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Reference frames for Bell inequality violation in the presence of superselection rules

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Abstract. Superselection rules (SSRs) constrain the allowed states and operations in quantum theory. They limit preparations and measurements and hence impact our ability to observe non-locality, in particular the violation of Bell inequalities. We show that a reference frame compatible with a particle number SSR does not allow observers to violate a Bell inequality if and only if it is prepared using only local operations and classical communication. In particular, jointly prepared separable reference frames are sufficient for obtaining violations of a Bell inequality. We study the size and non-local properties of such reference frames using superselection-induced variance. These results suggest the need for experimental Bell tests in the presence of superselection.

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Contents

1. Introduction

Symmetries impose powerful constraints on physics, leading Wick *et al* to suggest that the associated conserved quantities lead to additional restrictions on quantum theory, the so-called superselection rules (SSRs) [1]. They conjectured that superselection prevents the existence of coherent superpositions of charge, for example. However, Aharonov and Susskind [2] showed that the ability to observe superpositions depends on having a shared reference frame relative to which the system can be prepared and measured. More generally, elements of quantum theory require reformulation in the presence of SSRs. Quantum entanglement and various forms of non-locality are particular examples of phenomena that are affected by the presence of SSRs, and a vast body of literature already exists on these topics [3]–[22].

Here, we focus on Bell inequality violation in the presence of SSRs. We concentrate on the bi-partite case where *both* the entangled principal system and any ancilla/reference frame are subject to particle-number superselection. In particular, we examine the role of the measurement apparatus or reference frame used by the two observers (our prototypical Alice and Bob). We find that reference frames prepared using only local operations that satisfy SSRs and classical communication (SSR-LOCC) *cannot* reveal the non-locality of an entangled system. However, jointly prepared but separable reference frames can be used to violate a Bell inequality with an entangled principal system. By imposing separability of the reference frame, we deduce that violation is due to the measured entangled state; the reference allows the observers to carry out measurements that lead to a violation of the Bell inequality. In such cases, the reference is said to activate Bell violation.

Previous work has explored the issue of Bell inequality violation in the presence of SSRs, given suitable reference frames [15]–[20]. In contrast to some previous works, here the reference frame is both explicitly separable and compliant with the SSRs. We show in general that all references prepared jointly, and only such references, can activate Bell violation. We find minimal reference frames and relate the degree of Bell violation to the 'non-locality' in the

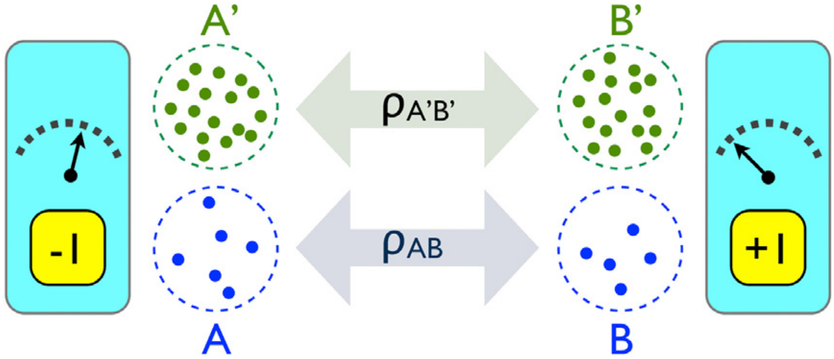

Figure 1. The Bell experiment in the presence of particle-number SSR. Two observers share in advance a reference system in a separable state $\rho_{A'B'}$. The experiment begins with an emission of a principal system in a state ρ_{AB}. According to the SSR both $\rho_{A'B'}$ and ρ_{AB} are incoherent mixtures of states with well-defined total number of particles. Alice (on the left) has now access to subsystems A and A' and similarly Bob (on the right) has access to B and B'. Both local subsystems are next measured by superselection-constrained observables, described by projections onto states with well-defined total number of particles in the subsystems. In a given experimental run, one of many observables is measured at each site, the choice of which is depicted by a tuneable knob (arrow) on the measuring device. Finally, one of many measurement results is obtained as depicted by a number on the yellow screen (e.g. -1 for Alice, $+1$ for Bob). In short, for this scenario a violation of a Bell inequality can be observed if and only if the reference state $\rho_{A'B'}$ is prepared jointly, and the more particles in the reference state the larger the violation.

reference as measured by superselection-induced variance (SIV) [5, 6]. This holds in particular for measurements of single-particle states and is a clear proof that such states can exhibit non-locality [7]–[18]. We also discuss related single-photon experiments [21, 22] and conclude that there is still the need for new experiments.

