On 132-representable graphs

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Abstract. A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. Word-representable graphs are the subject of a long research line in the literature, and they are the main focus in the recently published book "Words and Graphs" by Kitaev and Lozin. A word $w = w_1 \cdots w_n$ avoids the pattern 132 if there are no $1 \leq i_1 < i_2 < i_3 \leq n$ such that $w_{i_1} < w_{i_3} < w_{i_2}$. The theory of patterns in words and permutations is a fast growing area.

A recently suggested research direction is in merging the theories of word-representable graphs and patterns in words. Namely, given a class of pattern-avoiding words, can we describe the class of graphs represented by the words? We say that a graph is 132-representable if it can be represented by a 132-avoiding word. We show that each 132-representable graph is necessarily a circle graph. Also, we show that any tree and any cycle graph are 132-representable. Finally, we provide explicit 132-avoiding representations for all graphs on at most five vertices, and also describe all such representations, and enumerate them, for complete graphs.

Keywords: word-representable graph; pattern-avoiding word; circle graph; tree; cycle graph; complete graph

1 Introduction

Suppose that w is a word over some alphabet and x and y are two distinct letters in w. We say that x and y *alternate* in w if after deleting in w *all* letters *but* the copies of x and y we either obtain a word $xyxy\cdots$ (of even or odd length) or a word $yxyx\cdots$ (of even or odd length). For example, in the word 23125413241362, the letters 2 and 3 alternate. So do the letters 5 and 6, while the letters 1 and 3 do *not* alternate.

A graph $G=(V,E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. For example, the graph to the right in Figure 2.3 is word-representable and one of words representing it is *bcdad*. Some graphs are word-representable, others are not, and the minimum non-word-representable graph is the wheel W_5 shown to the left in Figure 2.2.

Word-representable graphs are the subject of a long line of research in the literature initiated in [7] by Kitaev and Pyatkin, and they are the main focus in the recently published book [6] by Kitaev and Lozin. A general program of research suggested in [6, p. 183] takes as the input a language defined, for example, through pattern-avoiding words, and outputs a description of the class of graphs represented by the language. For instance, as is discussed in [6, p. 183], the set of weakly increasing words (those *avoiding the pattern* 21) defines graphs whose vertices can be partitioned into a clique and an independent set, so that no edge connects the clique and the independent set.

In this paper, we study graphs defined by 132-avoiding words. Our research merges the theories of word-representable graphs [6] and patterns in words [2, 4], the latter being a very fast growing area. A word $w=w_1w_2\cdots w_n$ avoids the pattern 132 (resp., 123) if there are no indices $1\leq i_1<i_2<i_3\leq n$ such that $w_{i_1}<w_{i_3}<w_{i_2}$ (resp., $w_{i_1}<w_{i_2}<w_{i_3}$). We say that a graph G is 132-representable (resp., 123-representable) if there is a 132-avoiding (resp., 123-avoiding) word representing it. Note that for the last definition to make sense, labels of graphs are supposed to be taken from a totally ordered set. Also, when trying to 132-represent (123-represent) a graph, we are allowed to label the graph in any suitable way[1].

[1]There is no issue with labelling when considering word-representable graphs, since all labelings are equally good or bad. However, in the contexts when there is an order on labels, labelling graphs in a proper way may be essential for finding a representation.

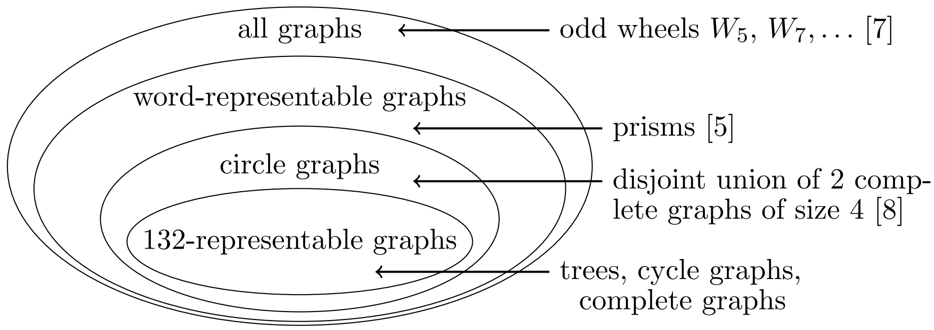

Figure 1.1: The place of 132-representable graphs in a hierarchy of graph classes

