Periodic orbits high above the ecliptic plane in the solar sail 3-body problem

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Overview

1. Solar Sails
   - Sail design
2. 3 Body Equations and Equilibria
   - Classical equilibria - the Lagrange points
   - Sail equilibria
3. Periodic Solutions
   - Lindstedt-Poincaré method
   - Families of periodic solutions
4. Applications
   - Polesitter
   - Invariant manifolds

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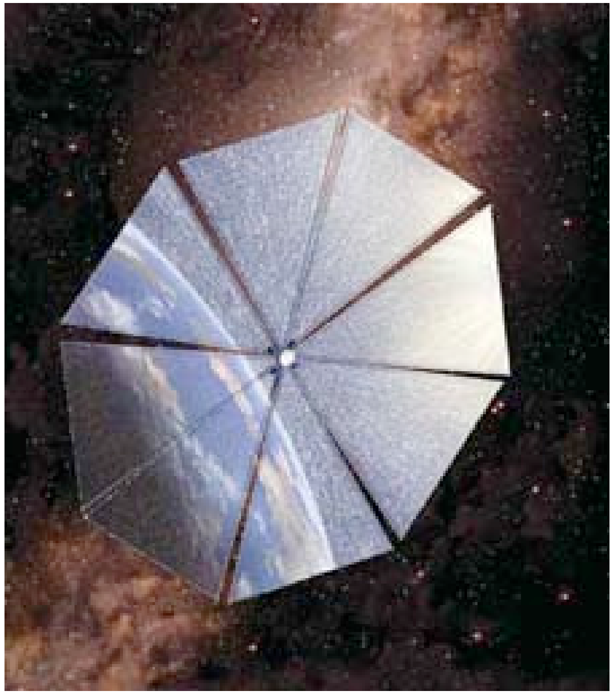
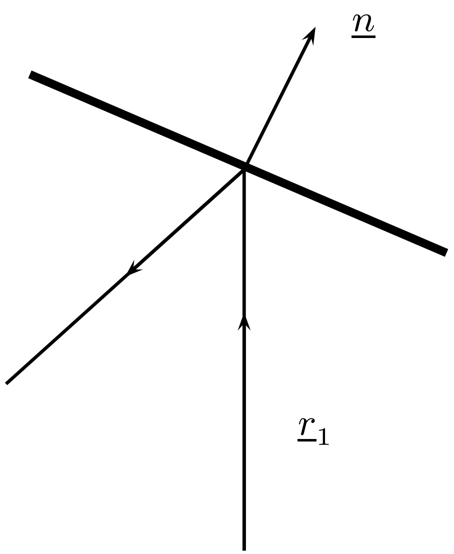
Sail design



- A solar sail is a spacecraft without an engine, and therefore needs no fuel. It is pushed along by the pressure of photons from the sun hitting the sail.
- Solar sails are typically large square sheets of a highly reflective film supported by booms, although other designs (discs, blades) are popular.
- The material of the sail must be very lightweight and thin, of the order of a couple of microns, and very large, the order of $(50m)x(50m)$.

Solar radiation pressure

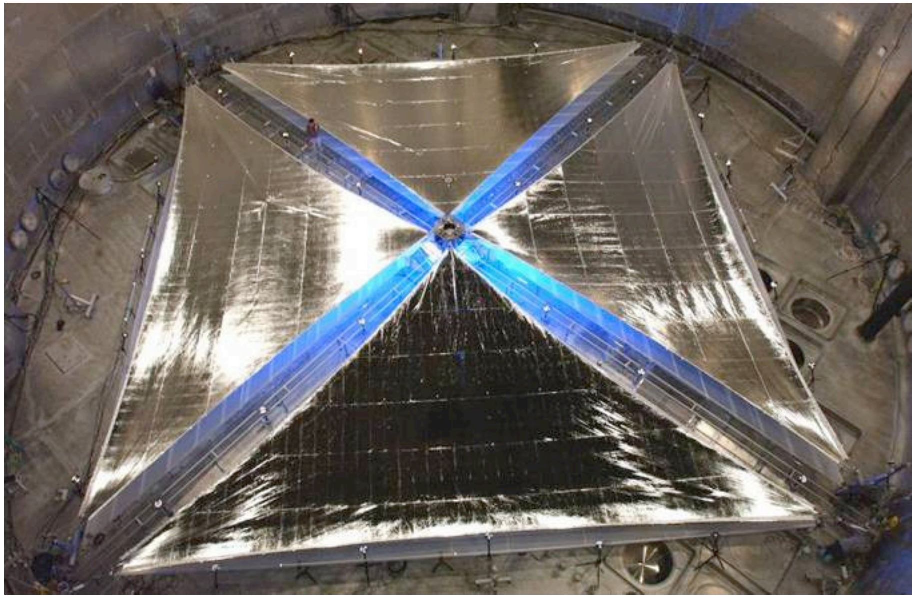


- Photons from the sun hitting the sail impart a small but constant radiation pressure.
- The acceleration on the sail due to this pressure is given by

$$\underline{a} = \beta \frac{M_s}{r_s^2} (\widehat{\underline{r}}_s . \underline{n})^2 \underline{n}.$$