2. Scenario

We begin with the description of a Bell experiment in the presence of particle-number superselection. Consider the situation as shown in figure 1. A source distributes to Alice and Bob an entangled pure state of N particles,

$$|\psi\rangle_{\mathrm{AB}} = \sum_{n=0}^{N} c_n \,|n\rangle_{\mathrm{A}} \,|N-n\rangle_{\mathrm{B}}, \tag{1}$$

where $|c_n|^2$ is the probability that Alice finds that she has n particles. Under particle-number superselection, all states and measurements commute with the particle-number operator $\hat{N}$. Ordinarily, all that Alice and Bob can do is simply count the number of particles each receives. In order to do more than this, they share in advance a joint reference frame in the state $\rho_{A'B'}$, assumed to be separable and also obeying particle-number superselection. Therefore, before state (1) is distributed to Alice and Bob they share no entanglement. By making joint SSR respecting measurements on each of their respective halves of the entangled system and reference frame ($\{A, A'\}$ for Alice, $\{B, B'\}$ for Bob), they hope to be able to demonstrate a Bell inequality violation.

3. Reference frames prepared locally

We first show that reference frames prepared using only local operations satisfying SSRs and classical communication (SSR-LOCC) cannot activate violation of any Bell inequality. The proof is straightforward.

All such reference states commute with *local* particle-number operators and therefore are of the form

$$\rho_{\mathrm{A'B'}}^{\mathrm{SSR-LOCC}} = \sum_{k,l} p_{kl} \, |k\rangle_{\mathrm{A'}}\langle k| \otimes |l\rangle_{\mathrm{B'}}\langle l| \,, \tag{2}$$

where k (l) counts particles in the reference frame of Alice (Bob). These states contain only classical correlations between fixed local number of particles as measured by quantum discord and similar quantities [23]–[26].

Consider for the moment that the reference frame is in the pure state $|k\rangle_{\mathrm{A'}}|l\rangle_{\mathrm{B'}}$. We can express the initial joint state of the system and reference frame as $\sum_n c_n \, |n, k\rangle_{\mathrm{AA'}} \, |N - n, l\rangle_{\mathrm{BB'}}$ grouping subsystems accessible to Alice and Bob, respectively. Note that every term of this superposition contains a different number of local particles, i.e. $n + k$ for Alice and $N - n + l$ for Bob. As local SSR observant measurements project onto states with a well-defined number of local particles, only one term in the superposition can contribute to the probability of a corresponding result. This, however, is exactly the same as making the measurements on a state in a mixture of $|n, k\rangle_{\mathrm{AA'}}|N - n, l\rangle_{\mathrm{BB'}}$ with probability $|c_n|^2$, and this separable state clearly admits a local hidden variable model. If one replaces $|c_n|^2$ in this model with $|c_n|^2 p_{kl}$ all measurement results obtained with a general mixed reference state (2) are reproduced. Therefore, no Bell inequality can be violated.

4. Reference frames prepared jointly

We now show that all reference frames that cannot be prepared via SSR-LOCC are useful for Bell violation. We begin with the characterization of such references. We can express all reference states in the general form

$$\rho_{\mathrm{A'B'}} = \sum_{N'} p_{N'} \rho_{N'}, \tag{3}$$

where $p_{N'}$ is the probability of N' particles in the reference frame and $\rho_{N'}$ is any state with a fixed total number of particles N', i.e. $\rho_{A'B'}$ is an arbitrary mixture of pure states of the form $|\phi\rangle_{\mathrm{A'B'}} = \sum_{i=0}^{N'} r_i \, |i\rangle_{\mathrm{A'}} \, |N' - i\rangle_{\mathrm{B'}}$. Since we assume that $\rho_{\mathrm{A'B'}}$ cannot be prepared via SSR-LOCC, it necessarily contains off-diagonal elements in the particle-number basis. All such states have a non-vanishing expectation of

$$\mathcal{V} = \mathrm{Re}[\mathrm{Tr}(R_+ \otimes R_- \rho_{\mathrm{A'B'}})], \tag{4}$$

