

Nonlinear Wave-Current Interaction in Water of Finite Depth

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Abstract:

The interaction of nonlinear progressive waves and a uniform current in water of finite depth is investigated analytically by means of the homotopy analysis method (HAM). With HAM, the velocity potential of the flow and the surface elevation are expressed by Fourier series and the nonlinear free surface boundary conditions are satisfied by continuous mapping. Unlike a perturbation method, the present approach does not depend on any small parameters; thus the solutions are suitable for steep waves and strong currents. To verify the HAM solutions, experiments are conducted in the wave-current flume of The Education Ministry Key Laboratory of Hydrodynamics at Shanghai Jiao Tong University (SJTU). It is found that the HAM solutions are in good agreement with experimental measurements. Based on the series solutions of the validated analytical model, the influence of water depth, wave steepness and current velocity on the physical properties of the coexisting wave-current field are studied in detail. The variation mechanisms of wave characteristics due to wave-current interaction are further discussed in a quantitative manner. The significant advantage of HAM in dealing with strong nonlinear wave-current interactions in the present study is clearly demonstrated in which the solution technique is independent of small parameters. A comparative study on

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29 wave characteristics further reveals the great potential of HAM to solve more complex wave-
30 current interaction problems leading to engineering applications in the offshore industry and
31 the marine renewable energy sector.

32 *Keywords:* Wave-current interaction; Nonlinear; Finite water depth; Homotopy analysis
33 method

34

35 **Introduction**

36 The co-existence of waves and currents is a common feature of most marine environments.
37 Nonlinear wave-current interaction is an important topic in both coastal and offshore
38 engineering. Giant waves (freak waves) have been registered in many regions of the oceans,
39 especially off the east coast of South Africa, where strong interactions between waves and
40 opposing currents exist (Mallory 1974; Kharif and Pelinovsky 2003). In these cases, the
41 opposing current significantly augments the wave height and steepness, resulting in
42 considerable hazards for ships and offshore structures. During the past several decades, wave-
43 current interaction has been the subject of numerous research efforts. Most of them are well
44 documented in the review articles by Peregrine (1976), Jonsson (1990) as well as Thomas and
45 Klopman (1997).

46 In many practical instances the current velocity varies significantly with depth, leading to the
47 creation of a velocity profile, for example, with a wind-driven current where the magnitude of
48 the current velocity varies exponentially with depth. Studies based on this type of current have
49 been reported in the literature (Thomas 1981; Thomas 1990; Swan et al. 2001; Swan and James
50 2001). In some other cases, however, it is reasonable to assume that the current velocity is

51 approximately uniform with depth. Examples of this type of current include large scale ocean
52 currents, and the majority of tidal flows where the time and length scales over which the current
53 varies are much larger than the wave period or wavelength. Rienecker and Fenton (1981)
54 presented the numerical solution for steady water waves progressing in constant water depth
55 based on the Fourier approximation method. In their model, the time mean Eulerian velocity,
56 i.e. the current velocity, can be taken into account. Later, this method is further simplified by
57 Fenton (1988) and applied to waves in both deep and shallow water conditions. Fenton (1985)
58 proposed a 5th-order perturbation solution for waves propagating on a uniform current in
59 constant water depth. For not-too-high waves and not-too-shallow water depths, the analytic
60 solution given by Fenton (1985) was in good agreement with the numerical solution by
61 Rienecker and Fenton (1981). However, it is worth noting that the perturbation solution
62 procedure by Fenton (1985) is rather complicated and difficult to extend to solve the more
63 complex interaction of multiple waves and a current. Umeyama (2011) also reported a 3rd-order
64 perturbation solution and experimental data for waves propagating on a following current. It is
65 important to point out that the experimental conditions in his work possess relatively weak
66 nonlinearity and low current velocities. Based on a Lagrangian coordinate system, Chen and
67 Chen (2014) also obtained a 5th-order perturbation series approximation for the interaction of
68 progressive waves and uniform currents. The focus of their research is on the wavy track of the
69 particle motion. Though there are several theoretical works on waves propagating on favorable
70 or adverse uniform currents, few analytical models describing the interaction between steep
71 waves and strong currents, as well as the effect of water depth, can be found in the literature.

72 Recently, an analytic approach named homotopy analysis method (HAM) has seen rapid

73 development. Different from the perturbation method, HAM does not depend on any small
74 parameter, so it is suitable for solving strong nonlinear problems. HAM was first applied to
75 water waves in infinite water depth by Liao and Cheung (2003). Later, Tao et al. (2007)
76 successfully extended Liao and Cheung (2003) to water of finite depth. Xu (2006) applied HAM
77 to investigate nonlinear wave and uniform current interaction in infinite water depths. It was
78 shown that the phase velocity of the waves in deep water obtained by HAM agrees well with
79 experimental measurements. In the framework of HAM, Cheng et al. (2009) investigated the
80 interaction of deep water waves and exponential shear currents. Liu et al. (2014) considered the
81 phase velocity effects of bi-chromatic wave interaction with exponentially sheared currents by
82 means of HAM. Examples can also be found in the literature demonstrating the effectiveness
83 of HAM to solve more complicated wave-wave interaction problems (Liao 2011; Xu et al. 2012;
84 Liu and Liao 2014; Lin et al. 2014).

85 The objective of the present study is to investigate the interaction between steep waves and
86 strong uniform currents in water of constant finite depth by HAM. In contrast to a perturbation
87 solution, the HAM series solution is independent of small parameters and thus possesses
88 considerable accuracy for strongly nonlinear problems. By including constant water depth in
89 the solution procedure, the present work further investigates the influence of water depth on the
90 nonlinear wave-current interaction problem in detail due to its significance in the shallow water
91 coastal region. To validate the effectiveness of the present approach, experiments are conducted
92 and the data are used to compare with the present HAM solution. The present paper is organized
93 as follows. The following section provides a description of governing equations and boundary
94 conditions; HAM is presented for a wave-current interaction problem; and the detailed solution

95 techniques are discussed. Following this section, the experimental setup and measurement
 96 techniques are described. Finally, detailed analytical results about how opposing currents and
 97 water depths influence the wave parameters of wave-current coexisting fields are presented.