One of the main results in this paper is in showing that any 132-representable graph is necessarily a circle graph. A result in [8] shows that 132-representable graphs are a strict subset of circle graphs. Also, we show that trees, cycle graphs and complete graphs are 132-representable. Thus, the place of 132-representable graphs in a hierarchy of graph classes is as shown in Figure 1.1, where we also indicate known facts that odd wheels are non-word-representable [7], while prisms are word-representable but not circle graphs [5]. Interestingly, the studies in [8] show that the class of 123-representable graphs, being different from the class of 132-representable graphs, is also a proper subclass of circle graphs, even though not all trees are 123-representable; all cycle graphs and complete graphs are 123-representable.

One should compare our results with the results on 12-*representable graphs* obtained in [3]. These graphs are an instance of u-representable graphs, a far reaching generalization of word-representable graphs, also introduced in [3], where u is a word over $\{1, 2\}$ different from $22 \cdots 2$. Similarly to the case of 132-representable graphs, labelling of graphs is important for 12-representation. A word w 12-represents a graph G, if for any labels x and y, $x < y$, xy is an edge in G if and only if after removing all letters in w but x and y, we will obtain a word of the form $yy \cdots yxx \cdots x$. Note that the notions of 132-representable graphs and 12-representable graphs are not directly related (in the former case the pattern is used to give a condition on words representing graphs, while in the latter case the pattern is used to define the representation itself). It was shown in [3] that any 12-representable graph is necessarily a comparability graph, while very few trees (called *double caterpillars*) and almost no cycle graphs (only cycle graphs on at most four vertices) are 12-representable.

This paper is organized as follows. In Section 2 we give necessary definitions, notation and results to be used in the paper. In Section 3 we derive a key property of words 132-representing graphs (see Theorem 3.4) and state its corollary, the main result in this paper, that any 132-representable graph is necessarily a circle graph (see Corollary 3.5). In Section 4 we not only establish 132-representability of trees and cycle graphs, but also describe and enumerate all 132-representants for complete graphs. Moreover, in Section 4 we discuss non-132-representable graphs and give explicit 132-representation of graphs on four and five vertices. Finally, in Section 5 we state a number of suggestions for further research.

2 Preliminaries

Graphs. We will now review a number of basic notions/notations in graph theory. In this paper, we deal with *simple graphs*, that is, graphs with no *loops* and no *multiple edges*.

The *degree* $d(v)$ of a vertex v in a graph G is the number of edges of G incident with v. The *complete graph* on n vertices is denoted by K_n. A *cycle graph* C_n is the graph on n vertices that consists of a single cycle. A *wheel graph* W_n is the graph on $n+1$ vertices obtained from C_n by adding an all-adjacent vertex (*apex*). The wheel graph W_5 is shown to the left in Figure 2.2.

A *prism* Pr_n is a graph consisting of two cycles $12\cdots n$ and $1'2'\cdots n'$, where $n \geq 3$, connected by the edges ii' for $i = 1, \ldots, n$. For example, Pr_4, also known as the *three-dimensional cube*, is shown to the right in Figure 2.2.

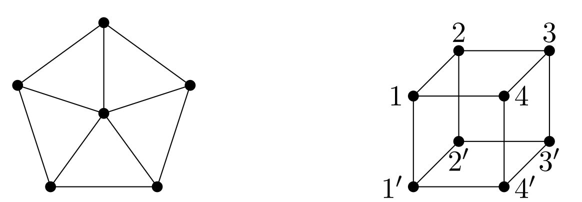

Figure 2.2: The wheel graph W_5 and the prism Pr_4

Finally, a *circle graph* is a graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords intersect. See Figure 2.3 for an example of a circle graph.