where $R_+ = \sum_{a=0}^{N-\Delta} |a+\Delta\rangle\langle a|$ and $R_- = \sum_{b=\Delta}^{N} |b-\Delta\rangle\langle b|$ for some $\Delta \geqslant 1$. To see this, note that $\mathcal{V}$ is proportional to the average value of the sum $\sum_{i=0}^{N'-\Delta} r_{i+\Delta}^* r_i$ over the pure states $|\phi\rangle_{A'B'}$ in the decomposition of $\rho_{\mathrm{A'B'}}$. Therefore, $\mathcal{V}$ vanishes if the sums vanish for all the pure states. For states $|\phi\rangle_{\mathrm{A'B'}}$ that have coherences in the particle-number basis, this only occurs if the signs of the products $r_{i+\Delta} * r_i$ alternate for some values of i leading to cancellation in the sum. In such a case however, we can always choose a larger Δ to skip the terms that lead to the cancellation. Thus, a state $\rho_{\mathrm{A'B'}}$ can be prepared via SSR-LOCC if and only if $\mathcal{V} = 0$ for all values of $\Delta \geqslant 1$.

As we now show, all reference states with the non-vanishing coherence parameter $\mathcal{V}$ enable observers to choose measurements that lead to a violation of the Clauser–Horne–Shimony–Holt (CHSH) inequality [27]. Consider an entangled principal system in the state

$$|\psi\rangle_{\mathrm{AB}} = \frac{1}{\sqrt{2}}(|2\rangle_{\mathrm{A}}\,|2+\Delta\rangle_{\mathrm{B}} + |2+\Delta\rangle_{\mathrm{A}}\,|2\rangle_{\mathrm{B}}). \tag{5}$$

We show in appendix that there always exist dichotomic local measurements compatible with the SSR whose outcomes on the joint state of the principal system and reference frame are correlated as

$$E(\alpha_k, \beta_l) = -\cos(2\alpha_k)\cos(2\beta_l) + \mathcal{V}\sin(2\alpha_k)\sin(2\beta_l), \tag{6}$$

where the angle α_k (β_l) parameterizes the kth (lth) setting of Alice (Bob). We insert this expression into the CHSH parameter

$$S \equiv E(\alpha_1, \beta_1) + E(\alpha_1, \beta_2) + E(\alpha_2, \beta_1) - E(\alpha_2, \beta_2), \tag{7}$$

and find values of α_k and β_l for which S is higher than the local realism bound $S \leqslant 2$. Namely, we choose $\alpha_1 = 0$, $\alpha_2 = \pi/4$, and parameterize the settings of Bob by a single angle $\beta \equiv \beta_1 = -\beta_2$, leading to

$$S = -2\cos(2\beta) + 2\mathcal{V}\sin(2\beta). \tag{8}$$

To find its maximum, note that S has the form of a scalar product between the vector $\vec{w} = (-2, 2\mathcal{V})$ and an arbitrary normalized vector $\vec{v} = (\cos(2\beta), \sin(2\beta))$. Therefore, there always exists an angle β such that $\vec{v}$ is parallel to $\vec{w}$ and the maximum of the scalar product is given by the length of $\vec{w}$:

$$S = 2\sqrt{1+\mathcal{V}^2}. \tag{9}$$

To summarize, all reference frames that cannot be prepared via SSR-LOCC have a non-vanishing coherence parameter $\mathcal{V}$ and consequently allow observers to carry out measurements on entangled states that lead to a violation of the CHSH inequality:

$$S > 2 \quad \text{for all } \mathcal{V} \neq 0. \tag{10}$$

An identical conclusion holds for the single-particle entangled principal system. The calculations are the same as long as the reference frame does not contain any vacuum.

5. Minimal separable reference frames

We have derived conditions for violating the CHSH inequality in the presence of particle-number SSR and we further study the properties of reference frames activating the violation. We show here that the minimal *separable* reference frame allowing violation contains two particles in total.

Relaxing for a moment the separability requirement, equation (4) shows that the minimal reference has only one particle in total. Namely, any state $|\phi\rangle_{A'B'} = r_0|0\rangle_{A'}|1\rangle_{B'} + r_1|1\rangle_{A'}|0\rangle_{B'}$ with $r_1 r_0 * \neq 0$ has a non-vanishing parameter $\mathcal{V}$. However, the application of the PPT criterion for entanglement [28, 29] reveals that $|\phi\rangle_{A'B'}$ is entangled for all $r_1 r_0 * \neq 0$. When entangled states are used as references, it is unclear whether the violation of a Bell inequality is due to the entanglement of the principal system or the reference frame.