98 **Theoretical Consideration**

99 *Governing Equations and Boundary Conditions*

100 **The Description of Wave-Current Interaction**

101 Consider the interaction between two-dimensional, nonlinear, progressive waves and a uniform
 102 current in water of finite depth. The fluid is assumed to be inviscid and incompressible, and the
 103 flow is irrotational. A Cartesian coordinate system (x, z) is adopted where the x-axis is
 104 positive in the direction of wave propagation, and the z-axis is positive vertically upwards from
 105 the still water level as shown in Fig. 1. The quantities $\varphi^*(x, z, t)$ and $\zeta(x, t)$ are defined as
 106 the velocity potential and the wave elevation, respectively. The fluid motion described by the
 107 velocity potential $\varphi^*(x, z, t)$ is governed by the Laplace equation:

$$108 \quad \nabla^2 \varphi^*(x, z, t) = 0, \quad -\infty < x < +\infty, \quad -d < z < \zeta(x, t) \quad (1)$$

109 and subject to two free surface boundary conditions:

$$110 \quad \frac{\partial \zeta}{\partial t} + \frac{\partial \varphi^*}{\partial x} \frac{\partial \zeta}{\partial x} - \frac{\partial \varphi^*}{\partial z} = 0, \quad z = \zeta(x, t) \quad (2)$$

$$111 \quad g\zeta + \frac{1}{2}(\nabla \varphi^*) \cdot (\nabla \varphi^*) + \frac{\partial \varphi^*}{\partial t} = \frac{1}{2}U_0^2, \quad z = \zeta(x, t) \quad (3)$$

112 and the following condition at the bottom:

$$113 \quad \frac{\partial \varphi^*}{\partial z} = 0, \quad z = -d \quad (4)$$

114 where $\nabla = (\partial / \partial x, \partial / \partial z)$, t denotes time, g is gravitational acceleration, d is the water
 115 depth and U_0 is the uniform current velocity. Since gravity capillary waves caused by surface
 116 tension are quite small compared to their wavelengths, the effect of surface tension is neglected.

117 By means of superposition for potential theory, the total velocity potential of the wave-current
 118 co-existing field is given by $\varphi^* = U_0 x + \varphi$, where φ denotes the wave velocity potential.

119 Combining Eqs. (2) and (3), the boundary condition becomes:

$$120 \quad \frac{\partial^2 \varphi^*}{\partial t^2} + g \frac{\partial \varphi^*}{\partial z} + \frac{\partial [(\nabla \varphi^*) \cdot (\nabla \varphi^*)]}{\partial t} + \frac{1}{2} (\nabla \varphi^*) \cdot \nabla [(\nabla \varphi^*) \cdot (\nabla \varphi^*)] = 0, \quad z = \zeta(x, t) \quad (5)$$

121 Substituting $\varphi^* = U_0 x + \varphi$ into Eqs. (1), (3), (4) and (5), the governing equation becomes:

$$122 \quad \nabla^2 \varphi(x, z, t) = 0, \quad -\infty < x < +\infty, \quad -d < z < \zeta(x, t) \quad (6)$$

123 which is subject to two nonlinear free surface conditions:

$$124 \quad g\zeta + U_0 \frac{\partial \varphi}{\partial x} + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi + \frac{\partial \varphi}{\partial t} = 0, \quad z = \zeta(x, t) \quad (7)$$

$$125 \quad \frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} + 2U_0 \frac{\partial^2 \varphi}{\partial x \partial t} + \frac{\partial(\nabla \varphi \cdot \nabla \varphi)}{\partial t} + U_0^2 \frac{\partial^2 \varphi}{\partial x^2} \\ + U_0 \frac{\partial(\nabla \varphi \cdot \nabla \varphi)}{\partial x} + \frac{1}{2} \nabla \varphi \cdot \nabla(\nabla \varphi \cdot \nabla \varphi) = 0, \quad z = \zeta(x, t) \quad (8)$$

126 and the following bottom boundary condition:

$$127 \quad \frac{\partial \varphi}{\partial z} = 0, \quad z = -d \quad (9)$$

128 Variable Transformation

129 The objective of this paper is to study the interaction between nonlinear progressive waves
 130 and a uniform current in an arbitrary, uniform water depth. Without loss of generality, assume
 131 that the wave-current co-existing field is made up of a current and a wave component with wave
 132 number k and corresponding angular frequency ω . It is convenient to define the phase
 133 function

$$134 \quad \theta = kx - \omega t + \theta_0 \quad (10)$$

135 where θ_0 denotes an arbitrary, constant phase for zero time at the origin of the (x, z)
 136 coordinate system. The above variable can be used to replace the variables x and t , and then
 137 the time, t , will not appear explicitly for a steady wave-current system. Thus, one can express
 138 the potential function $\varphi(x, z, t) = \phi(\theta, z)$, and the wave elevation $\zeta(x, t) = \eta(\theta)$ for the co-
 139 existing field of one train of progressive waves and a uniform current. With these definitions,

140 the governing equation becomes:

$$141 \quad \hat{\nabla}^2 \phi = k^2 \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad -d < z < \eta(\theta) \quad (11)$$

142 which is subject to the bottom boundary condition:

$$143 \quad \frac{\partial \phi}{\partial z} = 0, \quad z = -d \quad (12)$$

144 and the nonlinear free surface conditions:

$$145 \quad \eta = \frac{1}{g} \left[(\omega - U_0 k) \frac{\partial \phi}{\partial \theta} - f \right], \quad z = \eta(\theta) \quad (13)$$

$$146 \quad \omega^2 \frac{\partial^2 \phi}{\partial \theta^2} + g \frac{\partial \phi}{\partial z} - 2\omega \frac{\partial f}{\partial \theta} + \hat{\nabla} \phi \cdot \hat{\nabla} f - 2U_0 k \omega \frac{\partial^2 \phi}{\partial \theta^2} \\ + U_0^2 k^2 \frac{\partial^2 \phi}{\partial \theta^2} + 2U_0 k \frac{\partial f}{\partial \theta} = 0, \quad z = \eta(\theta) \quad (14)$$

147 where

$$148 \quad f = \frac{1}{2} \hat{\nabla} \phi \cdot \hat{\nabla} \phi \quad (15)$$

$$149 \quad \frac{\partial f}{\partial \theta} = \hat{\nabla} \phi \cdot \hat{\nabla} \left(\frac{\partial \phi}{\partial \theta} \right) \quad (16)$$

$$150 \quad \hat{\nabla} \phi \cdot \hat{\nabla} f = k^2 \frac{\partial \phi}{\partial \theta} \frac{\partial f}{\partial \theta} + \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial z} \quad (17)$$

151 and $\hat{\nabla} = (k \partial / \partial \theta, \partial / \partial z)$.