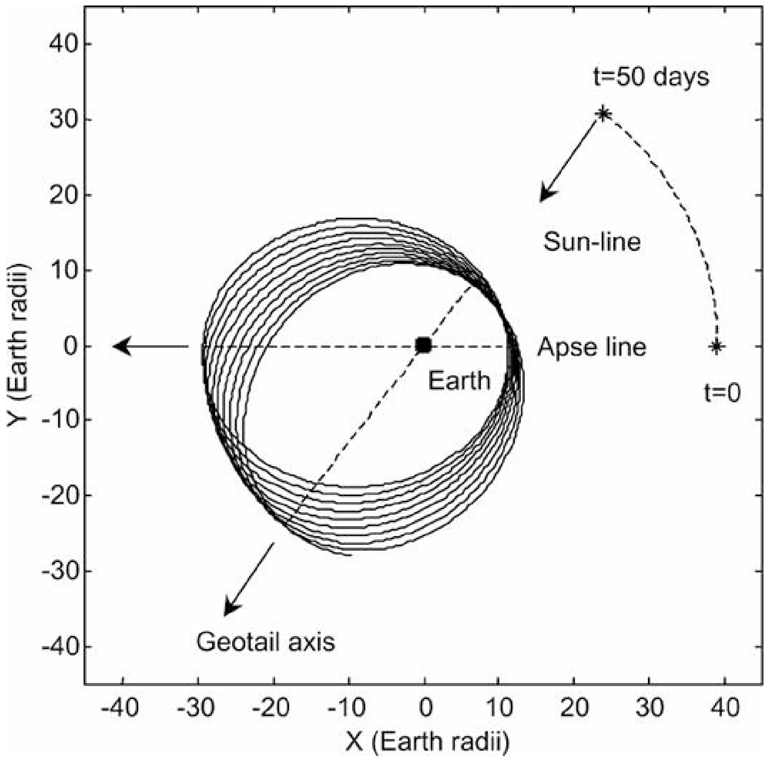
- Here β is the "sail lightness number", the ratio of the radiation pressure acceleration to gravitational acceleration.

Current designs

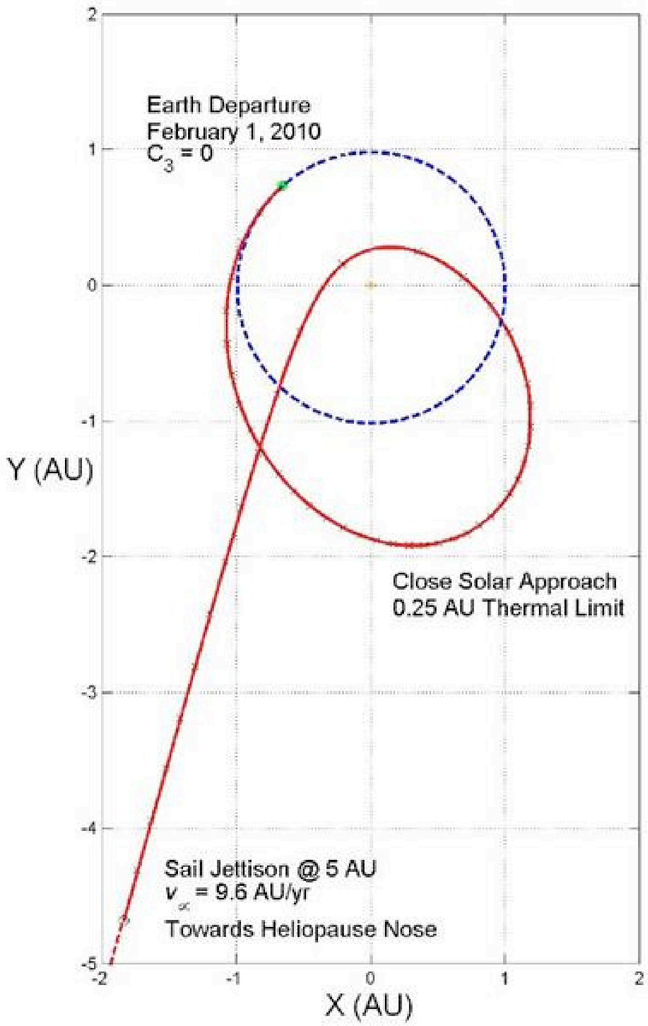


- Solar sails are currently being built with a lightness number $\beta \sim 0.2$.
- For the purpose of this presentation we only consider $\beta < 0.1$.

Possible missions



Geosail and Heliopause missions



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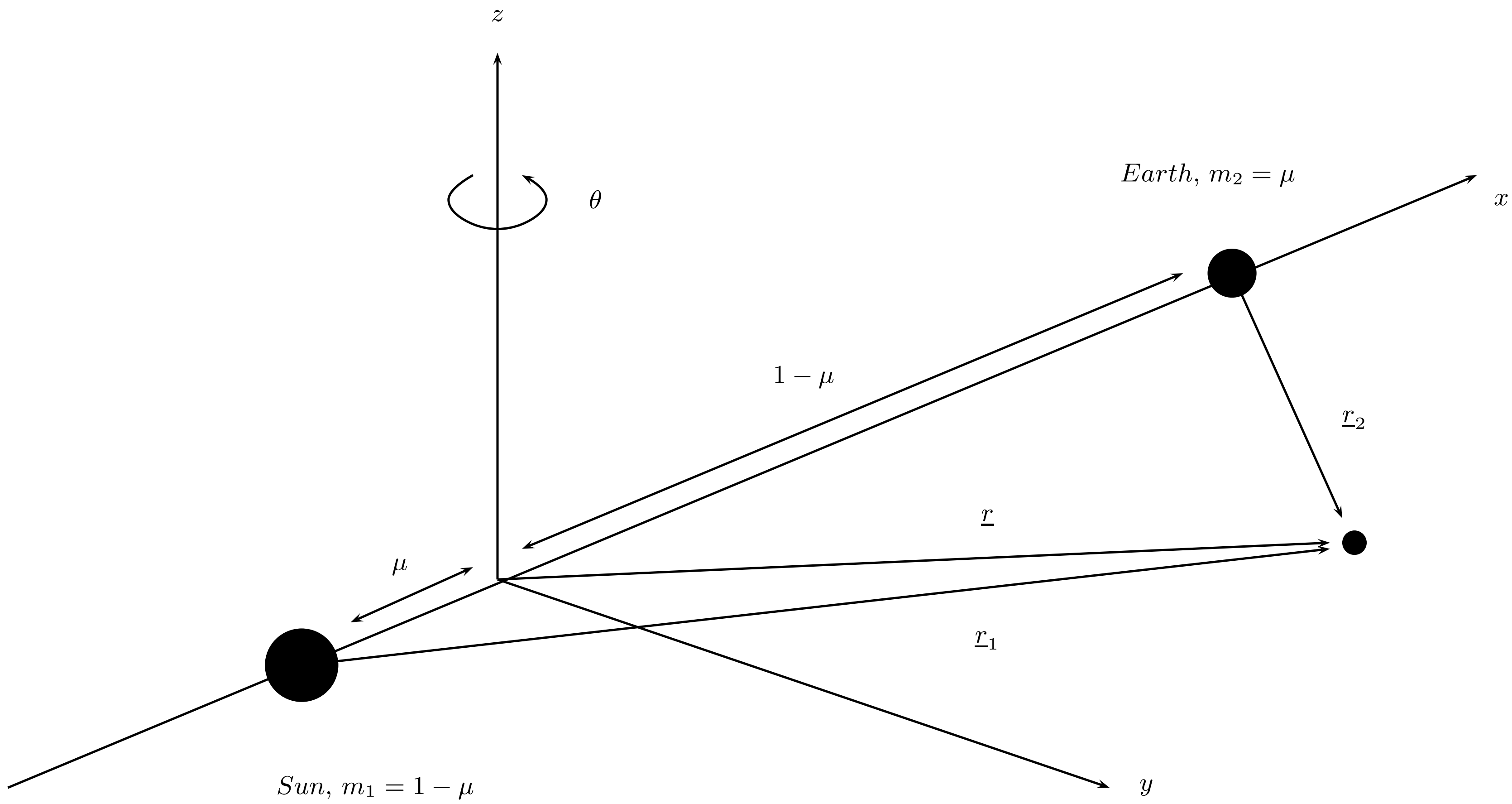
Classical equilibria - the Lagrange points