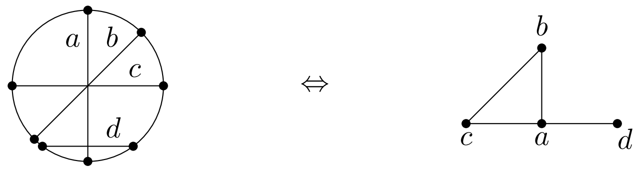

Figure 2.3: A circle with four chords and the corresponding circle graph

Words and permutations. For a finite word w, let $A(w)$ denote the set of letters occurring in w, and $\mathrm{red}(w)$ denote the word over $\{1, 2, \ldots, |A(w)|\}$ obtained by replacing the i-th smallest letter(s) by i. We call $\mathrm{red}(w)$ the *reduced form* of w. Also, for any $x \in A(w)$, let $n_w(x)$ denote the number of copies of x in w, and x_i denote the i-th occurrence of x in w from left to right. For example, if $w = 14661476212$, then $A(w) = \{1, 2, 4, 6, 7\}$, $\mathrm{red}(w) = 13441354212$, and say for $x = 6$, $n_w(6) = 3$. A word w is k-*uniform* if each letter in w occurs exactly k times.

Suppose that x and y are two distinct letters in $A(w)$. As is defined above, we say that x and y *alternate* in w if after deleting in w all letters but the copies of x and y we either obtain a word $xyxy\cdots$ (of even or odd length) or a word $yxyx\cdots$ (of even or odd length). In particular, if w has a single occurrence of x and a single occurrence of y, then x and y alternate in w.

A word or permutation $w = w_1w_2\cdots w_n$ *avoids* the pattern 132 if there are no indices $1 \leq i_1 < i_2 < i_3 \leq n$ such that $w_{i_1} < w_{i_3} < w_{i_2}$. For example, the word 31458 avoids the pattern 132, while 3474 is not 132-avoiding (the subsequence 374 in this word forms the pattern 132). It is a well-known fact (e.g. see [4, p. 32]) that the number of 132-avoiding permutations of length n is given by the *n-th Catalan number* $c_n = \frac{1}{n+1}\binom{2n}{n}$.

A subword of w formed by consecutive letters is called a *factor* of w. For example, 6651 and 41 are factors of 26651141. Finally, we let $[n] = \{1, 2, \ldots, n\}$.

Word-representable graphs. A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet $A(w) = V$ such that x and y alternate in w if and only if $xy \in E$ for each $x \neq y$ (that is, x and y are connected by an edge). In this context, we say that w *represents* G and w is a *word-representant* for G.

In this paper we assume that elements in V come from a totally ordered alphabet, which is important for the following definition. A word-representable graph G is 132-*representable* if, possibly after relabelling the graph, there exists a 132-avoiding word w that represents G. In this context, w is called a 132-*representant* for G.

For example, if $w = 43451251$, then the subword induced by the letters 1 and 2 is 121, and hence the letters 1 and 2 alternate in w, so that the respective vertices are connected in G. On the other hand, the letters 1 and 3 do not alternate in w, because removing all other letters we obtain 311; thus, 1 and 3 are not connected in G. Figure 2.4 shows the graph represented by w. Moreover, since w is 132-avoiding, G is 132-representable and w is a 132-representant of G.

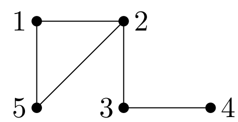

Figure 2.4: A 132-representable graph G

We note that labelling of a graph is important when dealing with 132-representation, which is not the case with just word-representation since all labellings are equally good or bad. For example, the fact that the (unlabelled) graph A in Figure 2.5 is 132-representable is given by the labelled version B of it and the 132-avoiding word 43212341. However, if we would label A to obtain the graph C in Figure 2.5, then no 132-avoiding representation of it exists. Indeed, suppose that a 132-representant w for C exists. Then at least two letters in $\{1, 2, 3\}$, say x and y, $x < y$, must be repeated at least twice in w, or else there would be at least one unwanted edge in $\{12, 13, 23\}$. Further, because 4 is an apex, there are x's and y's on both sides of a 4 in w (the 4 must alternate with x and y), which leads to an occurrence $x4y$ of the pattern 132; contradiction.

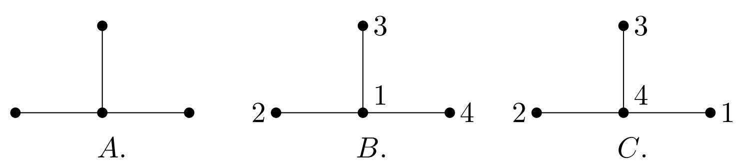

Figure 2.5: Significance of proper labelling

The following result is of special importance to us.

