides AB and AC , as well as the size of the *included angle* $\angle BAC$, we can calculate the length of BC and the size of the angles at B and C . Here are four ways of proving two triangles congruent.

- SSS (side-side-side)
- SAS (side-angle-side; two sides and the *included angle*)
- AAS (angle-angle-side; two angles and the *corresponding side*)
- RHS (right angle-hypotenuse-side).

SAS was the one discussed above. To illustrate AAS consider $\angle BAC = \angle RPQ = 45^\circ$ and $\angle ABC = \angle PRQ = 60^\circ$. Note that A, B and C correspond to P, R and Q respectively. Hence BC and QR are corresponding sides. We say that $\triangle ABC$ is congruent to $\triangle PQR$ and write $\triangle ABC \equiv \triangle PQR$.

(Note that we use $\triangle PRQ$ and *not* $\triangle PQR$ because B, C correspond to R, Q respectively in Figure 2.)



Figure 2: triangles given AAS

In contrast, knowing the three angles of a triangle is not enough to specify a triangle uniquely (although it would be unique up to similarity, rather than congruence). Likewise two pairs of angles and a pair of non-corresponding sides are not sufficient.

In days of yore, before the advent of vectors, the proofs in school geometry books were based fairly and squarely on the use of the four cases of congruence listed above. We are talking here of books like those by Walker and Millar (12s 3d in old money) or White and Morrison. It is instructive to recall what such proofs looked like. (I'll only include two in detail now, but those of a frail disposition may wish to jump ahead.) Let us consider the (internal) bisectors of the angles of a triangle. As a starter for ten, we need

Theorem 1 The locus of a point equidistant from two intersecting straight lines is the bisector of the angle between them.

There is actually quite a lot to be done here as we shall now see.

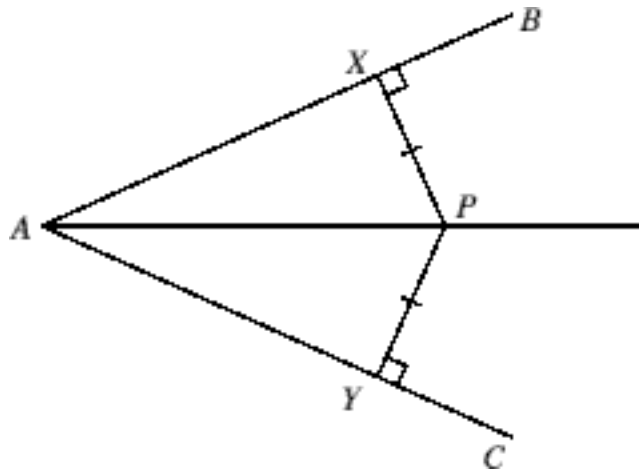


Figure 3: P equidistant from AB and AC

Part 1 We shall prove that any point which is equidistant from two intersecting lines lies on the bisector of the angle between them. Let us set things up in Figure 3.

Given:- Straight lines AB and AC meeting at A ;

P any point equidistant from AB and AC .

RTP (Required to prove):- P lies on the bisector of $\angle BAC$.

Construction:- Join PA . Draw PX, PY perpendicular to AB, AC respectively.

Proof:- In $\triangle APX$ and $\triangle APY$,

1. $PX = PY$ (given; P is equidistant from AB and AC)
2. $AP = AP$ (AP is a common side of the two triangles)
3. $\angle AXP = \angle AYP$ (right angles, by construction).

$\therefore \triangle APX \equiv \triangle APY$ (RHS).

$\therefore \angle XAP = \angle YAP$.

$\therefore P$ is on the bisector of $\angle BAC$.

Don't relax yet. We are only half finished!