For this reason we consider a reference frame with at most two particles:

$$\rho_{A'B'} = p_{00}\,|00\rangle_{A'B'}\langle 00| + p_{11}\,|11\rangle_{A'B'}\langle 11| + p_\phi\,|\phi\rangle_{A'B'}\langle\phi|\,, \tag{11}$$

where as before $|\phi\rangle_{A'B'} = r_0|0\rangle_{A'}|1\rangle_{B'} + r_1|1\rangle_{A'}|0\rangle_{B'}$. Since the definition of equation (4) involves only the real part of the off-diagonal elements, we choose the coefficients r_0 and r_1 to be real, i.e. $\mathcal{V} = p_\phi r_0 r_1$, and we have used $\Delta = 1$ in equation (4). The application of the PPT criterion reveals that the state $\rho_{A'B'}$ is separable if and only if

$$p_{00}\,p_{11} \geqslant p_\phi^2 r_0^2 r_1^2 = \mathcal{V}^2. \tag{12}$$

Therefore, for all separable reference frames activating the violation, $\mathcal{V} \neq 0$, there must be some mixture of the two-particle state ($p_{11} > 0$). Note that the same argument applies to p_{00} and one concludes that separable reference states of the form (11) enabling the violation must contain some vacuum. This is a consequence of the fact that an arbitrary mixture of any pure two-qubit entangled state with 'colored noise' $|11\rangle\langle 11|$ is always entangled [30, 31].

6. Local and global twirling

A useful mathematical tool that illustrates and generalizes the results presented so far is the twirling operation. Twirling $\mathcal{T}$ eliminates the coherences that are not compatible with SSR:

$$\mathcal{T}(\rho) \equiv \sum_n \Pi_n \rho \Pi_n, \tag{13}$$

where Π_n is a projector on a subspace with a fixed number of particles n. This operation describes the lack of a reference frame enabling access to the phase information of the probability amplitudes.

The usefulness of twirling is best illustrated by considering states that can be prepared via SSR-LOCC. A simple proof demonstrates that they cannot activate CHSH violation. A SSR-LOCC reference frame commutes with local particle number operators and therefore it is invariant under the action of local twirlings, $\rho_{A'B'}^{\mathrm{SSR-LOCC}} = (\mathcal{T}_{A'} \otimes \mathcal{T}_{B'})(\rho_{A'B'}^{\mathrm{SSR-LOCC}})$. This implies for the coherence parameter

$$\begin{aligned}\mathcal{V} \sim\ & \mathrm{Tr}\left\{R_+ \otimes R_- \rho_{A'B'}^{\mathrm{SSR-LOCC}}\right\}\\ =\ & \mathrm{Tr}\left\{(R_+ \otimes R_-)(\mathcal{T}_{A'} \otimes \mathcal{T}_{B'})\rho_{A'B'}^{\mathrm{SSR-LOCC}}\right\}\\ =\ & \mathrm{Tr}\left\{\mathcal{T}_{A'}(R_+) \otimes \mathcal{T}_{B'}(R_-)\rho_{A'B'}^{\mathrm{SSR-LOCC}}\right\} = 0,\end{aligned} \tag{14}$$

where the last equality follows from the fact that $\mathcal{T}_{A'B'}(R_\pm) = 0$ because $R_\pm$ contain only off-diagonal elements in the particle-number basis.

Note that the operator $R_+ \otimes R_-$ conserves total particle number and therefore is invariant under global twirling, $\mathcal{T}(R_+ \otimes R_-) = R_+ \otimes R_-$. All states satisfying SSR are of the form

$\rho^{\rm SSR}_{\rm A'B'} = \mathcal{T}(\rho_{\rm A'B'})$, where now $\rho_{\rm A'B'}$ need not have a fixed number of particles. Therefore,

$$\begin{aligned}\mathrm{Tr}\{R_+ \otimes R_- \rho^{\rm SSR}_{\rm A'B'}\} &= \mathrm{Tr}\{R_+ \otimes R_- \mathcal{T}(\rho_{\rm A'B'})\} = \mathrm{Tr}\{\mathcal{T}(R_+ \otimes R_-)\rho_{\rm A'B'}\} \\ &= \mathrm{Tr}\{R_+ \otimes R_- \rho_{\rm A'B'}\},\end{aligned} \tag{15}$$

where we have used the cyclic property of trace. This means that in order to calculate $\mathcal{V}$ for an SSR respecting reference frame $\rho^{\rm SSR}_{\rm A'B'}$, we can use in equation (4) any state $\rho_{\rm A'B'}$ whose twirling gives $\rho^{\rm SSR}_{\rm A'B'}$.