Theorem 2.1 ([1]). *A graph G is word-representable and its representation requires at most two copies of each letter if and only if G is a circle graph.*

Another relevant result is as follows.

Theorem 2.2 ([5]). *Prisms cannot be represented using at most two copies of each letter, but can be represented using at most three copies of each letter.*

3 132-representants

In this section, we discuss some properties of 132-representants.

We first present a simple, but useful theorem.

Theorem 3.1. *Let G be a 132-representable graph, and x be a vertex in G such that $d(x) \geq 2$. Then for any 132-representant w of G, we have $n_w(x) \leq 2$.*

Proof. Since $d(x) \geq 2$, there exist vertices a and b, $a > b$, in G that are adjacent with x.

Suppose that there are at least three copies of x in w. Then by the definition of a 132-representant, there exists a subsequence xw_1xw_2x in w, where for $i = 1, 2$, w_i is a factor of w containing exactly one a, one b, and no x. There are three cases to consider, all of which contradict the requirement that w is 132-avoiding:

- $x > a > b$: bxa is a 132 pattern in w where $b \in w_1$ and $a \in w_2$;
- $a > b > x$: xab is a 132 pattern in w where $a \in w_1$ and $b \in w_2$;
- $a > x > b$: bax is a 132 pattern in w where $b \in w_1$ and $a \in w_2$.

Hence, at most two copies of x can appear in w. ∎

As consequences of Theorem 3.1, we obtain the following results.

Corollary 3.2. *If each vertex in a graph G is of degree at least 2, then any 132-representant for G is of length at most $2n$.*

Corollary 3.3. *Let w be a 132-representant for a graph G. If $d(x) = 1$ and the vertex a connected to x has degree at least 2, then x occurs at most three times in w.*

Proof. Let w denote a 132-representant for G. Since $d(a) \geq 2$, by Theorem 3.1 a occurs at most twice in w. Combining with the fact that a and x alternate in w, we have that x occurs at most three times in w. ∎

The following theorem generalizes Theorem 3.1.

Theorem 3.4. *If a graph G is 132-representable, then there exists a 132-avoiding word w representing G such that for any letter x in w, $n_w(x) \leq 2$.*

Proof. Let w be a 132-representant for G. If all the vertices in G have degree at least 2, then by Theorem 3.1 every letter appears in w at most twice. Hence it suffices to consider the case where there exists a vertex x in G such that $d(x) = 1$. Let a be the vertex connected to x. We consider two cases.

- $d(a) \geq 2$. By Corollary 3.3, the letter x occurs at most three times in w. To prove the theorem, we assume that there are three copies of x in w and then we will construct a new 132-avoiding word w' which also represents G but contains only two copies of x. By Theorem 3.1, there are exactly two copies of a in w. In what follows, according to our notation, x_i denotes the i-th x and a_j the j-th a in w from left to right, where $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

  Suppose that $a > x$. If there are no letters between the a's except for x then a is connected only to x in G; contradiction with $d(a) \geq 2$. Thus there is a letter $b \neq x$ between a_1 and a_2 in w. If $b > a > x$, then $x_1 b a_2$ will be the pattern 132; if $a > b > x$, then $x_1 a_1 b$ will form the pattern 132; if $a > x > b$, then $b a_2 x_3$ will form the pattern 132; in either case, there is a contradiction with the definition of w. Thus we must have $a < x$.

  We next construct a new 132-avoiding word w' from w. Since there is no element t smaller than a to the left of a_1 in w (or else, $t x_2 a_2$ would be the 132-pattern), we obtain that a is a left-to-right minimum in w (that is, no letter to the left of a is less than a). We delete all three x's and replace a_1 by the factor $a^+ a_1 a^+$ to obtain the new word w', where $a < a^+ < a + 1$. By construction of w', if it contains an occurrence of the pattern 132 then this occurrence cannot involve a^+ and thus it would give an occurrence of the pattern in w; contradiction. Moreover, a is the only letter in w' alternating with a^+, and thus w' 132-represents G' obtained from G by replacing the label x by a^+.