152 *HAM for the Wave-Current Interaction*

153 **The Solution Expressions**

154 By satisfying the Laplace Eq. (11) and bottom condition Eq. (12), the velocity potential

155 $\phi(\theta, z)$ can be expressed by a set of base functions

$$156 \quad \left\{ \sin(m\theta) \frac{\cosh[mk(z+d)]}{\cosh(mkd)} \middle| m \geq 1 \right\} \quad (18)$$

157 in the form:

$$158 \quad \phi(\theta, z) = \sum_{m=1}^{+\infty} b_m \Psi_m(\theta, z) \quad (19)$$

159 where

$$160 \quad \Psi_m(\theta, z) = \sin(m\theta) \frac{\cosh[mk(z+d)]}{\cosh(mkd)} \quad (20)$$

161 and b_m ($m=1,2,\dots$) are coefficients. This provides us with a rule for the solution expression
162 for $\phi(\theta, z)$ (Liao, 2003). It should be noted that Eq. (19) automatically satisfies the governing
163 equation (11) and the bottom boundary condition (12). Accordingly, the wave elevation can be
164 expressed by a set of base functions:

$$165 \quad \{\cos(m\theta) | m \geq 1\} \quad (21)$$

166 in the form:

$$167 \quad \eta(\theta) = \sum_{m=1}^{+\infty} a_m \cos(m\theta) \quad (22)$$

168 where a_m are coefficients to be determined.

169 **Zeroth-Order Deformation Equation**

170 In the framework of HAM (Liao, 2003), there is great freedom to choose the linear auxiliary
171 operator. According to the linear part of the nonlinear boundary conditions (13) and (14), two
172 linear auxiliary operators are chosen as:

$$173 \quad L_1[(\cdot)] = (\cdot) \quad (23)$$

$$174 \quad L_2[\phi] = \bar{\omega}^2 \frac{\partial^2 \phi}{\partial \theta^2} + g \frac{\partial \phi}{\partial z} \quad (24)$$

175 where

$$176 \quad \bar{\omega} = \sqrt{gk \tanh(kd)} \quad (25)$$

177 Based on the nonlinear boundary conditions, two nonlinear operators can be defined as:

$$178 \quad N_1[\eta, \phi, \omega] = \eta - \frac{1}{g} \left[(\omega - U_0 k) \frac{\partial \phi}{\partial \theta} - f \right] \quad (26)$$

179
$$N_2[\phi, \omega] = \omega^2 \frac{\partial^2 \phi}{\partial \theta^2} + g \frac{\partial \phi}{\partial z} - 2\omega \frac{\partial f}{\partial \theta} + \hat{\nabla} \phi \cdot \hat{\nabla} f$$

(27)

$$- 2U_0 k \omega \frac{\partial^2 \phi}{\partial \theta^2} + U_0^2 k^2 \frac{\partial^2 \phi}{\partial \theta^2} + 2U_0 k \frac{\partial f}{\partial \theta}$$

180 Then the *zer*oth-order deformation equation can be constructed as:

181
$$\hat{\nabla}^2 \check{\phi}(\theta, z; q) = 0, \quad -d < z \leq \check{\eta}(\theta; q)$$

(28)

182 which subject to the bottom boundary condition:

183
$$\frac{\partial \check{\phi}(\theta, z; q)}{\partial z} = 0, \quad z = -d$$

(29)

184 and the two nonlinear boundary conditions on $z = \check{\eta}(\theta; q)$:

185
$$(1-q)L_1[\check{\eta}(\theta; q)] = qc_0 N_1[\check{\eta}(\theta; q), \check{\phi}(\theta, z; q), \check{\omega}(q)]$$

(30)

186
$$(1-q)L_2[\check{\phi}(\theta, z; q) - \phi_0(\theta, z)] = qc_0 N_2[\check{\phi}(\theta, z; q), \check{\omega}(q)]$$

(31)

187 where $q \in [0, 1]$ is an embedding parameter; c_0 is the so-called nonzero convergence-control
 188 parameter; $\phi_0(\theta, z)$ is the initial estimate of the potential function; and $\check{\phi}(\theta, z; q)$, $\check{\eta}(\theta; q)$
 189 and $\check{\omega}(q)$ are the mapping functions, respectively.

190 When $q = 0$, the *zer*oth-order deformation Eqs. (28)-(31) have the solution:

191
$$\check{\phi}(\theta, z; 0) = \phi_0(\theta, z)$$

(32)

192
$$\check{\eta}(\theta; 0) = 0$$

(33)

193 When $q = 1$, the *zer*oth-order deformation Eqs. (28)-(31) are equivalent to the original Partial
 194 Differential Equations (PDEs) (11)-(14), respectively, provided that:

195
$$\check{\phi}(\theta, z; 1) = \phi(\theta, z)$$

(34)

196
$$\check{\eta}(\theta; 1) = \eta(\theta)$$

(35)

197
$$\check{\omega}(1) = \omega$$

(36)

198 Thus, as the embedding parameter q increases from 0 to 1, $\check{\phi}(\theta, z; q)$ and $\check{\eta}(\theta; q)$ deform
 199 continuously from initial estimates $\phi_0(\theta, z)$ and 0 to become the exact solutions of the original

200 problem, respectively. Similarly, $\check{\omega}(q)$ deforms continuously from ω_0 to the exact
 201 frequency ω .