The CR3BP in a rotating coordinate frame:



Classical equilibria - the Lagrange points

- In the inertial frame, the equations of motion of the third body are:

$$\frac{d^2\underline{r}}{dt^2} = -\nabla V, \qquad V = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right).$$

- However, in the rotating frame two additional forces are introduced: the Corioli force and the centrifugal force,

$$\frac{d^2\underline{r}}{dt^2} + \left\{2\underline{\theta} \times \frac{d\underline{r}}{dt} + \underline{\theta} \times (\underline{\theta} \times \underline{r})\right\} = -\nabla V.$$

- Equilibrium points now exist in this coordinate system.

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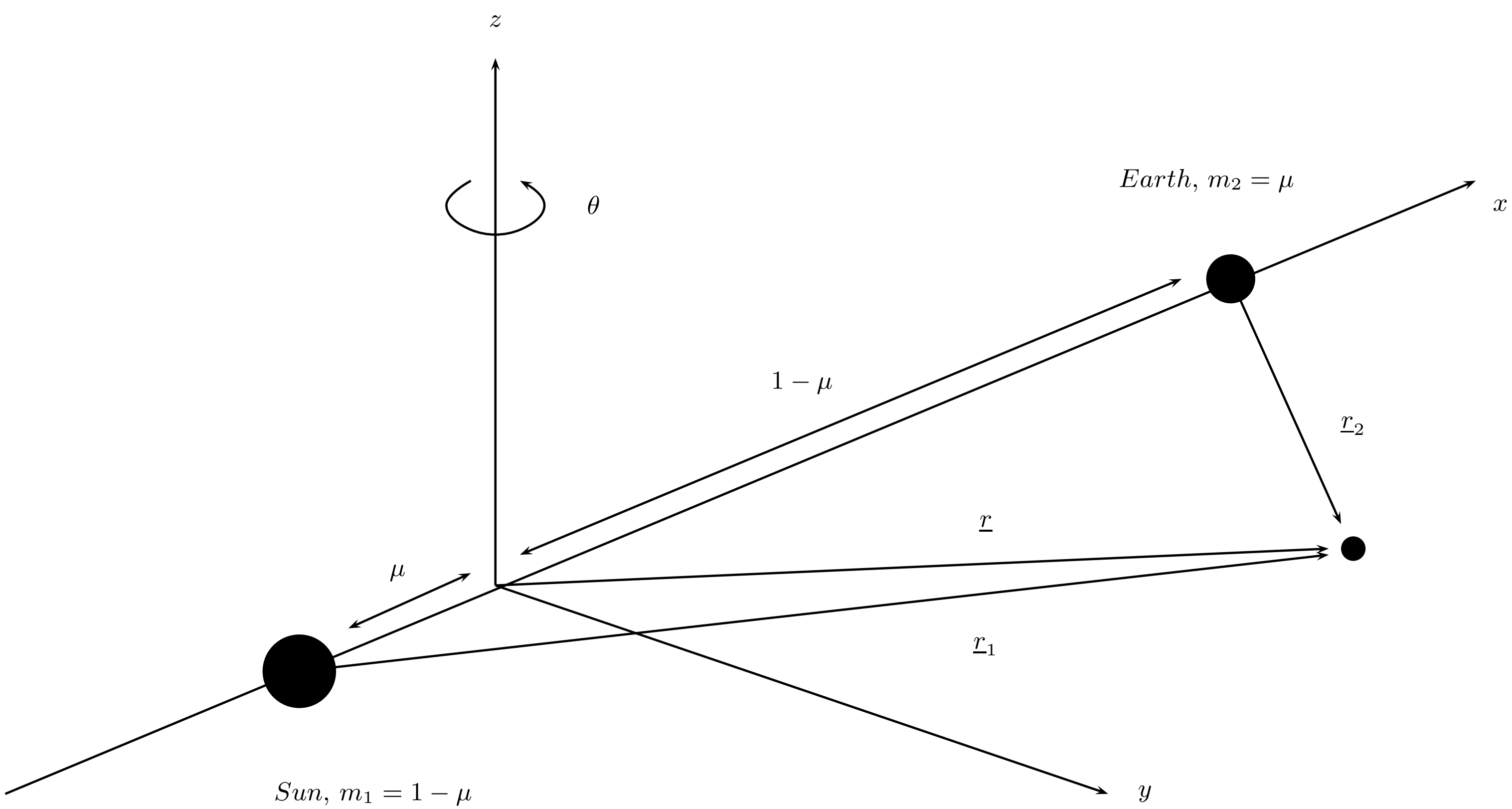
Classical equilibria - the Lagrange points

There are five equilibrium points in the rotating coordinate frame, called Lagrange points. All are in the plane of the primaries' mutual orbit.