7. Separable reference for maximal violation

Twirling allows further study of separable reference frames. Note that the minimal reference frame of two particles we have derived in equation (11) activates the violation but does not lead to maximal violation. Indeed, the highest value of $\mathcal{V}$ for entangled reference states (11) is $\frac{1}{2}$ and for separable reference states equation (12) implies $\mathcal{V} \leqslant \frac{1}{4}$, whereas the maximal violation of the CHSH inequality allowed by quantum theory $S = 2\sqrt{2}$ [32] requires $\mathcal{V} = 1$. Note also that the Tsirelson bound of $S = 2\sqrt{2}$ implies that $|\mathcal{V}| \leqslant 1$.

Here we show that there are separable reference frames allowing maximal violation of the CHSH inequality with an entangled state. First note that all separable states satisfying SSR are of the form $\rho^{\rm SSR}_{\rm sep} = \mathcal{T}(\rho_{\rm sep})$, with $\rho_{\rm sep} = \sum_j p_j |a'_j\rangle\langle a'_j| \otimes |b'_j\rangle\langle b'_j|$ where $|a'_j\rangle|b'_j\rangle$ need not have a fixed number of particles. This follows from the fact that global twirling is an LOCC operation (but not SSR-LOCC) and as such cannot produce entanglement. We now derive the separable reference frames that maximize the coherence parameter $\mathcal{V}$. Since equation (4) is linear in $\rho_{\rm A'B'}$, $\mathcal{V}$ is maximal for a pure product state $|a'\rangle|b'\rangle$. Moreover, due to the fact that only the real part enters (4), it is sufficient to consider states with real coefficients $|a'\rangle = \sum_{n=0}^{N} \mathfrak{a}_n\, |n\rangle$ and $|b'\rangle = \sum_{m=0}^{M} \mathfrak{b}_m\, |m\rangle$ with $\mathfrak{a}_n, \mathfrak{b}_m \in \mathbb{R}$. For such pure states $\mathcal{V} = f_N g_M$ where

$$f_N \equiv \langle a'|R_+|a'\rangle = \sum_{n=0}^{N-1} \mathfrak{a}_n \mathfrak{a}_{n+1}, \tag{16}$$

$$g_M \equiv \langle b'|R_-|b'\rangle = \sum_{m=0}^{M-1} \mathfrak{b}_m \mathfrak{b}_{m+1}, \tag{17}$$

with N and M denoting the dimensionality of the reference frames of Alice and Bob, respectively, i.e. the maximal number of particles in their reference frames, and we put $\Delta = 1$. To find the maximum of $\mathcal{V}$, it is now sufficient to optimize f_N, because g_M has the same form and optimization over Bob's state is independent of that over Alice's. Note that for states with real coefficients $\langle a'|R_+|a'\rangle = \langle a'|R_-|a'\rangle$ and therefore $f_N = \frac{1}{2}\langle a'|\,(R_+ + R_-)\,|a'\rangle$. The only non-vanishing elements of matrix $R_+ + R_-$ are a strip of identities above and below its diagonal, and therefore it is a Hermitian matrix. The maximal value of f_N is attained for $|a'\rangle$ being the eigenvector of $R_+ + R_-$ with the highest eigenvalue. The amplitudes of the optimal state read

$$\mathfrak{a}_n = \sqrt{\frac{2}{N+2}} \sin\left(\frac{\pi(n+1)}{N+2}\right), \quad \text{with } n = 0, 1, \ldots, N, \tag{18}$$

and its maximal eigenvalue gives

$$\max f_N = \cos\left(\frac{\pi}{N+2}\right). \tag{19}$$

For small reference frames containing at most one particle on both sides $N = M = 1$, we find $\mathcal{V} \leqslant \frac{1}{4}$ in agreement with the results of section 5 on minimal reference frames. If the reference frames of Alice and Bob are both unbounded, equation (19) implies that $\mathcal{V} \to 1$ and the violation of the CHSH inequality approaches its maximum. Practically, for $N = M \approx 30$ particles in each reference frame, $\mathcal{V} \approx 0.99$.

The reference frames for violation of Bell inequality were also studied in [33] in the context of directional reference frames, finding that in the limit of unbounded reference frame, maximal violation can be achieved. We stress that in our case the corresponding limit is twofold: to maximally violate CHSH inequality the reference frame has to be prepared jointly, and it should be unbounded.

8. Non-locality of reference states

Let us now discuss the relation between $\mathcal{V}$ and the non-locality of reference frames as captured by SIV [5]. We show that violation of the CHSH inequality is a witness of non-zero SIV in the reference frame and that the amount of SIV in small reference frames gives an upper bound on the CHSH violation.