Part 2

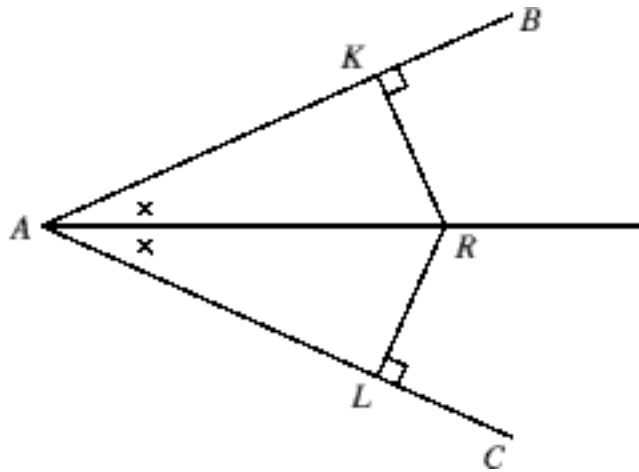


Figure 4: R on the bisector of $\angle BAC$

Given:- Straight lines AB and AC meeting at A ,

R any point on the bisector of $\angle BAC$.

RTP:- R is equidistant from AB and AC .

Construction:- Draw RK , RL perpendicular to AB , AC respectively as in Figure 4.

We want to prove that $RK = RL$.

Proof:- In $\triangle KAR$ and $\triangle LAR$

1. $\angle KAR = \angle LAR$ (given)

2. $\angle AKR = \angle ALR$ (right angles, by construction)

3. $AR = AR$ (common side)

$\therefore \triangle KAR \equiv \triangle LAR$ (AAS; AR is opposite the right angle in both triangles).

$\therefore RK = RL$.

$\therefore R$ is equidistant from AB and AC .

Only now can we add Q.E.D. (*quod erat demonstrandum*) if we wish!

Theorem 2 The internal bisectors of the three angles of a triangle are concurrent. The point of concurrence is called the *incentre* and is the centre of the *inscribed circle* which touches the three sides of the triangle.

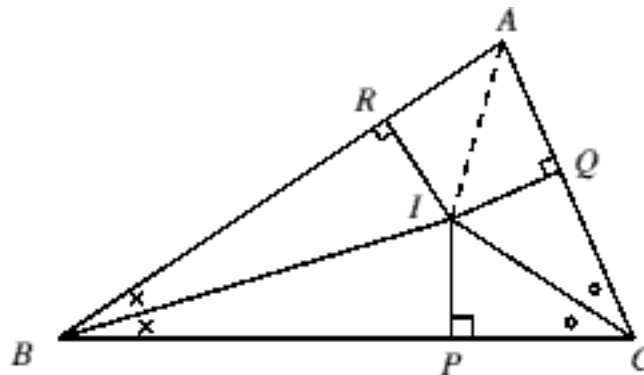


Figure 5: concurrence of internal bisectors

Given:- $\triangle ABC$.

RTP:- The bisectors of $\angle ABC$, $\angle BCA$ and $\angle CAB$ are concurrent.

Construction:- Draw IB and IC , the bisectors of $\angle ABC$ and $\angle BCA$, respectively which meet at I , as shown in Figure 5.

Draw IP , IQ , IR perpendicular to AB , BC , CA respectively.

Join IA .

We shall show that IA bisects $\angle BAC$.

Proof:- IB is the bisector of $\angle ABC$.

$\therefore IB$ is the locus of points equidistant from BA and BC (Theorem 1, Part 2).

$\therefore IP = IR$.

Similarly $IP = IQ$.

$\therefore IQ = IR$.

$\therefore I$ is equidistant from AB and AC .

$\therefore I$ lies on the bisector of $\angle BAC$ (Theorem 1, Part 1).

Also, since $IP = IQ = IR$, I is the centre of a circle which passes through P , Q , R and therefore touches the three sides of $\triangle ABC$.

Q.E.D.

Commentary

The old style was nothing if not disciplined. In order we have

- a clear statement of the result
- a diagram with relevant points labelled
- a clear statement of what we are given
- a clear statement of what we are trying to prove
- some constructions, if required
- a proof based on congruent triangles (and using results proved previously).

Note the use of ' \therefore ' for 'therefore' rather than the all-pervasive ' \Leftrightarrow ' which is frequently misused and abused nowadays. Theorem 1 also highlights the difference between a statement and its converse (i.e. $P \Rightarrow Q$ versus $Q \Rightarrow P$).

Note Those of a frail disposition can rejoin us here!

Triangle Centres

There are four well-known (?) sets of three lines, associated with a triangle $\triangle ABC$, which meet at a point.