202 The Maclaurin series of $\check{\phi}(\theta, z; q)$, $\check{\eta}(\theta; q)$ and $\check{\omega}(q)$, with respect to the embedding
 203 parameter q , read as:

$$204 \quad \check{\phi}(\theta, z; q) = \sum_{m=0}^{+\infty} \phi_m(\theta, z) q^m \quad (37)$$

$$205 \quad \check{\eta}(\theta; q) = \sum_{m=0}^{+\infty} \eta_m(\theta) q^m \quad (38)$$

$$206 \quad \check{\omega}(q) = \sum_{m=0}^{+\infty} \omega_m q^m \quad (39)$$

207 where

$$208 \quad \phi_m(\theta, z) = \frac{1}{m!} \left. \frac{\partial^m \check{\phi}(\theta, z; q)}{\partial q^m} \right|_{q=0} \quad (40)$$

$$209 \quad \eta_m(\theta) = \frac{1}{m!} \left. \frac{\partial^m \check{\eta}(\theta; q)}{\partial q^m} \right|_{q=0} \quad (41)$$

$$210 \quad \omega_m = \frac{1}{m!} \left. \frac{\partial^m \check{\omega}(q)}{\partial q^m} \right|_{q=0} \quad (42)$$

211 Assuming that c_0 is properly chosen so that the Maclaurin series (37), (38) and (39) converge
 212 at $q = 1$, then the so-called homotopy-series solutions are obtained as:

$$213 \quad \phi(\theta, z) = \phi_0(\theta, z) + \sum_{m=1}^{+\infty} \phi_m(\theta, z) \quad (43)$$

$$214 \quad \eta(\theta) = \sum_{m=1}^{+\infty} \eta_m(\theta) \quad (44)$$

$$215 \quad \omega = \omega_0 + \sum_{m=1}^{+\infty} \omega_m \quad (45)$$

216 High-Order Deformation Equation

217 Substituting the series in Eqs. (37)-(39) into the *zeroth*-order deformation equations and

218 equating the like-power of q , the so-called m th-order deformation equations are:

$$219 \quad \hat{\nabla}^2 \phi_m(\theta, z) = 0 \quad (46)$$

$$220 \quad \frac{\partial \phi_m(\theta, z; q)}{\partial z} = 0, \quad z = -d \quad (47)$$

$$221 \quad \bar{L}_2[\phi_m(\theta, z)] = R_m^\varphi(\theta; c_0) \quad (48)$$

$$222 \quad \eta_m(\theta) = R_m^\eta(\theta; c_0) \quad (49)$$

223 where

$$224 \quad R_m^\varphi(\theta; c_0) = c_0 \Delta_{m-1}^\phi + \chi_m S_{m-1} - \bar{S}_m \quad (50)$$

$$225 \quad R_m^\eta(\theta; c_0) = c_0 \Delta_{m-1}^\eta + \chi_m \eta_{m-1} \quad (51)$$

$$226 \quad \Delta_m^\varphi = \sum_{n=0}^m \left(\sum_{l=0}^n \omega_l \omega_{n-l} \right) \bar{\phi}_{m-n}^2 - 2U_0 k \sum_{n=0}^m \omega_n \bar{\phi}_{m-n}^2 + U_0^2 k^2 \bar{\phi}_m^2 \\ + g \bar{\phi}_{z,m}^0 + 2U_0 k \Gamma_{m,1} - 2 \sum_{n=0}^m \omega_n \Gamma_{m-n,1} + \Lambda_m \quad (52)$$

$$227 \quad \Delta_m^\eta = \eta_m - \frac{1}{g} \left[\sum_{n=0}^m \omega_n \bar{\phi}_{m-n}^1 - U_0 k \bar{\phi}_m^1 - \Gamma_{m,0} \right] \quad (53)$$

228 $\bar{L}_2[\phi] = L_2[\phi]_{z=0}$ and $m \geq 1$. The definitions of S_m , \bar{S}_m , χ_m , Λ_m , $\bar{\phi}_{z,m}^0$, $\Gamma_{m,i}$,

229 $\bar{\phi}_m^i$ ($i = 0, 1, 2$) and their detailed derivations can be found in Liao (2011).

230 **The Initial Estimate**

231 Without loss of generality, assume that the wave number k and the coefficient a_1 in the
 232 wave profile of Eq. (22) are given in the wave-current system. The wave frequency ω is
 233 unknown and to be calculated. Liao (2003) has demonstrated that there is great freedom to
 234 choose the initial estimate in HAM. Based on the dispersion relation of the linear wave-current
 235 interaction, the initial estimate of the frequency ω is set as: $\omega_0 = (1 + \varepsilon)(\bar{\omega} + U_0 k)$, where ε
 236 is a very small, positive real value. The rest of this subsection considers the choice of the initial
 237 estimate for $\phi(\theta, z)$.

238 The auxiliary linear operator in Eq. (24) has the property

$$239 \quad L_2[\Psi_m] = \lambda_m \cdot \Psi_m \quad (54)$$

240 where Ψ_m is defined by Eq. (20) and

$$241 \quad \lambda_m = g |mk| \tanh(|mk|d) - (m\bar{\omega})^2 \quad (55)$$

242 Therefore, the inverse operator L_2^{-1} is defined as

$$243 \quad L_2^{-1}[\Psi_m] = \frac{\Psi_m}{\lambda_m}, \quad \lambda_m \neq 0 \quad (56)$$

244 Note that the inverse operator L_2^{-1} has definition only for non-zero values of λ_m . When

245 $\lambda_m = 0$, therefore,

$$246 \quad g |mk| \tanh(|mk|d) = (m\bar{\omega})^2 \quad (57)$$

247 In this paper, there is only $\lambda_1 = 0$. Thus, an initial estimate for $\phi_0(\theta, z)$ can be chosen as

$$248 \quad \phi_0(\theta, z) = b_1 \cdot \Psi_1(\theta, z) \quad (58)$$

249 where b_1 is an unknown constant to be determined later.