The SIV of a pure state $|\phi\rangle$ is defined as the variance of local number of particles

$$\tfrac{1}{4}V(\phi) \equiv \langle\phi|N_{\mathrm{A}}^2 \otimes I|\phi\rangle - \langle\phi|N_{\mathrm{A}} \otimes I|\phi\rangle^2. \tag{20}$$

The factor of 4 is introduced for normalization: one unit of SIV is defined for the state $\frac{1}{\sqrt{2}}(|n, n+1\rangle + |n+1, n\rangle)$. Since SIV is symmetric with respect to permutation of the parties, one can equally consider the variance of the local particle number on Bob's side (N_{B}). Pure states that have non-zero SIV cannot be prepared via SSR-LOCC and this is the property of reference frames we are interested in. However, such pure states are always entangled, whereas we insist on separability of the reference frame. Therefore, we must consider mixed states. Just like entanglement, for mixed states another measure of SIV is required. We shall use the variance of formation defined as [5]

$$V_{\mathrm{F}}^{\mathrm{SSR}}(\rho) = \min_{\{p_i, \phi_i\}} \sum_i p_i V(\phi_i), \tag{21}$$

where the ensembles of pure states $\{\phi_i\}$ obey SSRs. As a measure of the off-diagonal terms in the density matrix, we can consider $\mathcal{V}$, or equivalently Bell inequality violation, as a witness of non-zero SIV. Moreover, it was shown in [6] that for states (11) the variance of formation reads $V_{\mathrm{F}}^{\mathrm{SSR}}(\rho) \geqslant 4p_\phi^2 r_0^2 r_1^2 = 4\mathcal{V}^2$ with real r_0 and r_1. Accordingly, one can directly relate SIV to $\mathcal{V}$ as

$$|\mathcal{V}| \leqslant \frac{\sqrt{V_{\mathrm{F}}^{\mathrm{SSR}}(\rho)}}{2}. \tag{22}$$

Therefore, the corresponding states with vanishing SIV have also vanishing $\mathcal{V}$.

9. Experiments

Our last topic is the experimental verification of Bell inequality violations under SSRs and the need for new experiments. We relate this by commenting on current experiments related to the Bell inequality and single-photon non-locality [7, 21, 22]. Although not intended to violate a

Bell inequality under an SSR, these experiments may be seen as such for (an induced) photon-number SSR [19]. In the proposal of [7], a single photon is directed onto a balanced beam-splitter producing (it is hoped) a non-local state of one photon. In each output port of this first beam-splitter there is another balanced beam-splitter with a (reference) coherent state directed at its second input port (see figure 1 of [7]). One considers correlations between the number of photons registered in detectors placed in the output ports of the second set of beam-splitters.

The experimental realizations [21, 22] differ from the proposal [7] in that the secondary beam-splitters may be unbalanced. Note that in principle the possible measurement results are unbounded, and therefore the CHSH inequality cannot be applied. It turns out that the correlation functions violate the CHSH inequality only for a small mean number of photons in the coherent states, in which case the events of having many photons in the detectors are rare and the CHSH inequality becomes applicable. This, however, opens up an effective detection loophole that allows for a local hidden variable model.

Let us denote by r_a (r_b) the reflectivity of the first (second) beam-splitter supplied with a coherent state. The corresponding transmittances are: $t_n = 1 - r_n$ with $n = a, b$. It is assumed that both coherent states have the same mean number of photons $\bar{n}$ and relative phase $\varphi = \alpha - \beta$. The correlation function between the number of photons measured behind the two beam-splitters reads

$$E_{\varphi} = \frac{(r_a - t_a)(r_b - t_b)(\bar{n} - 1) + 4\sqrt{r_a r_b t_a t_b}\sin\varphi}{\bar{n} + 1}. \tag{23}$$

Using this expression in the CHSH parameter, one finds that the proposal of [7] is optimal in the sense that it is best to choose balanced beam-splitters $r_a = r_b = \frac{1}{2}$. Any other values of r_a and r_b lead to smaller values of the CHSH parameter. In particular, the assumption of [21] that after the beam-splitter a photon may have equal likelihood to have come from a single-photon 'beam' or a coherent state, i.e. $r\bar{n} = t$, leads to no violation for all values of $\bar{n}$. For the balanced beam-splitters the inequality is violated if $\bar{n} < \sqrt{2} - 1$, which translates into the critical probability of vacuum in the coherent state $P_{\text{vac}} \approx \frac{2}{3}$. Using such coherent states it is quite rare to measure two photons in a setup and one may utilize this effective detection loophole to explain the observed results with, e.g., the model of Gisin [34].