- The medians meet at the *centroid*, G
- The altitudes meet at the *orthocentre*, H
- The (internal) angle bisectors meet at the *incentre*, I
- The perpendicular bisectors of the sides meet at the *circumcentre*, O .

O is the centre of the *circumcircle* of $\triangle ABC$, i.e. the unique circle which passes through the three vertices of the triangle.

The letters G , H , I and O will be used for these special points from now on.

As mentioned earlier, we have no right to expect *three* lines to be concurrent. Yet here we have four such 'flukes'. However, these are not all independent of each other.

Theorem 3 Given that the perpendicular bisectors of the sides of any triangle are concurrent, it follows that the altitudes of any triangle are concurrent.

Proof:- We shall adopt a streamlined version of the style used in Theorem 1 and 2.

Let $\triangle ABC$ be any triangle. We wish to prove that its altitudes are concurrent.

Through each vertex draw a line parallel to the opposite side (i.e. lines through A, B, C parallel to BC, CA, AB respectively). Let these intersect at A', B', C' as shown in Figure 6. It is not hard to prove that $ACBC'$ and $ABCB'$ are parallelograms so that $C'A = AB'$, both being equal to BC .

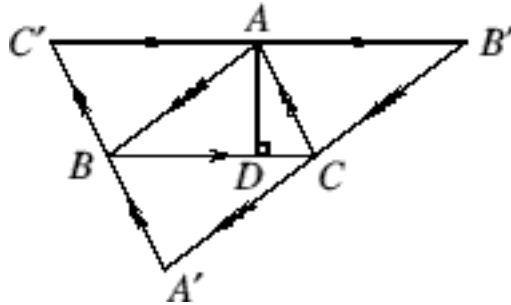


Figure 6: altitude of $\triangle ABC$

Let AD be the altitude of $\triangle ABC$ through A . Since BC and $C'B'$ are parallel, AD is perpendicular to $C'B'$. From above, it follows that AD is the perpendicular bisector of $C'B'$. By a similar argument applied to the other two altitudes of $\triangle ABC$, we see that the altitudes of $\triangle ABC$ are the perpendicular bisectors of the sides of $\triangle A'B'C'$. By assumption, the perpendicular bisectors of $\triangle A'B'C'$ are concurrent. Hence the altitudes of $\triangle ABC$ are concurrent.

Q.E.D.

Here is another interconnection.

Definition Let D, E, F be the feet of the altitudes from A, B, C respectively in $\triangle ABC$ as shown in Figure 7. Then $\triangle DEF$ is called the *pedal triangle* of $\triangle ABC$.

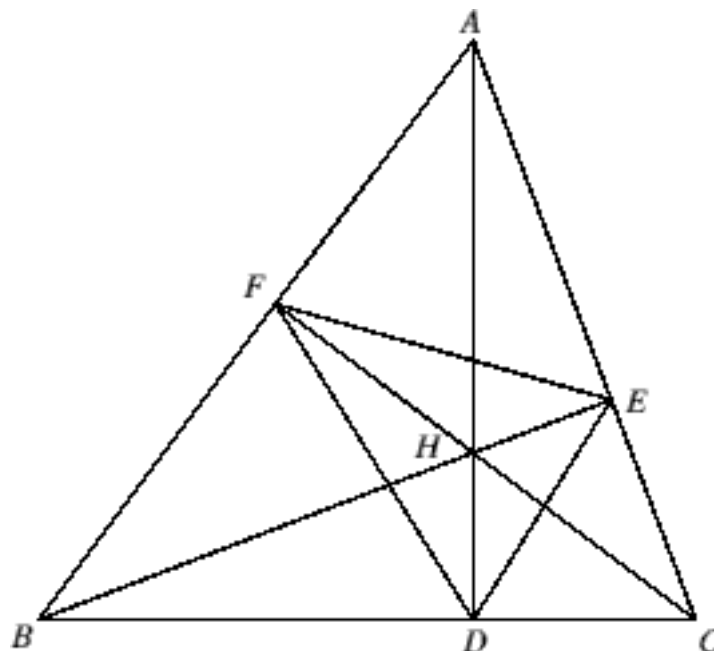


Figure 7: pedal triangle of $\triangle ABC$

Figure 7 contains a lot of rich structure. At this stage, the dreaded cyclic quadrilateral surfaces. (For younger readers in the same boat as the postgraduates, a cyclic quadrilateral is a quadrilateral whose four vertices all lie on a circle.) In the diagram, H is the orthocentre of $\triangle ABC$. We know that

$$\angle BFH + \angle BDH = 90^\circ + 90^\circ = 180^\circ.$$

$\therefore BFHD$ is a cyclic quadrilateral (opposite angles supplementary).