- $d(a) = 1$, which means that the edge xa is disconnected from the rest of the graph. Let w' denote the word obtained from w by deleting a and x. Clearly, w' is 132-avoiding. But then the 132-avoiding word $n(n-1)n(n-1)w'$, where n and $n-1$ are larger than any other letter in $A(w')$, represents the graph G' obtained from G by replacing the labels a and x by n and $n-1$ (in any order).

We can repeat the procedure described above for any other vertices of degree 1 in G to obtain the desired result. ∎

One of the main results in this paper is the following statement.

Corollary 3.5. *Any* 132*-representable graph is a circle graph.*

Proof. Let G be a 132-representable graph. By Theorem 3.4, there exists a 132-representant w of G that contains at most two copies of each letter. By Theorem 2.1 G is a circle graph. ∎

Note that we do not know whether each circle graph is 132-representable or not.

4 132-representable graphs

In this section, we will show that trees, cycles, and complete graphs are 132-representable.

4.1 Trees and cycle graphs

Theorem 4.1. *Trees are* 132*-representable.*

Proof. We proceed by induction on the number of vertices with an additional condition. The tree with only one vertex can be represented by 1. Suppose that we can represent a tree with less than n vertices by a 132-avoiding word and the label of the root has only one occurrence and the label of the non-root vertex has exactly two occurrences in the corresponding word.

Given a tree T with n vertices, label it in pre-order, that is, starting from the root traverse the subtrees from left to right recursively. See the graph to the left in Figure 4.6 for an example. Suppose that the root has r children, which means that T has r subtrees, whose roots are children of the root of

T. Denote the r trees by T_i for $1 \leq i \leq r$ from left to right and suppose that the root of T_i is labeled by n_i. Note that $2 \leq n_1 < n_2 < \cdots < n_r \leq n$, so that for $1 \leq i \leq r$, T_i has $n_{i+1} - n_i$ vertices, where $n_{r+1} = n + 1$. Hence T_i is a tree having less than n vertices. By induction hypothesis, T_i is 132-representable and it can be represented by a 132-avoiding word $w(T_i)$ with only one copy of n_i and two copies of any other letter. Let $w = w(T_r)w(T_{r-1}) \cdots w(T_1)1n_1n_2 \cdots n_r$. It is easy to see that w represents T, and in particular, the root labeled by 1 is only connected to its children. Moreover, since for $1 \leq i < j \leq r$ the labels of T_i are smaller that the labels of $w(T_j)$, we get that w is 132-avoiding. We are done. ∎

Example 4.2. Let T be a tree as follows. It is clearly that T has three subtrees T_1, T_2 and T_3. By Theorem 4.1, there is $w(T_2) = 5$. Moreover, we have $w(T_1) = 43234$ and $w(T_3) = 87678$, which can be obtained by applying the inductive argument again. Hence $w(T) = 87678.5.43234.1256$, where the dots showing parts of $w(T)$ should be ignored. It is obvious that $w(T)$ is 132-avoiding and it represents T.

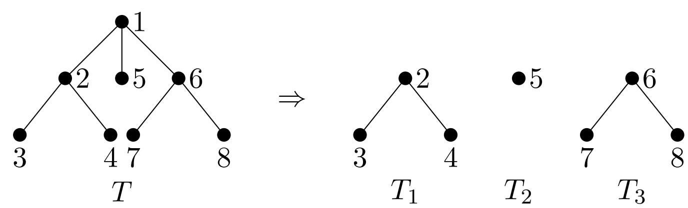

Figure 4.6: A tree T of size 8 and its subtrees

Theorem 4.3. *Cycle graphs C_n are 132-representable.*

Proof. Let $n \geq 3$. A path graph P_n (see Figure 4.7) is a tree, and, by the proof of Theorem 4.1, it can be represented by the 132-avoiding word

$$w = n(n-1)n(n-2)(n-1)(n-3)(n-2) \cdots 45342312.$$

Let w' be the word obtained from w by deleting the first n in w. Then it is easy to see that w' represents C_n. ∎

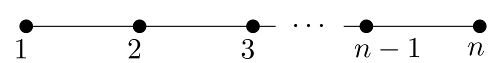

Figure 4.7: A path graph P_n

Example 4.4. 132-representants for C_4 and C_5, based on Corollary 4.3, are given in Figure 4.8.