We therefore hope that this research will stimulate further experiments testing Bell violation in the presence of SSRs. Ideally one would use systems with natural SSRs such as massive particles or charges. However, studies of partial superselection can also be performed through controlled decoherence, as decoherence between different subspaces can be seen as a type of SSR.

10. Conclusions

We have studied the effects of restrictions imposed by SSRs on Bell inequality violation. We found that the violation primarily depends on how a reference frame is prepared and only secondarily on its size. Even unbounded reference frames do not lead to Bell violation if they are prepared via SSR-LOCC and therefore are strictly classically correlated according to quantum discord and similar measures. This condition was shown to be necessary and sufficient for the violation; that is, reference frames enable the violation of a Bell inequality if and only if they cannot be prepared via SSR-LOCC. The violation can be achieved with separable reference frames explicitly consistent with particle-number superselection, the minimal such reference containing up to two particles. We linked the violation to the amount of non-locality in the reference frame as captured by SIV. It would be interesting to study how other subfields of quantum theory, e.g. quantum tomography, are modified in the presence of superselection.

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Appendix. Derivation of correlation formula (6)

Assume for the moment that the reference system is in a pure state

$$|\phi\rangle_{A'B'} = \sum_{i=0}^{N'} r_i |i\rangle_{A'} |N' - i\rangle_{B'}. \tag{A.1}$$

Since it cannot be prepared via SSR-LOCC, we have $r^*_{i+\Delta} r_i \neq 0$ for some i and $\Delta \geqslant 1$.

Consider the principal system in a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|2\rangle_{\mathrm{A}}\,|2+\Delta\rangle_{\mathrm{B}} + |2+\Delta\rangle_{\mathrm{A}}\,|2\rangle_{\mathrm{B}})$ such that the initial state of the principal system and reference together reads

$$|\psi\phi\rangle = \sum_i \frac{r_i}{\sqrt{2}} (|2, i\rangle_{\mathrm{AA'}} |2+\Delta, N'-i\rangle_{\mathrm{BB'}} + |2+\Delta, i\rangle_{\mathrm{AA'}} |2, N'-i\rangle_{\mathrm{BB'}}), \tag{A.2}$$

where we grouped in kets subsystems accessible to Alice and Bob, respectively.

We present local dichotomic measurements compatible with particle-number SSR, which lead to the correlation function (6) of the main text. Alice measures a local observable parameterized by angle α:

$$\mathcal{A} = \sum_{a=-\Delta}^{N'} |\alpha(a)\rangle\langle\alpha(a)| - \sum_{a=-\Delta}^{N'} |\bar{\alpha}(a)\rangle\langle\bar{\alpha}(a)|\,, \tag{A.3}$$

where the eigenstates are defined as follows:

$$|\alpha(a)\rangle = \cos\alpha\,|1, a+\Delta+1\rangle_{\mathrm{AA'}} + \sin\alpha\,|2, a+\Delta\rangle_{\mathrm{AA'}}\,, \tag{A.4}$$

$$|\bar{\alpha}(a)\rangle = \sin\alpha\,|1, a+\Delta+1\rangle_{\mathrm{AA'}} - \cos\alpha\,|2, a+\Delta\rangle_{\mathrm{AA'}} \tag{A.5}$$

for $a = -\Delta, \ldots, -1$;

$$|\alpha(a)\rangle = \cos\alpha\,|2+\Delta, a\rangle_{\mathrm{AA'}} + \sin\alpha\,|2, a+\Delta\rangle_{\mathrm{AA'}}\,, \tag{A.6}$$

$$|\bar{\alpha}(a)\rangle = \sin\alpha\,|2+\Delta, a\rangle_{\mathrm{AA'}} - \cos\alpha\,|2, a+\Delta\rangle_{\mathrm{AA'}} \tag{A.7}$$

for $a = 0, \ldots, N' - \Delta$;

$$|\alpha(a)\rangle = \cos\alpha\,|2+\Delta, a\rangle_{\mathrm{AA'}} + \sin\alpha\,|3+\Delta, a-1\rangle_{\mathrm{AA'}}\,, \tag{A.8}$$

$$|\bar{\alpha}(a)\rangle = \sin\alpha\,|2+\Delta, a\rangle_{\mathrm{AA'}} - \cos\alpha\,|3+\Delta, a-1\rangle_{\mathrm{AA'}}\,, \tag{A.9}$$

for $a = N' - \Delta + 1, \ldots, N'$. These observables are compatible with the SSR because all the eigenstates contain a fixed total number of particles $2 + a + \Delta$. The reason behind the three different cases is that the reference subsystem cannot contain more than N' particles and less than zero. They are also chosen to form an orthonormal set of vectors. To obtain the observables of Bob one just needs to replace $A \to B$, $A' \to B'$, $\alpha \to \beta$ and $a \to b$.