$\therefore \angle FBH = \angle FDH$ (angles in same segment of circle $BFHD$).

Similarly $\angle ECH = \angle EDH$ since $CEHD$ is a cyclic quadrilateral.

Also $\angle FBE = \angle FCE$ since $BCEF$ is a cyclic quadrilateral.

$\therefore \angle FBH = \angle ECH$ (relabelling angles).

$\therefore \angle FDH = \angle FBH = \angle ECH = \angle EDH$.

$\therefore HD$ bisects $\angle FDE$.

We have therefore proved the following result.

Theorem 4 The altitudes of a triangle are the angle bisectors of the corresponding pedal triangle.

Euler Line

Next let us investigate connections between various triangle centres. To get the full flavour, we shall work with a *scalene* triangle, i.e. one in which the sides all have different lengths. (If the original triangle is isosceles or equilateral, diagrams will degenerate. For example, if $\triangle ABC$ is equilateral, G , H , I and O coincide.)

Use of some package such as *Geometer's Sketchpad* or *Cabri Geomètre* may suggest various results. For example, it looks as though O , G and H lie on a line and that G trisects OH . Can this be true?

Theorem 5 The centroid G , the orthocentre H and the circumcentre O of a (scalene) triangle lie on a straight line, called the *Euler Line* of the triangle. Further, in vector notation,

$$\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OH}.$$

Proof:-

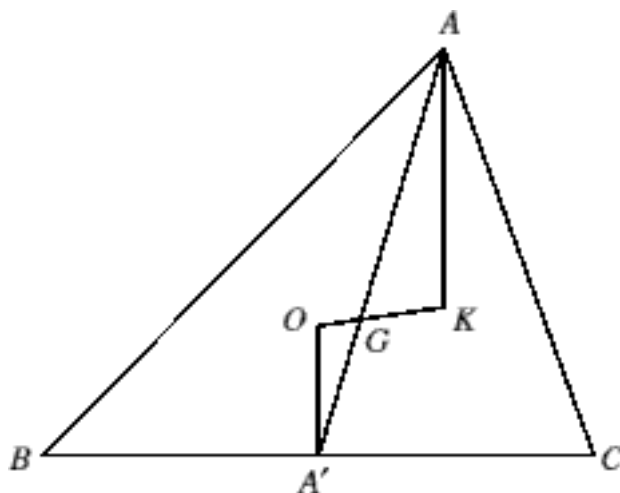


Figure 8: Euler line of $\triangle ABC$

Let A' be the mid-point of BC , so that OA' is the perpendicular bisector of BC and AA' is the median through A .

It is a standard result that $\overrightarrow{AG} = 2\overrightarrow{GA'}$.

Let K be the point on the line through O and G such that $\overrightarrow{GK} = 2\overrightarrow{OG}$, as shown in Figure 8.

We shall prove that K is actually H .

In $\triangle A'GO$ and $\triangle AGK$

$$1. \frac{A'G}{GA} = \frac{OG}{GK} = \frac{1}{2} \text{ (from above)}$$

$$2. \angle A'GO = \angle AGK \text{ (vertically opposite angles).}$$

$\therefore \triangle A'GO$ and $\triangle AGK$ are similar.

$$\therefore \angle OA'G = \angle GAK.$$

$\therefore OA'$ is parallel to AK .

$\therefore AK$ (extended) is perpendicular to BC (since $OA' \perp BC$).

$\therefore K$ lies on the altitude of $\triangle ABC$ through A .

Similarly K lies on the altitudes of $\triangle ABC$ through B and C .