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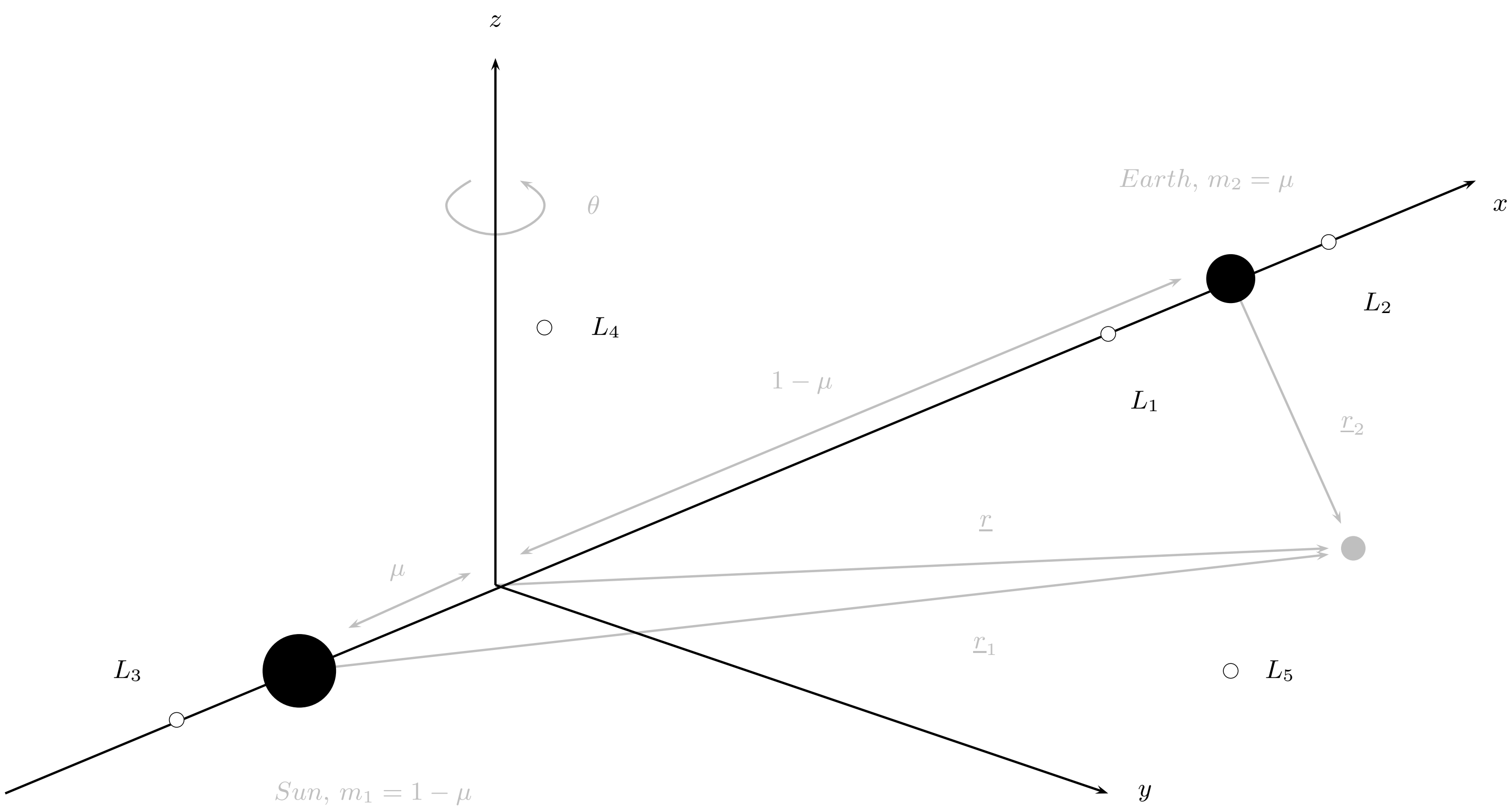

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Sail equilibria

- In the rotating coordinate frame the equations of motion of the solar sail are

$$\frac{d^2\underline{r}}{dt^2} + 2\underline{\theta} \times \frac{d\underline{r}}{dt} = \underline{a} - \underline{\theta} \times (\underline{\theta} \times \underline{r}) - \nabla V \equiv \underline{F},$$