250 ***Solution Procedure***

251 Considering the rule for solution expressions (19) and (22) and the property of the auxiliary

252 linear operator L_2 in Eq. (54), the right-hand side of Eq. (48) can be expressed as

$$253 \quad R_m^\varphi = \sum_{i=1}^{I_m} \tilde{b}_{m,i} \sin(i\theta) \quad (59)$$

254 where $\tilde{b}_{m,i}$ are coefficients and I_m is related to the right-hand side of Eq. (48). According to

255 the property of the auxiliary linear operator,

$$256 \quad \tilde{b}_{m,1} = 0 \quad (60)$$

257 has to be enforced to avoid the so-called secular terms. Therefore, using Eq. (57), it is

258 convenient to obtain the solution of Eq. (48):

259
$$\phi_m(\theta, z) = \sum_{i=2}^{I_m} \bar{b}_{m,i} \Psi_i(\theta, z) + \bar{b}_{m,1} \Psi_1(\theta, z) \quad (61)$$

260 where $\bar{b}_{m,1}$ is an unknown coefficient to be determined in the $(m+1)$ th-order deformation
 261 equation. Similarly, according to Eq. (49), there is

262
$$\eta_m(\theta) = \sum_{i=1}^{I_m} \bar{a}_{m,i} \cos(i\theta) \quad (62)$$

263 When $m=1$ using Eq. (60), the unknown coefficient b_1 in Eq. (58) can be obtained for
 264 the initial estimate $\phi_0(\theta, z)$. When $m \geq 2$, since the coefficient of the primary wave
 265 component is given, there are

266
$$\sum_{n=1}^m \bar{a}_{n,1} = a_1 \quad (63)$$

267 Thus, Eqs. (60) and (63) provide a set of algebraic equations for $\bar{b}_{m-1,1}$ and ω_{m-1} ($m \geq 2$)
 268 and make the problem closed. The high-order deformation equations can be solved by means
 269 of the symbolic computation software—Mathematica 7. At the M^{th} -order approximations, we
 270 have:

271
$$\left\{ \begin{array}{l} \phi(\theta, z) \approx \phi_0(\theta, z) + \sum_{m=1}^M \phi_m(\theta, z) \\ \eta(\theta) \approx \sum_{m=1}^M \eta_m(\theta) \\ \omega \approx \omega_0 + \sum_{m=1}^M \omega_m \end{array} \right. \quad (64)$$

272 ***Optimal Convergence-Control Parameters***

273 For the m^{th} -order approximations $\phi(\theta, z)$ and $\eta(\theta)$, there is still one unknown
 274 parameter c_0 , which is used to guarantee the convergence of the approximation series. In order
 275 to choose an optimal c_0 , two averaged residual square errors of the boundary conditions are
 276 defined as:

277
$$E_m^\eta = \frac{1}{(1+I_k)} \sum_{i=0}^{I_k} \left(N_1 [\phi(\theta, z), \eta(\theta)] \Big|_{\theta=i\Delta\theta} \right)^2 \quad (65)$$

278
$$E_m^\phi = \frac{1}{(1+I_k)} \sum_{i=0}^{I_k} \left(N_2 [\phi(\theta, z)] \Big|_{\theta=i\Delta\theta} \right)^2 \quad (66)$$

279 where I_k is the number of discrete points and $\Delta\theta = \pi / I_k$. In this paper, $I_k = 20$ is used.

280 Defining the total averaged residual square error as $E_m^T = E_m^\phi + E_m^\eta$, then by solving

281 $dE_m^T / dc_0 = 0$, the optimal value of c_0 can be obtained, which corresponds to the minimum

282 value of E_m^T .

283 **Experimental Set-up**

284 To provide better understanding on the interaction between waves and currents and validate

285 the developed analytical model, wave flume tests were carried out at the Education Ministry

286 Key Laboratory of Hydrodynamics at Shanghai Jiao Tong University, China. The details of the

287 facility, measurement apparatus and test conditions are described as follows.

288 **Wave Flume**

289 The experiments are conducted in a glass-walled wave flume 60.0 m long and 0.8 m wide

290 with a fixed water depth of 0.5 m. The schematic of the experimental setup is shown in Fig. 2.

291 The flume is equipped with a hydraulically driven piston-type wave maker, while wave

292 absorbers are equipped on the other end to absorb the incident wave energy. The following and

293 opposing circulating currents are generated by a pump located near the wave maker.

294 The time series of water surface elevations are recorded by three capacitance wave gauges,

295 which are represented by filled circles in Fig. 2. These gauges are placed along the flume with

296 a spacing of 0.5 m. The absolute accuracy of these wave gauges is on the order of ± 1 mm.

297 Before the wave gauges are used, they are calibrated to ensure their accuracy during the tests.

298 The duration of each record is 120 s. The sampling frequency is 50 Hz.

299 Both current and wave particle velocity measurements are made using a Nortek acoustic
300 Doppler velocimeter (ADV) with velocity range 1m/s, sampling rate 200 Hz, and specified
301 accuracy of 1 mm/s. Detailed measurements of the vertical current profile and wave particle
302 profile along the centreline of the flume are carried out.

303 *Wave-Current Condition*

304 The experimental conditions are listed in Table 1. The depth-averaged following and
305 opposing current velocities are approximately 0.135 m/s and -0.139 m/s, respectively, in the
306 current-only cases. Runs W1-W4 are for waves without a current, while Runs WFC1-WFC4
307 are waves from W1-W4 superimposed on a following current, and Runs WOC1-WOC3 are
308 waves from W1-W3 superimposed on an opposing current. The specified wave periods input
309 to the wave-making system for all cases are set to 1 s, while the corresponding wave periods
310 measured are almost constant around 1 s. These consistent values indicate that the assumption
311 of a constant wave period during wave-current interaction is a reasonable one for this theoretical
312 study. On the other hand, it is noted that differences between measured wave heights and
313 specified ones will not affect the experiment results as the measured wave heights will be used
314 in post-processing.