Figure 4.8: 132-representants for C_4 and C_5

4.2 Complete graphs

In the following theorem we shall describe and enumerate all 132-representants for K_n.

Theorem 4.5. *For $n \geq 1$, a complete graph K_n is 132-representable. Moreover, for $n \geq 3$, there are*

$$2 + c_{n-2} + \sum_{i=0}^{n} c_i$$

different 132-representants for K_n, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. Finally, K_1 can be represented by a word of the form $11\cdots1$ and K_2 by a word of the form $1212\cdots$ (of even or odd length) or $2121\cdots$ (of even or odd length).

Proof. Clearly, K_1 can only be represented by a word of the form $11\cdots1$, and K_2 can only be represented by a word of the form $1212\cdots$ (of even or odd length) or $2121\cdots$ (of even or odd length). Each of these words is 132-avoiding.

Let $n \geq 3$. Suppose that w is a 132-representant for K_n. According to the definition of a complete graph, for any $1 \leq i < j \leq n$, we have that i and j alternate in w. Since $d(n) \geq 2$, by Theorem 3.1, there are two cases to consider.

Case 1. There are exactly two copies of n in w, and $w = w_1 n w_2 n w_3$, where w_k is a word over $[n-1]$ for $k = 1, 2, 3$. Since for $1 \leq i \leq n-1$, i and n

alternate in w, there is exactly 1 copy of i in w_2, which means that w_2 is in fact a permutation of length $n-1$. Moreover, for $1 \leq i \leq n-2$, i must not appear in w_1, or $i, n, n-1$ will form the pattern 132. Thus, $w_1 = n-1$ or $w_1 = \epsilon$, the empty word. Similarly, we have that $w_3 = 1$ or $w_3 = \epsilon$. Thus, there are four subcases to consider and in each subcase, we just need to consider the form of w_2.

Subcase 1.1. $w_1 = n-1$ and $w_3 = 1$. Thus 1 is to the left of $n-1$ in w_2, since 1 and $n-1$ alternate in w. For $2 \leq i \leq n-2$, i must be between 1 and $n-1$ in w_2 since i alternates with 1 and $n-1$. Moreover, for $2 \leq i < j \leq n-2$, they are in increasing order in w_2, or $1, j, i$ will form a 132 pattern. Hence, we obtain that $w = (n-1)nw'1$ where w' is the increasing permutation $12 \cdots n$, and this case contributes one representation.

Subcase 1.2. $w_1 = n-1$ and $w_3 = \epsilon$. For $1 \leq i \leq n-2$, i is to the left of $n-1$ in w_2, since i and $n-1$ alternate in w. Hence $w = (n-1)nw'(n-1)n$ where w' is any 132-avoiding permutation over $[n-2]$. Thus, this case contributes c_{n-2} representations.

Subcase 1.3. $w_1 = \epsilon$ and $w_3 = 1$. Similarly to the Subcases 1.1 and 1.2, we obtain that $w = nw'1$ where w' is the increasing permutation $12 \cdots n$, and this case contributes one representation.

Subcase 1.4. $w_1 = \epsilon$ and $w_3 = \epsilon$. Here, $w = nw_2n$ where w_2 is a 132-avoiding permutation over $[n-1]$. Thus, this case contributes c_{n-1} representations.

Case 2. There is only one copy of n in w. For $1 \leq i < j \leq n-1$, suppose that there are exactly two copies of i and j in w (by Theorem 3.1 there can be at most two copies of each letter). Since K_n is a complete graph, we have that n lies between i_1 and i_2 in w, and n also lies between j_1 and j_2 in w, where recall that, e.g. i_1 and i_2 denote the first and the second occurrences of i, respectively, in the word. Then i_1, n, j_2 will form the pattern 132; contradiction. Using Theorem 3.1, there are two subcases to consider.

Subcase 2.1. Every element in $A(w)$ has only one occurrence in w. Thus, w is a 132-avoiding permutation over $\{1, 2, \ldots, n\}$. Thus, this case contributes c_n representations.