where

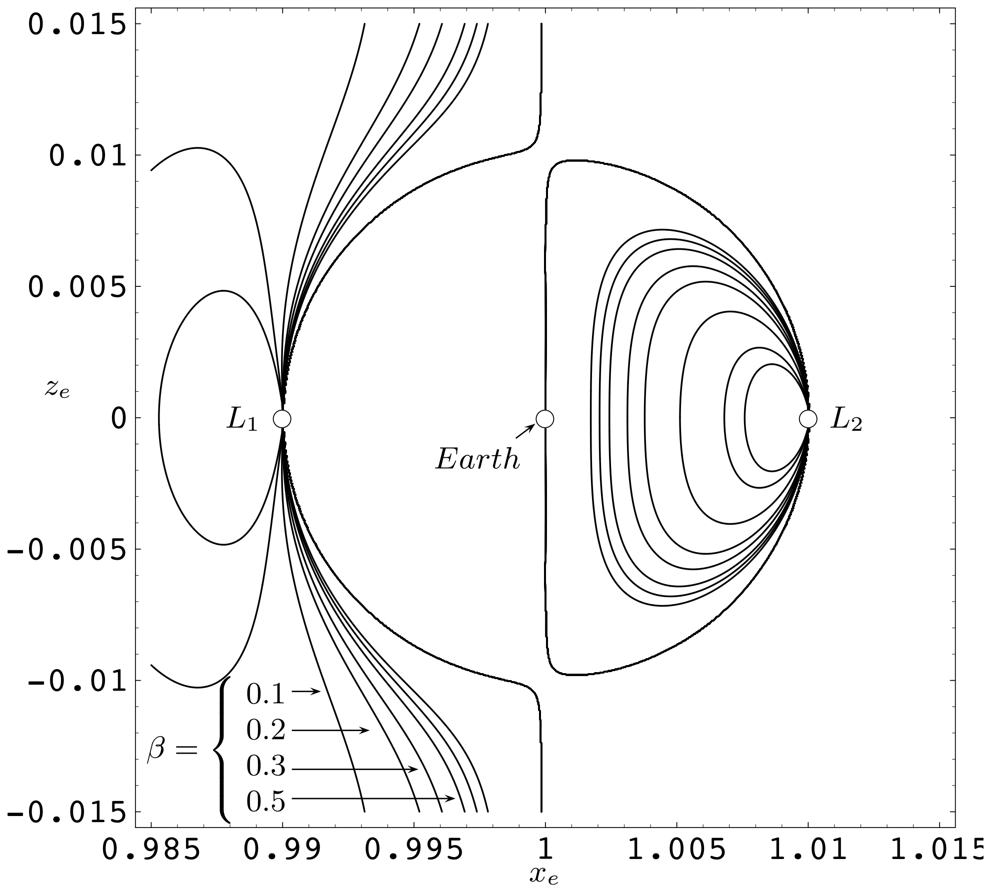
$$\underline{a} = \beta \frac{1-\mu}{r_1^2} (\widehat{\underline{r}}_1 . \underline{n})^2 \underline{n}.$$

- At equilibrium, $\dot{r}$ and $\ddot{r}$ vanish, so an equilibrium point is a zero of $\underline{F}$.
- We seek equilibria in the x-z plane, thus we let

$$\underline{n} = \cos(\gamma)\widehat{\underline{x}} + \sin(\gamma)\widehat{\underline{z}}.$$

Sail equilibria

We find continuous surfaces of equilibria:



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Periodic solutions

- We use the Lindstedt-Poincaré method to find periodic solutions.
- We Taylor expand around an equilibrium point

$$\begin{pmatrix} \ddot{x} - 2\dot{y} \\ \ddot{y} + 2\dot{x} \\ \ddot{z} \end{pmatrix} = \delta r^a (\partial_a \underline{F})|_e + \frac{1}{2}\delta r^a \delta r^b (\partial_a \partial_b \underline{F})|_e$$

$$+\frac{1}{6}\delta r^a \delta r^b \delta r^c (\partial_a \partial_b \partial_c \underline{F})|_e + O(\delta \underline{r}^4)$$

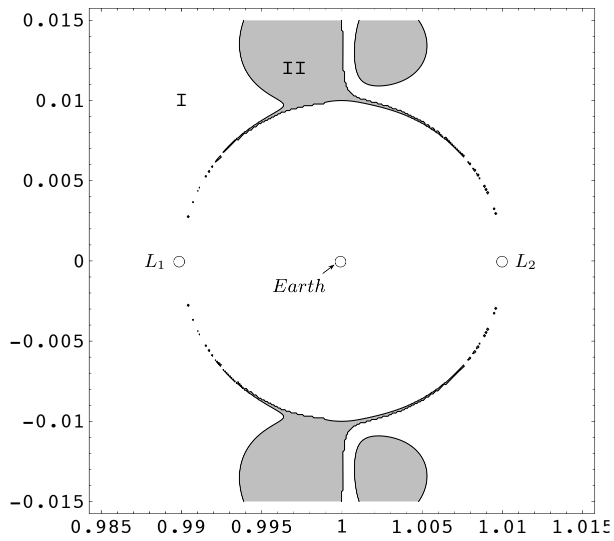
- At linear order

$$\begin{aligned} \ddot{x} - 2\dot{y} &= ax + bz \\ \ddot{y} + 2\dot{x} &= \quad cy \\ \ddot{z} \quad &= dx + ez \end{aligned}$$

Periodic solutions

- There are two distinct regions, depending on the eigenvalues of the linear system.
- In region I, the linear spectrum is

$$I : \{ \pm \lambda_1 i, \pm\lambda_2 i, \pm\lambda_r \}$$



- Thus periodic solutions exist at linear order, by suppressing unwanted modes:

$$x, y, z = A\cos(\lambda_1 t) + B\sin(\lambda_1 t) + C\cos(\lambda_2 t) + D\sin(\lambda_2 t) + Ee^{\lambda_r t} + Fe^{-\lambda_r t}$$

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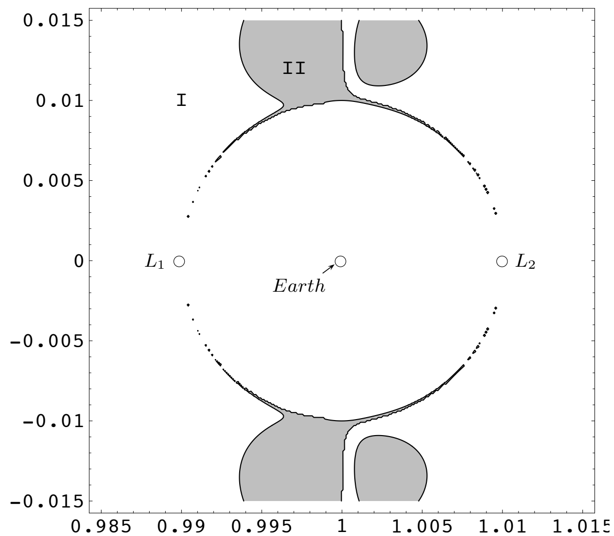

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Lindstedt-Poincaré method

- We let ε be a small parameter (typically the amplitude of a periodic orbit), and we let