315 In the present experiments the measuring section is located 15 m off the wave maker. At this
316 location it is possible to generate the required test conditions for the duration of sufficient wave
317 cycles. During this period regular waves coexist with the current, and the relevant experimental
318 data are recorded before the incident wave train is disrupted by reflected waves travelling in the
319 opposite direction. Fig. 3(a-d) shows a typical time history of the wave-only surface elevations.

320 As can be seen in Fig. 3(a-d), the time histories of wave elevations for the wave-only Cases
321 W1-W4 (around 20 to 70 seconds) appear to be quite stable, even for Cases W3
322 ($a_1k \approx 0.25$, $kd \approx 2.0$) and W4 ($a_1k \approx 0.30$, $kd \approx 1.9$), with relatively high wave steepness and
323 low water depths. For these values of the parameters a_1k and kd , Mclean (1982) pointed out
324 that Stokes wave trains without current interaction are unstable to 3D perturbations. However,
325 it is clear that the 3D perturbation effect in Cases W3 and W4 (around 20 to 70 seconds) is not
326 evident. This indicates that the 3D effect in the present experiments conducted in the wave
327 flume with the given configuration (60 m \times 0.8 m) is rather weak. Ma et al. (2010) also reported
328 that experiments conducted in a wave flume with the same configuration (60 m \times 0.8 m) can
329 ensure two-dimensionality of the wave field. Fig. 4(a-d) shows a typical time history
330 comparison between wave-only cases and wave-current coexisting cases. It is worth noting that
331 the relatively stable parts of the time histories of wave elevations for the coexisting wave-
332 current Cases WFC3 ($a_1k \approx 0.17$, $kd \approx 1.8$) and WOC3 ($a_1k \approx 0.28$, $kd \approx 2.4$) occur around 40
333 to 46 seconds in Fig. 4(c). For Case WFC4 ($a_1k \approx 0.21$, $kd \approx 1.7$) this relatively stable portion
334 occurs around 20 to 26 seconds in Fig. 4(d); these stable sections were extracted from the initial
335 phase of the complete time history of wave elevations (up to 100 seconds) for the corresponding
336 cases as shown in Fig. 5(c1-c2, d). As seen in Fig. 5(c1-c2, d), the latter parts of the time
337 histories of wave elevations for Cases WFC3 and WOC3 (around 55 to 95 seconds), as well as
338 WFC4 (around 40 to 90 seconds), appear to be unstable. Since it is not the focus of the
339 present study, no special wave gauges were arranged to obtain sufficient data to study
340 the instability due to 3D perturbation effects in the experiment. In the present paper, the
341 experimental measurements are used to make a comparison with the 2D HAM solution

342 without any perturbation. Only the relatively stable parts of the time histories (after the initial
343 phases) for Cases WFC3 and WOC3 (around 40 to 50 seconds), as well as Case WFC4 (around
344 15 to 30 seconds), as shown in the box in Fig. 5(c1-c2, d), were utilized in post-processing.
345 These stable sections were extracted and compared to those of the wave-only Cases W3 and
346 W4 as shown in Fig. 4(c-d). In addition, Fig. 5(a1-a2, b1-b2) was obtained from
347 experimental measurements for a range of low wave steepness, i.e., Cases W1
348 ($ak \approx 0.1, kd \approx 2.1$) and W2 ($ak \approx 0.17, kd \approx 2.0$) with following and opposing currents; as can
349 be seen in the figure, the complete time histories of wave elevations from 20 up to 70
350 seconds remain stable. This indicates that during the recording period (around 20 to 70
351 seconds), 3D perturbation effects are not evident for Cases W1 and W2 with following
352 and opposing currents, i.e., Cases WFC1 ($ak \approx 0.07, kd \approx 1.8$), WOC1 ($ak \approx 0.13, kd \approx 2.5$),
353 WFC2 ($ak \approx 0.12, kd \approx 1.7$) and WOC2 ($ak \approx 0.24, kd \approx 2.4$).

354 **Results and Discussion**

355 The analytical model, which is proposed as a solution of the interaction between nonlinear
356 waves and a uniform current, has been validated by comparing analytical results against
357 experimental data in the following subsection. Based on the accurate homotopy series solutions,
358 the variation in flow characteristics due to the nonlinear interaction between steep waves and
359 strong opposing currents is further examined in detail, together with the influence of water
360 depth.

361 ***Validation of the Analytical Model***

362 To validate the analytical model for nonlinear wave-current interactions, the analytical
363 solutions are compared with the experimental measurements of wavelength and wave steepness.

364 As shown in Table 2, the relative water depth of each test case in the experiment is
365 approximately 0.3, corresponding to an intermediate water depth condition. For the same wave
366 period and wave height presented in Table 2, it can be seen that the wavelength and wave
367 steepness obtained by HAM are in good agreement with the experimental data for waves with
368 and without a current. As shown in Table 2, the experiment measured wavelength for Case W4
369 is approximately 1.647 m, while the HAM obtained wavelength is about 1.640 m. The relative
370 error between them is 0.4%. Further, Fig. 6 shows wave steepness values from the experimental
371 data and analytical solution. It can be seen that a notable discrepancy exists for Case WOC3,
372 i.e., the experimental data and analytical solution for wave steepness are 0.090 and 0.085
373 respectively. The relative error between these values is 5.6%, which indicates that even for this
374 case, the agreement between the analytical solution and the experimental data is acceptable. As
375 shown in Table 2, the maximum total averaged residual square error $E_m^T (m = 20)$ approaches
376 the magnitude of 10^{-5} , which further demonstrates that all the series approximation solutions
377 are convergent and possess a high level of accuracy.