Subcase 2.2. There is only one letter i, $1 \leq i \leq n-1$, in $A(w)$ that occurs twice in w. Any letter in $A(w)$ distinct from i must lie between i_1 and i_2 in w. Since w is 132-avoiding, we obtain that $w = i(i+1) \cdots nw'i$ where

w' is any 132-avoiding permutation over $\{1, 2, \ldots, i-1\}$. Thus, this case contributes $\sum_{i=1}^{n-1} c_{i-1} = \sum_{i=0}^{n-2} c_i$ representations. ∎

By Theorem 4.5, the initial values for the number of 132-representants for K_n, starting from $n = 3$, are

$$12, 27, 72, 213, 670, 2190, 7349, 25146, 87364, 307310, 1092200, 3915866, \ldots.$$

Example 4.6. For $n = 3$, we can see that all 12 132-representants for K_3, ordered as in the proof of Theorem 4.5, are 231231; 23123; 31231; 3123, 3213; 123, 231, 213, 312, 321; 1231, 2312.

A direct corollary of Theorem 4.5 is the following statement.

Corollary 4.7. *For $n \geq 3$ and a 132-representant w for K_n, the length of w is either n, or $n+1$, or $n+2$, or $n+3$.*

4.3 Non-132-representable graphs and 132-representation of small graphs

Each non-word-representable graph is clearly non-132-representable. In this subsection we will show that the minimum (with respect to the number of vertices) non-word-representable graph, the wheel graph W_5 given in Figure 2.2, is actually a minimum non-132-representable graph. We do not know whether there exist other non-132-representable graphs on six vertices (no other non-word-representable graphs on six vertices exist). As for non-132-representable but word-representable graphs, an example of those is prisms Pr_n, where $n \geq 3$. The latter follows from Theorems 2.2 and 3.4.

We note that the complement of a 132-representable graph is not necessarily a 132-representable graph. Indeed, for example, the 132-avoiding word 6645342312 defines a 132-representable graph, which is disjoint union of a cycle and the isolated vertex 6. However, the complement of this graph is the wheel graph W_5, which is not word-representable.

The following lemma allows us to restrict ourselves to considering graphs without isolated vertices when studying 132-representation.

Lemma 4.8. *Let G be a graph and G' be a graph obtained from G by adding an isolated vertex. Then G is 132-representable if and only if G' is 132-representable.*

Proof. If G' is 132-represented by w then removing from w the letter corresponding to the isolated vertex we obtain a word 132-representing G.

Conversely, suppose that G is 132-represented by w and n is larger than any letter in w. Then we label the isolated vertex by n and note that the word nnw 132-represents G'. ∎

Lemma 4.8 cannot be generalized to adding to a graph a new connected 132-representable component instead of an isolated vertex. This follows from the fact established in [8] that disjoint union of two complete graphs K_4 is non-132-representable, while K_4 is 132-representable. However, such a generalization can be done in a special case as recorded in the following simple, but useful lemma.

Lemma 4.9. *Let G_1, $G_2, \ldots, G_k$ be connected components of a graph G that can be* 132*-represented by* 2*-uniform words w_1, $w_2, \ldots, w_k$, respectively. Then G is* 132*-representable (by a* 2*-uniform word).*

Proof. For $1 \leq i \leq k$, let $a_i = |A(w_i)|$ denote the number of vertices in G_i, and let $\mathrm{red}^*(w_i)$ denote the word obtained from $\mathrm{red}(w_i)$ by replacing each element j, $1 \leq j \leq a_i$, by $j + \sum_{m=1}^{i-1} a_m$. Then the 2-uniform word