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + O(\varepsilon^4).$$

- The idea is, if the frequency of a periodic solution in the linear system is λ, then this method lets you find approximations to periodic solutions in the nonlinear system with frequency $\omega\lambda$.
- We define a new time coordinate, $\tau = \omega t$, and let $x_n = x_n(\tau)$ etc. Then the following statements are equivalent:

The periodic solution has frequency $\omega\lambda$ in t-seconds $\Leftrightarrow$ The periodic solution has frequency λ in τ-seconds

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Lindstedt-Poincaré method

- We find the solutions for x and z are trigonometric cosine series and y is a sine series. For example when n is odd

$$x_n = p_{n3}\cos(3T) + \ldots + p_{nn}\cos(nT),$$

and when n is even

$$x_n = p_{n0} + p_{n2}\cos(2T) + \ldots + p_{nn}\cos(nT).$$

- The problem reduces to solving systems of algebraic equations for the coefficients p_{ni} and ω_i.
- For large amplitude periodic solutions high above the ecliptic plane, we need to include up to the 7th term in the series (7th order approximation).

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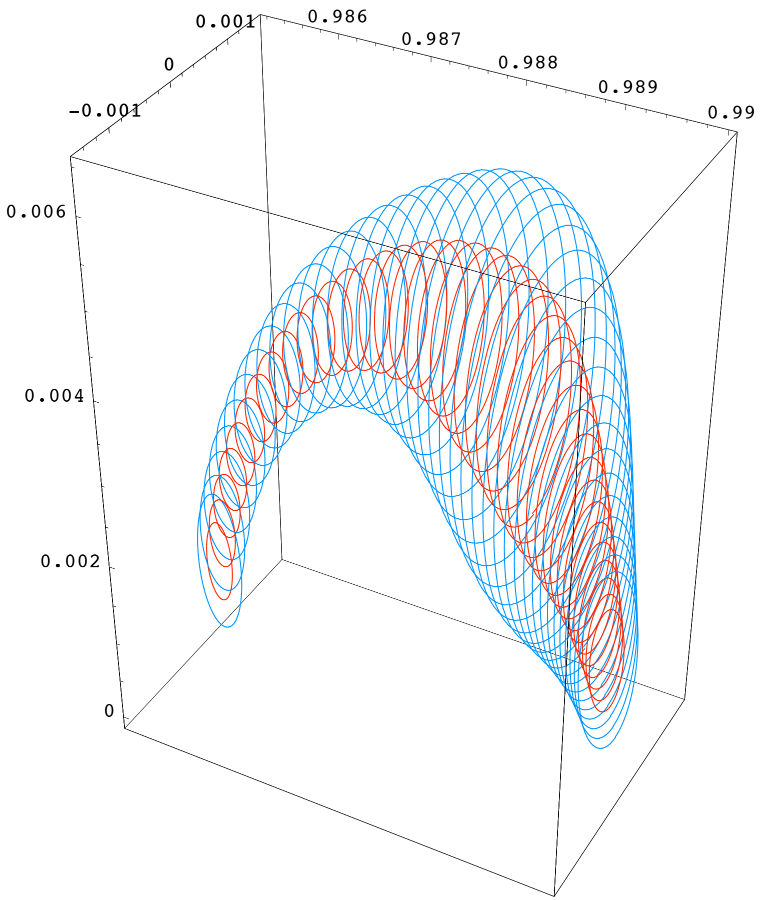
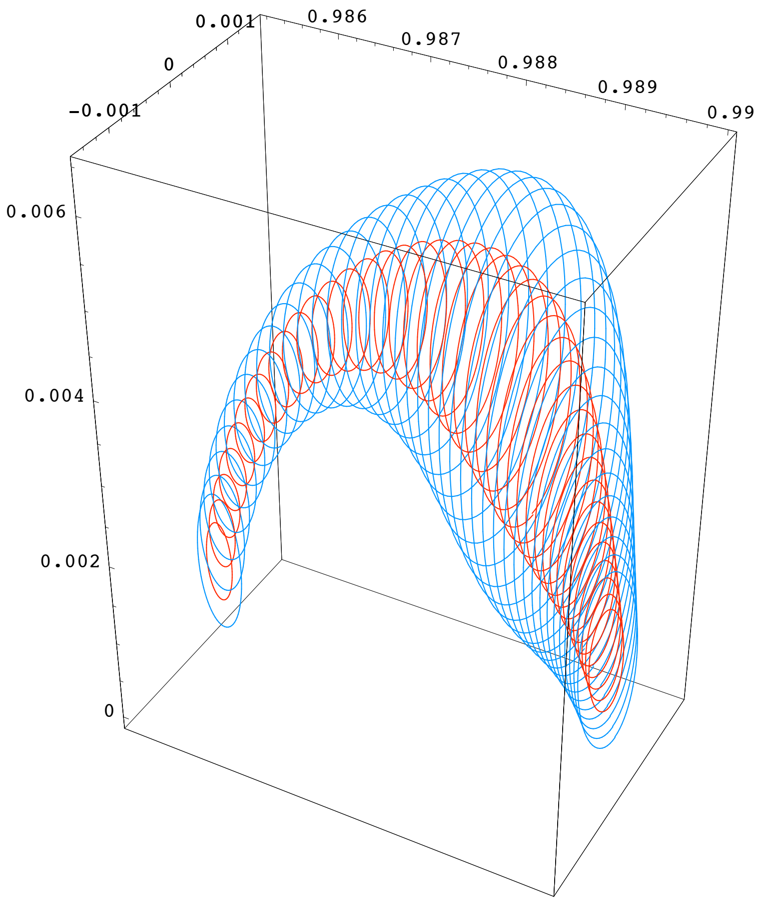
Families of periodic solutions



- The Lindstedt-Poincaré series provides our first guess for the initial data of a periodic solution of the full nonlinear system.
- We then use a differential corrector to fine tune the initial data.

The picture on the left shows some of the periodic orbits possible for a solar sail with $\beta = 0.03$.