378 It is of interest to validate the effectiveness of the present model for the prediction of wave
379 kinematics. The HAM solutions of horizontal velocities of water particles at the crest and trough
380 are compared to the corresponding experimental data. In addition, the present HAM solutions
381 and experimental measurements are also compared to numerical results obtained by the Fourier
382 approximation method (Fenton, 1988). Fig. 7 shows the comparison of horizontal particle
383 velocities at wave crest and trough between theoretical solutions and experiments (for cases
384 WFC1-4 and cases WOC1-3). It can be observed that the present solutions agree well with the
385 numerical results obtained by the Fourier approximation method. It is worth noting that the

386 current distributions measured in the experiments have boundary layers near the bottom,
387 resulting in a weak influence on the water wave dynamics (see Fig. 7). Therefore, the
388 discrepancy in the wave kinematics near the bottom is mainly attributed to the shear current
389 that occurs due to the bottom boundary effect. However, it will not influence the effectiveness
390 of the present analytical model to predict wave characteristics near the free surface.

391 The comparisons presented above indicate that the present analytical model is capable of
392 producing reliable predictions for nonlinear wave-current interaction in water of finite depth.
393 In the next subsection, we will further investigate the interaction of steep waves and a strong
394 opposing current and the influence of the opposing current and water depth on the wave
395 characteristics.

396 ***Study of Wave-Current Interaction***

397 To examine the influences of a strong opposing current and water depth on the free surface
398 and wave steepness, further analytical calculations with the validated model are presented in
399 this section, and two sets of the wave-current parameters are listed in Table 3 and Table 4,
400 respectively.

401 **The Influence of an Opposing Current**

402 For a given initial wave period, the influence of an opposing current on the free surface of a
403 nonlinear wave is considered by varying the opposing current velocities from -0.15 m/s to -0.4
404 m/s at an interval of 0.05 m/s (Table 3). Fig. 8 shows the free surface profiles for waves
405 coexisting with different opposing currents at the instantaneous time of $t=0$ at a water depth
406 of 0.5 m, in which a_1 is set to 0.05 m and the wave period $T=1.01$ s is kept constant
407 throughout. The results are non-dimensionalised as η/d and x/d . It can be observed that,

408 for a given amplitude parameter a_1 , the opposing current tends to narrow both the crest and
 409 corresponding trough to condense the wavelength. For example, the wavelength for the wave-
 410 only Case C1 is 1.600 m, which is approximately 1.19 times that of the wave-current coexisting
 411 Case C2 ($L=1.343$ m) with minimum opposing current velocity, and it is approximately 1.81
 412 times that of Case C7 ($L=0.883$ m) with maximum opposing current velocity. The variation in
 413 wavelength is evident, which demonstrates that an opposing current leads to a significant
 414 decrease in wavelength. Further, the elevation near the crest increases significantly while the
 415 elevation near trough appears almost unchanged. To clearly see the tendency of wave
 416 characteristics in Fig. 8, the variations in non-dimensional wavelength L/d , wave crest height
 417 H_1/a_1 and wave trough height H_2/a_1 are plotted against non-dimensional current velocity
 418 $|U_0|/(gd)^{1/2}$ in Fig. 9. From Fig. 9, one can see that L/d decreases significantly from 3.2
 419 to 1.7 as $|U_0|/(gd)^{1/2}$ increases from 0 to 0.18. It is interesting to note that H_1/a_1 increases
 420 significantly over the range of larger current velocity values. For $|U_0|/(gd)^{1/2}$ values ranging
 421 from 0 to 0.1, values of H_1/a_1 are almost constant around 1.15. As $|U_0|/(gd)^{1/2}$ increases
 422 beyond 0.1, from 0.1 to 0.18, H_1/a_1 increases instantly from 1.15 to 1.35, indicating that a
 423 stronger opposing current tends to significantly increase the wave crest height. However, the
 424 corresponding wave trough height tends to remain approximately constant throughout (also see
 425 Table 3).

426 To further demonstrate the influence of an opposing current, wave steepness H/L and the
 427 non-dimensional current velocity $|U_0|/C_0$ for Cases C1-C7 are calculated; the results are
 428 listed in Table 5 and plotted in Fig. 10, in which C_0 is the phase velocity of the linear wave at
 429 the water depth of $d = 1.7$ m which corresponds to a deep water wave condition in this paper.

430 Table 5 also presents the 5th perturbation solution of H/L for the corresponding wave
431 condition based on Fenton (1985). It can be seen that, when the opposing current velocity is
432 low, the wave steepness values obtained by HAM and the perturbation method are almost the
433 same. When the opposing current velocity increases, however, there exists a small discrepancy
434 between wave steepness values H/L . It is worth noting that the present model can provide
435 an estimate of the accuracy by computing the total averaged residual square errors E_m^T of the
436 30th HAM solution for each case as shown in Table 5. Besides, compared to the perturbation
437 technique in Fenton (1985), the present model is much easier to apply and extend to solve more
438 complex wave-current interaction problems. As shown in Fig. 10, one can see that at a water
439 depth of 0.5 m, H/L increases up to 0.125 as $|U_0|/C_0$ increases, which indicates that the
440 wave possesses relatively strong nonlinearity due to the effect of an opposing current. To further
441 compare the influence of an opposing current on the wave steepness H/L at different water
442 depths, the plots of H/L against the non-dimensional opposing current velocity $|U_0|/C_0$ at
443 different water depths are also presented in Fig. 10. Again, it is seen in Fig. 10 that the H/L
444 values at each water depth increase consistently as $|U_0|/C_0$ increases. Moreover, the wave
445 steepness increases as the water depth decreases. However, for increases in water depth beyond
446 0.8 m, the wave steepness tends to be independent of the water depth. It is also observed that
447 the water depth effect in the Cases with lower current velocity is more pronounced than that in
448 the Cases with higher current velocity, as shown in Fig. 10. For example, for $|U_0|/C_0 = 0.095$,
449 when the water depth varies from 0.4 m to 1.7 m, the variation in the wave steepness is about
450 0.004, which is about 5% of H/L at $d = 0.4$ m. However, for the maximum opposing
451 current velocity value of $|U_0|/C_0 = 0.253$ in this paper, the corresponding variation in wave

452 steepness is about 0.002, which is only about 1.6% of H/L at $d = 0.4$ m.