$\therefore K$ is the orthocentre H of $\triangle ABC$.

$$\therefore O, G, H \text{ are collinear and } \overrightarrow{OG} = \frac{1}{2}\overrightarrow{GK} = \frac{1}{3}\overrightarrow{OH}.$$

Q.E.D.

Theorem 5 provides us with another wonderful surprise and one which escaped all the

Greek geometers. (Euler discovered this in the 18th century.) We shall meet the Euler line again in a moment.

Meanwhile, let us turn our attention to circles related to a triangle. We have already met the incircle and the circumcircle, but that's only a start.

The 9-point circle

Given that three non-collinear points determine a circle uniquely, the likelihood of nine special points lying on a circle seems remote indeed. However, we are in for another wonderful surprise.

Theorem 6 Given $\triangle ABC$, there is a circle passing through the following nine points:

A', B', C' the mid-points of BC, CA, AB respectively

D, E, F the feet of the altitudes AD, BE, CF respectively

U, V, W the mid-points of AH, BH, CH respectively.

Further, the centre N of the circle is the mid-point of OH and the radius of the circle is $\frac{1}{2}R$, where R is the radius of the circumcircle of $\triangle ABC$ shown in Figure 9.

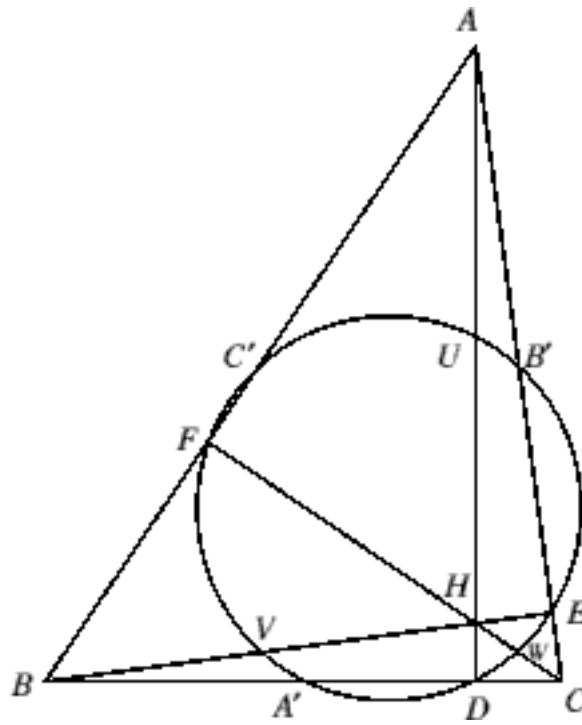


Figure 9: 9-point circle

Proof:- To start with we shall use vectors and take O , the circumcentre, as origin (a shrewd choice!).

Let $\vec{OA} = \mathbf{x}$, $\vec{OB} = \mathbf{y}$, $\vec{OC} = \mathbf{z}$.

By definition of the circumcircle,

$$|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = R \quad (*)$$

By a standard result $\vec{OG} = \frac{1}{3}(\mathbf{x} + \mathbf{y} + \mathbf{z})$.

$\therefore \vec{OH} = \mathbf{x} + \mathbf{y} + \mathbf{z}$ (Theorem 5).

Also $\vec{ON} = \frac{1}{2}(\mathbf{x} + \mathbf{y} + \mathbf{z})$ (N is the mid-point of OH).

Now $\vec{OA'} = \frac{1}{2}(\mathbf{y} + \mathbf{z})$ (A' is the mid-point of BC).

$\therefore \vec{NA'} = \vec{OA'} - \vec{ON} = -\frac{1}{2}\mathbf{x}$.

$$\therefore NA = \frac{1}{2}|\mathbf{x}| = \frac{1}{2}R \text{ (by(*)).}$$

$$\text{Similarly } NB = NC = \frac{1}{2}R.$$

$\therefore A, B$ and C lie on the circle with centre N and radius $\frac{1}{2}R$.