$$w = \mathrm{red}^*(w_k)\mathrm{red}^*(w_{k-1}) \cdots \mathrm{red}^*(w_1)$$

132-represents G. ∎

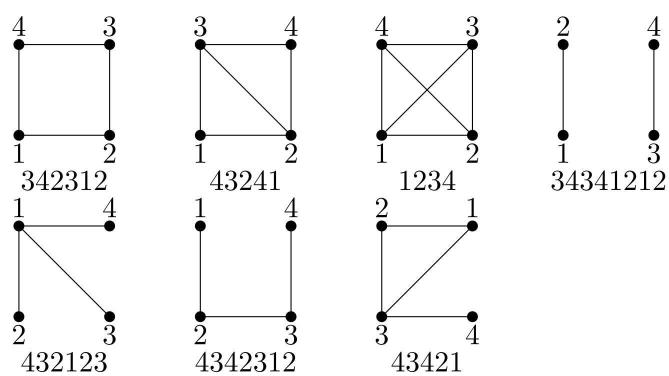

Figure 4.9: 132-representants for graphs on four vertices

By Lemma 4.8, we exclude isolated vertices from our considerations in the rest of this subsection. Moreover, graphs on up to three vertices are either trees or the cycle graph C_3, and thus they are 132-representable.

Further, there are seven graphs on four vertices which can be 132-represented as shown in Figure 4.9. Finally, there are 23 graphs on five vertices that have no isolated vertices, and these graphs can be 132-represented as in Figure 4.10. Note that Lemma 4.9 was used (in a straightforward way) to 132-represent graphs in Figures 4.9 and 4.10 that have two connected components.

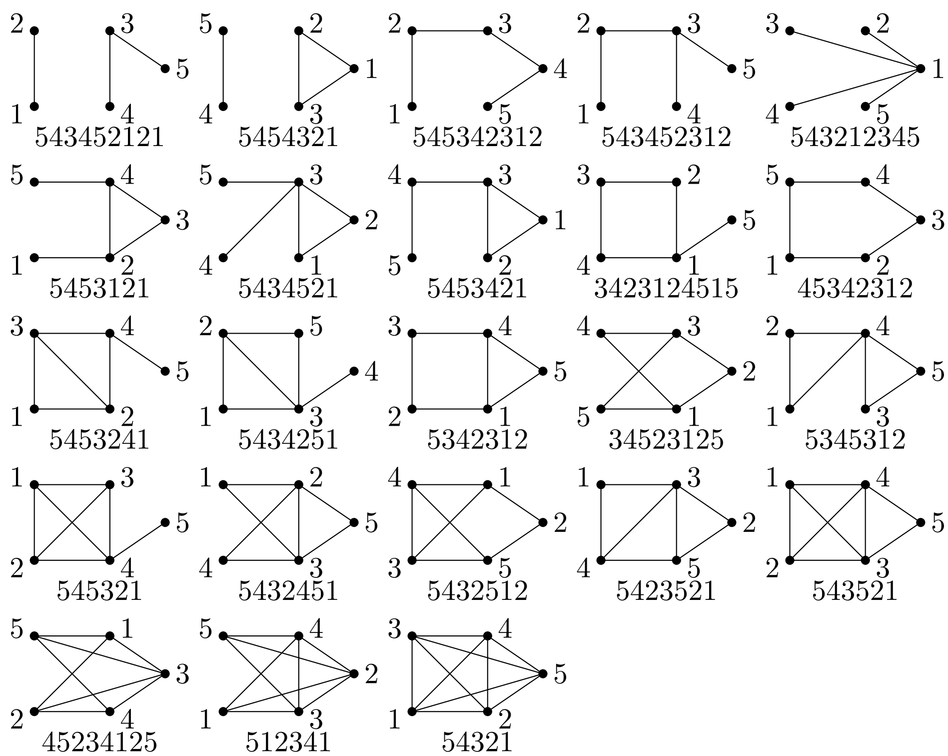

Figure 4.10: 132-representants for graphs on five vertices (with no isolated vertices)

5 Concluding remarks

This paper just scratches the surface of a big research direction dealing with representing graphs by pattern-avoiding words. Our studies were extended to 123-representation of graphs in [8], where more results on 132-representable graphs were obtained as well. Further steps may be in considering longer patterns and/or patterns of other types (e.g. those described in [2, 4]) while defining words to be used to represent graphs, and asking the question on which classes of graphs can be represented in this way. Simultaneous avoidance of patterns, like avoiding the patterns 132 and 231 at the same time, can be considered as well.

To conclude, we state the following question, solving which by exhaustive search would involve finding appropriate labelling of graphs and then considering all words over six letter alphabet that have at most two occurrences of each letter.

Question: Is the wheel graph W_5 the only non-132-representable graph on six vertices?

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