We reverse the equations for the eigenvectors and write the initial state of the principal system and the reference as

$$|\psi\phi\rangle = \sum_i \frac{r_i}{\sqrt{2}}\{(\sin\alpha|\alpha_{i-\Delta}\rangle - \cos\alpha|\bar{\alpha}_{i-\Delta}\rangle)(\cos\beta|\beta_{N'-i}\rangle + \sin\beta|\bar{\beta}_{N'-i}\rangle) + (\cos\alpha|\alpha_i\rangle + \sin\alpha|\bar{\alpha}_i\rangle)(\sin\beta|\beta_{N'-i-\Delta}\rangle - \cos\beta|\bar{\beta}_{N'-i-\Delta}\rangle)\}. \tag{A.10}$$

The probabilities of the results corresponding to different eigenvectors are

$$P_{ab} \equiv |\langle\alpha(a)\beta(b)|\psi\phi\rangle|^2 = \tfrac{1}{2}|r_{a+\Delta}\sin\alpha\cos\beta + r_a\cos\alpha\sin\beta|^2\delta_{b,N'-a-\Delta},$$
$$P_{a\bar{b}} \equiv |\langle\alpha(a)\bar{\beta}(b)|\psi\phi\rangle|^2 = \tfrac{1}{2}|r_{a+\Delta}\sin\alpha\sin\beta - r_a\cos\alpha\cos\beta|^2\delta_{b,N'-a-\Delta},$$
$$P_{\bar{a}b} \equiv |\langle\bar{\alpha}(a)\beta(b)|\psi\phi\rangle|^2 = \tfrac{1}{2}|-r_{a+\Delta}\cos\alpha\cos\beta + r_a\sin\alpha\sin\beta|^2\delta_{b,N'-a-\Delta},$$
$$P_{\bar{a}\bar{b}} \equiv |\langle\bar{\alpha}(a)\bar{\beta}(b)|\psi\phi\rangle|^2 = \tfrac{1}{2}|-r_{a+\Delta}\cos\alpha\sin\beta - r_a\sin\alpha\cos\beta|^2\delta_{b,N'-a-\Delta}.$$

Note that for every a there is only one b for which the corresponding probability does not vanish, and it is easy to verify that indeed $\sum_{a,b=-\Delta}^{N'}(P_{ab} + P_{a\bar{b}} + P_{\bar{a}b} + P_{\bar{a}\bar{b}}) = 1$. Finally, the correlation function is the average of the product of dichotomic local results

$$E_\phi(\alpha,\beta) = \sum_{a,b=-\Delta}^{N}(P_{ab} + P_{\bar{a}\bar{b}} - P_{\bar{a}b} - P_{a\bar{b}}). \tag{A.11}$$

Plugging in the formulae for the probabilities, we obtain

$$E_\phi(\alpha,\beta) = -\cos(2\alpha)\cos(2\beta) + \mathcal{V}\sin(2\alpha)\sin(2\beta), \tag{A.12}$$

where $\mathcal{V} = \sum_{a=0}^{N'-\Delta}\mathrm{Re}(r^*_{a+\Delta}r_a)$. Alternatively $\mathcal{V}$ can be expressed using operators $R_+ = \sum_{a=0}^{N'-\Delta}|a+\Delta\rangle\langle a|$ and $R_- = \sum_{b=\Delta}^{N'}|b-\Delta\rangle\langle b|$ with the help of which $\mathcal{V} = \mathrm{Re}(\langle\phi|\,R_+\otimes R_-\,|\phi\rangle)$. This calculation holds for *arbitrary* pure state $|\phi\rangle_{A'B'}$ of the reference. Therefore, for the reference in an arbitrary mixed state $\rho_{A'B'} = \sum_\phi p_\phi\,|\phi\rangle_{A'B'}\langle\phi|$ the correlations formula reads

$$E(\alpha,\beta) = \sum_\phi p_\phi E_\phi(\alpha,\beta), \tag{A.13}$$

and therefore it is of the same form as equation (A.12), but with the modified coherence parameter

$$\mathcal{V} = \mathrm{Re}[\mathrm{Tr}(R_+\otimes R_-\rho_{A'B'})]. \tag{A.14}$$

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