$$\text{Next } \overrightarrow{OU} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OH}) = \frac{1}{2}(2\mathbf{x} + \mathbf{y} + \mathbf{z}).$$

$$\therefore \overrightarrow{NU} = \overrightarrow{OU} - \overrightarrow{ON} = \frac{1}{2}\{(2\mathbf{x} + \mathbf{y} + \mathbf{z}) - (\mathbf{x} + \mathbf{y} + \mathbf{z})\} = \frac{1}{2}\mathbf{x}.$$

$$\therefore NU = \frac{1}{2}|\mathbf{x}| = \frac{1}{2}R.$$

$$\text{Similarly } NV = NW = \frac{1}{2}R.$$

$\therefore U, V$ and W lie on the circle with centre N and radius $\frac{1}{2}R$.

$$\text{From above } \overrightarrow{NA'} = -\frac{1}{2}\mathbf{x} \text{ and } \overrightarrow{NU} = \frac{1}{2}\mathbf{x}.$$

$\therefore N$ is the mid-point of $A'U$.

However $A'U$ subtends a right-angle at D .

$\therefore A'U$ is a diameter of the circumcircle of $\Delta A'DU$ and

N is the centre of this circle.

$$\therefore ND = NA' = NU = \frac{1}{2}R.$$

$$\text{Similarly } NE = NF = \frac{1}{2}R.$$

$\therefore D, E$ and F lie on the circle with centre N and radius $\frac{1}{2}R$.

For your further amazement, we can state

Theorem 7 The 9-point circle of $\triangle ABC$ is also the 9-point circle of each of $\triangle AHB$, $\triangle BHC$ and $\triangle CHA$.

Proof:- Consider $\triangle AHB$ in Figure 9. It is enough to prove that the 9-point circle of $\triangle ABC$ is the unique circle passing through the mid-points of the sides AH , HB and BA . However, these are the points, U , V and C' in the notation of Theorem 6 and do indeed lie on this circle.

The same applies to $\triangle BHC$ and $\triangle CHA$.

Three more circles now come into the picture. We consider one in detail, with the other two being similar. We shall bisect $\angle A$ of $\triangle ABC$ internally as before. However, we now bisect $\angle B$ and $\angle C$ externally as shown in Figure 10.

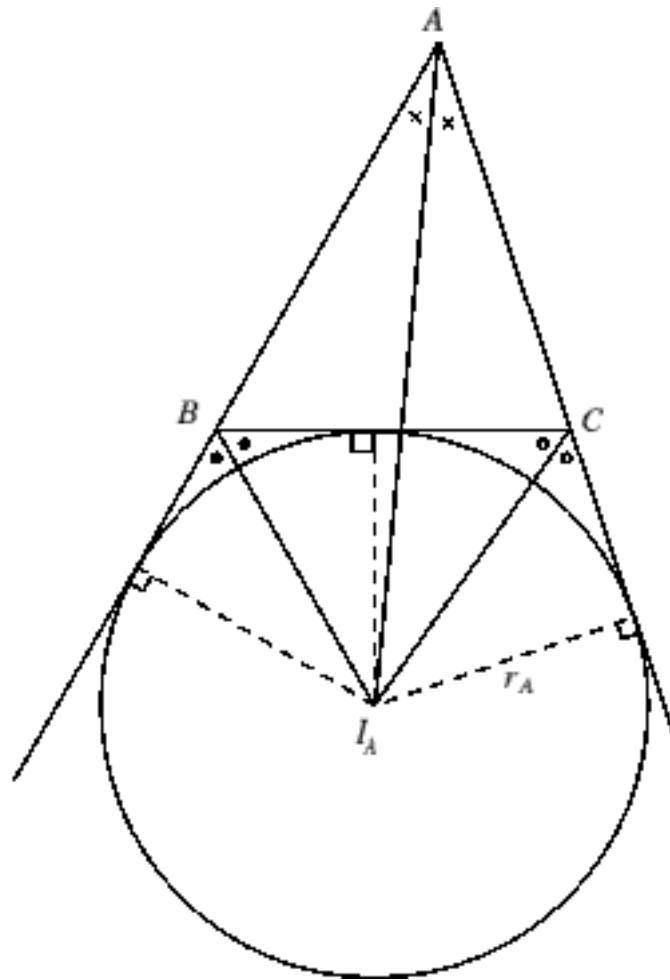


Figure 10 : escribed circle opposite A

Theorem 8 The internal bisector of $\angle A$ and the external bisectors of $\angle B$ and $\angle C$ are concurrent at a point I_A , which is the centre of a circle which touches BC internally and the sides AB and AC externally (in the sense that AB and AC are extended beyond B and C).