Families of periodic solutions



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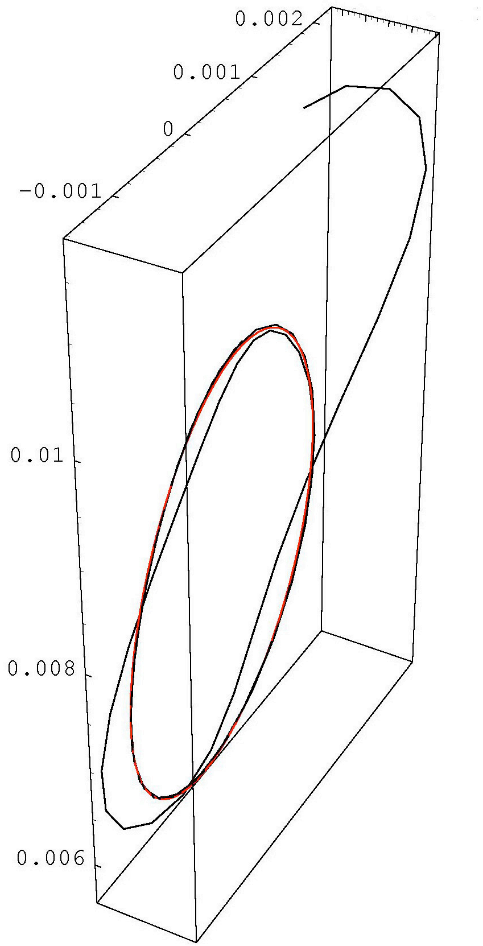
Stability - Control



- The orbital stability of these periodic solutions is found by calculating the eigenvalues of the monodromy matrix.
- The spectrum is typically $\{1, 1, \lambda_r, 1/\lambda_r, \lambda_c, \bar{\lambda}_c\}$
- However, we can easily control to the nominal orbit with variations in the orientation of the sail normal.

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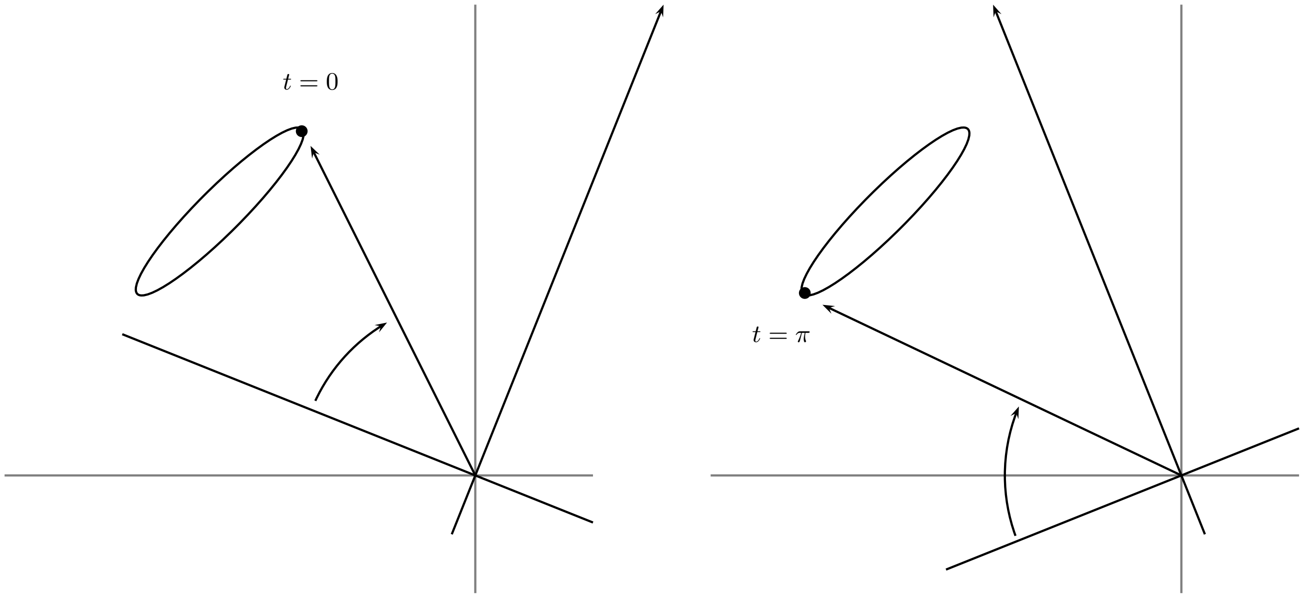
Polesitter