453 Fig. 11 shows the variation of non-dimensional wavelength L/L_0 against non-dimensional
454 opposing current velocity $|U_0|/C_0$ at different water depths d , in which C_0 and L_0 are
455 the corresponding phase velocity and wavelength of the linear wave at a water depth of $d = 1.7$
456 m. It is clearly seen that at each water depth L/L_0 decreases as $|U_0|/C_0$ increases. With the
457 same current strength and water depths ranging from 0.4 m to 0.8 m, a decrease in water depth
458 also leads to a decrease in wavelength; however, for water depths beyond 0.8 m, the wavelength
459 tends to be independent of the water depth. It is important to note that the water depth effect on
460 L/L_0 also dominates over the range of lower opposing current velocities and the value of
461 L/L_0 for each water depth tends to approach an identical value as $|U_0|/C_0$ increases. For
462 instance, for $|U_0|/C_0 = 0.095$, when the water depth varies from 0.4 m to 1.7 m, the variation
463 in L/L_0 is about 0.038, which is about 4.6% of L/L_0 at $d = 0.4$ m. However, for the
464 maximum opposing current velocity $|U_0|/C_0 = 0.253$, the corresponding variation in L/L_0
465 is about 0.003, which is only about 0.57% of L/L_0 at $d = 0.4$ m. It is clearly seen that, as the
466 opposing current velocity increases, the percent variation in the non-dimensional wavelength
467 due to changes in water depth is much smaller than variations in wave steepness. That is why
468 the value of L/L_0 for each water depth tends to approach an asymptotic value (about 0.55) as
469 $|U_0|/C_0$ increases.

470 **The Influence of Water Depth**

471 To investigate how decreases in water depth influence wave steepness for the case of waves
472 coexisting with an opposing current, the initial wave period is also kept constant throughout.
473 Then the influence of water depth on the wave steepness under different opposing current

474 velocities is considered by varying the water depth from 0.1 m to 0.7 m at an interval of 0.1 m
475 (for larger water depths) and 0.05 m (for smaller water depths) as shown for $U_0 = -0.3$ m/s in
476 Table 4. For a constant wave period (0.76 s), the wavelengths presented in Table 4 are used as
477 the input to the present model. From Table 4, it is clearly seen that for water depths ranging
478 from 0.1 m to 0.3 m, the variation in water depth leads to a relatively significant variation in
479 wavelength. However, for water depths beyond 0.3 m, the variation in water depth only results
480 in small variations in wavelength. This is a further demonstration that the water depth has an
481 evident effect on the wavelength starting from an intermediate water depth condition.

482 The HAM solutions for wave steepness H/L for Cases D1-D9 are plotted against relative
483 water depth d/L in Fig. 12, and the values are listed in Table 6 with the corresponding total
484 averaged residual square errors E_m^T . As shown in Table 6, for $d/L < 0.5$, a decrease in d/L
485 leads to a distinct increment in H/L , while for $d/L > 0.5$ (which corresponds to a deep-
486 water wave condition in the present study), the influence of the variation in d/L on the
487 H/L is less evident. On the other hand, although the value of E_m^T for Case D9 is relatively
488 high (6.360×10^{-3}), it is noted that the nonlinearity in this case is very high resulting from a
489 stronger opposing current and shallower water depth. To further investigate the influence of
490 water depth on wave steepness under different opposing current velocities, additional
491 calculations are carried out and the results are plotted in Fig. 12. It can be seen in Fig. 12 that
492 for each opposing current, a transitional point exists and divides the curve into two parts. By
493 connecting all the transitional points, it is clearly seen that on the left side of the transition line,
494 the values of H/L increase significantly as d/L decreases. On the right side of the
495 transition line, however, H/L is independent of d/L . As shown in Fig. 12, H/L

496 increases to a higher value (approximately 0.12) as the opposing current velocity increases.

497 This is a further demonstration of the influence of the opposing current on the wave steepness.

498 **Conclusion**

499 In this paper, an analytical approximation of nonlinear wave-current interaction in water

500 of finite depth is derived using the homotopy analysis method. Series approximation solutions

501 are obtained and compared to experimental and available numerical results; they demonstrate

502 that the present method not only gives highly accurate results of wave parameters for the

503 interaction between steep waves and a strong opposing current, but the method also produces

504 excellent results for wave kinematics. Based on the validated analytical model, the interaction

505 between waves and a strong opposing current is investigated to clarify the influence of the

506 strong opposing current and water depth on the wave profile, wavelength and wave steepness.

507 The following key conclusions can be drawn from the present study:

508 (1) The accuracy and convergence of the series approximation solutions obtained by the

509 proposed method are verified by estimating the errors of the exact kinematic and dynamic free

510 surface boundary conditions, and by comparing the present experimental measurements and to

511 an available numerical solution. This demonstrates the proposed homotopy analysis method is

512 a very effective technique to study nonlinear waves interacting with a strong current in finite

513 water depths.

514 (2) An opposing current leads to significant decreases in wavelength and tends to narrow

515 both the crest and trough. The wave crest elevation increases as the opposing current velocity

516 increases, and the wave trough elevation tends to remain constant throughout.

517 (3) The wave steepness H/L at each water depth increases consistently as the non-

518 dimensional opposing current velocity $|U_0|/C_0$ increases. It is also observed that the water
519 depth effect in the case of smaller opposing current velocity is more pronounced than that in
520 the case of larger opposing current velocity.

521 (4) At each water depth the non-dimensional wavelength L/L_0 decreases as the non-
522 dimensional opposing current velocity $|U_0|/C_0$ increases. The water depth effect on L/L_0
523 dominates over the range of smaller opposing current velocities, and the value of L/L_0 for
524 each water depth tends to approach an asymptotic value as U_0/C_0 increases.

525 (5) Under the existence of an opposing current, a decrease in relative water depth d/L
526 leads to an increase in wave steepness H/L . Two regimes exist and are separated by a
527 transition line: on the left side of the transition line, the value of H/L increases significantly
528 as d/L decreases, while on the right side of the transition line, the value of H/L is
529 independent of d/L .

530 The method presented in this paper can be applied to solve more complex scenarios of
531 nonlinear wave interaction with strong currents in water of finite depths leading to engineering
532 applications in the coastal and offshore industries.

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