Definition In the notation of Theorem 8, I_A is called the *excentre* of $\triangle ABC$ opposite A and the corresponding circle is called the *escribed circle* opposite A .

There are excentres and escribed circles opposite vertices B and C as well.

Relations between the various circles

We now have 6 circles and it is of interest to see how they interact with each other. The following table is useful.

Circle	Centre	Radius
Circumcircle	O	R
Inscribed	I	$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
Escribed, opposite A	I_A	$r_A = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
Escribed, opposite B	I_B	$r_B = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$
Escribed, opposite C	I_C	$r_C = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$
Nine-point	N	$\frac{1}{2}R$

Table 1

Calculation of the radii of the inscribed and escribed circles involves the use of trigonometric identities and is left as an exercise for the reader. Here are two more such exercises.

- $OI^2 = R^2 - 2Rr$, so that $R \geq 2r$.
- $\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}$.

However, rather than dwell on algebraic manipulation, we turn to yet another remarkable result, proved by Karl Feuerbach in 1822.

Theorem 9 (Feuerbach's Theorem)

The 9-point circle of a triangle touches the inscribed circle and each of the three escribed circles.

What right have we to expect this? None at all. We shall not give a proof but instead invite the reader to get a sheet of A3 paper and draw a large diagram.

Challenge Starting with a large scalene triangle, construct every special point and circle mentioned so far, using only a straight edge (with no scale) and a pair of compasses. If you succeed, frame the drawing and hang it on the wall in your classroom.

While we are in this neck of the woods, look at the diagram below in which the angles of an arbitrary triangle ABC have been *trisected*. What do you think is true of $\triangle PQR$?

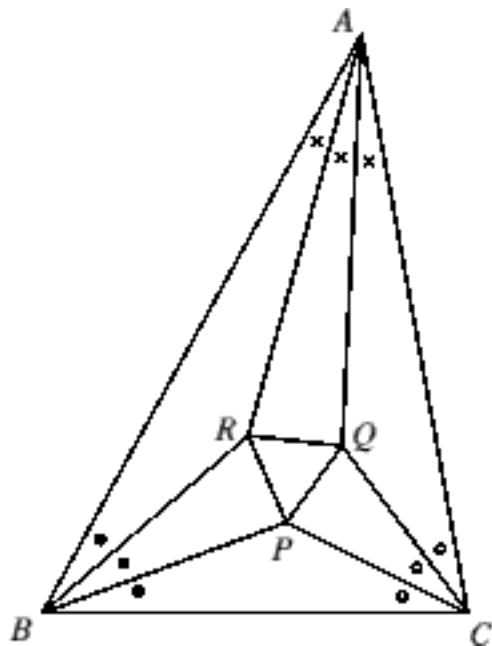


Figure 11: Morley's theorem

Theorem 10 (Morley's Theorem; 1899)

(i) $\triangle PQR$ is equilateral.

(ii) Each side of $\triangle PQR$ has length $8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}$, where R is the circumradius of $\triangle ABC$.

The formula for the side-length is reminiscent of those in Table 1. However, there is one new feature. You can bisect an angle using a straight edge and compasses but you cannot trisect an angle in this way. The reason must wait for another time.

Conclusion

We have explored a number of remarkable surprises which we have no right to expect.

Actually this is only the tip of the iceberg. For example, look again at Theorem 9. The 9-point circle of $\triangle ABC$ contains 4 more interesting points, where it touches the inscribed and escribed circles of $\triangle ABC$. Moreover, Theorem 7 tells us that we'll get another 4 interesting points from each of $\triangle AHB$, $\triangle BHC$ and $\triangle CHA$. So our 9-point circle now has 25 interesting points lying on it!

As for interesting points inside a triangle, G , H , I , N and O are a mere drop in the ocean. Find hundreds more 'triangle centres' at the website

<http://cedar.evansville.edu/~ck6>

which is lovingly maintained by Clark Kimberling.

The next time you see a triangle, just think of all that's going on inside it!