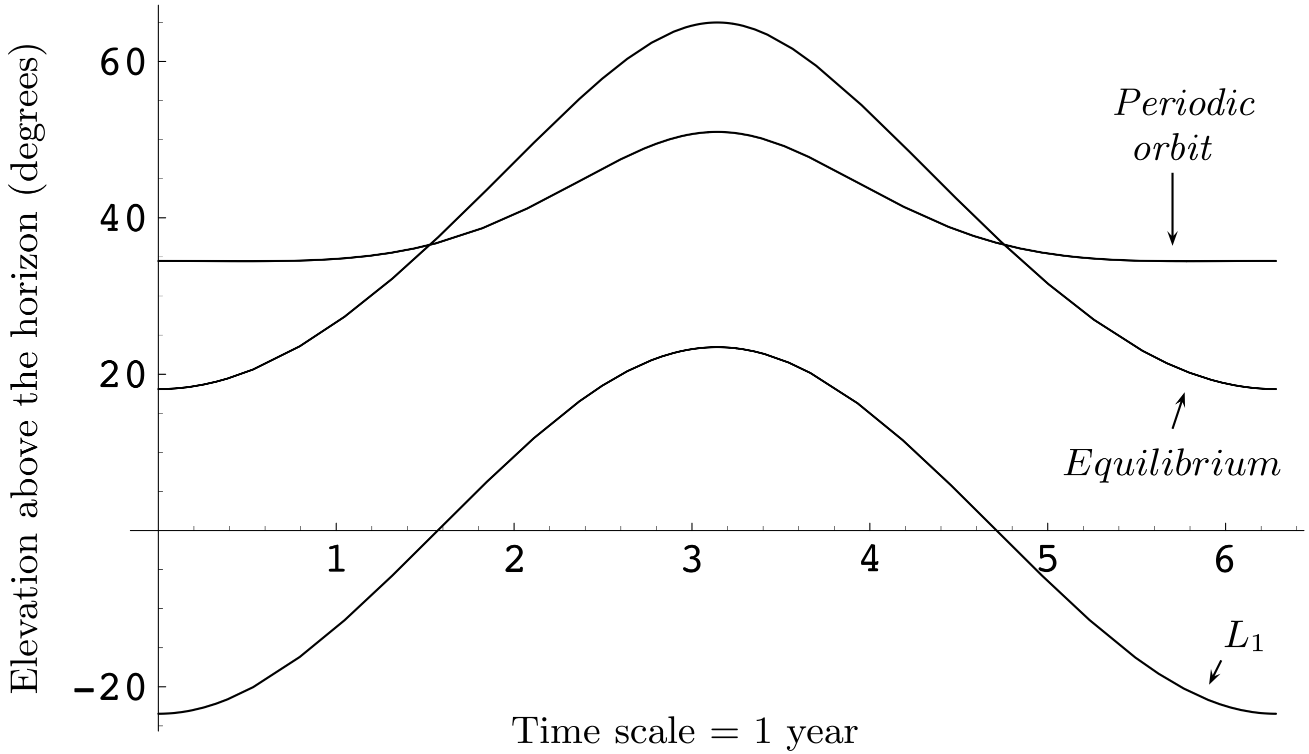
- We may use these orbits to provide a constant view of one of the poles.



Wintertime, Northern Hemisphere *Summertime, Northern Hemisphere*

View from the pole

By timing the orbit well we can narrow the angle subtended by the sail:



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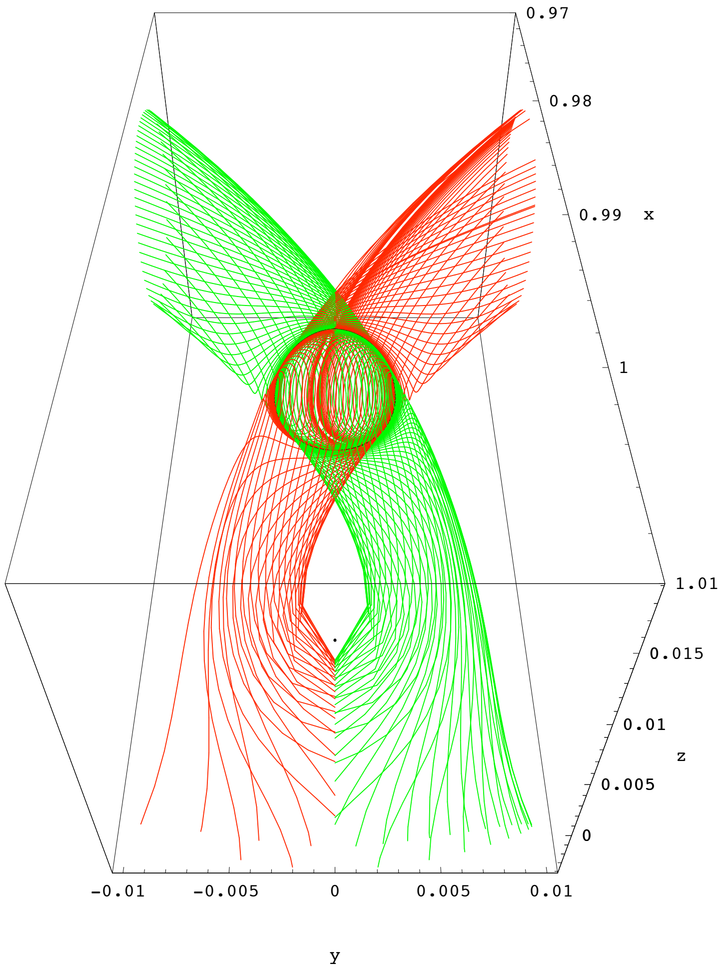
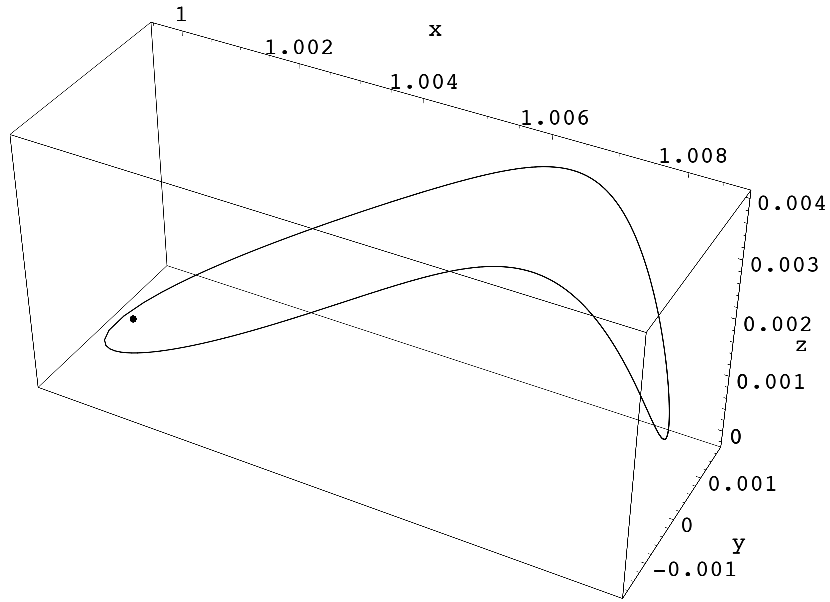
Invariant manifolds



- Associated with each periodic orbit are a set of invariant manifolds.
- We find these by integrating in the direction of the stable and unstable eigenvectors of the monodromy matrix.
- These manifolds provide us with a mechanism to transfer between various periodic orbits.

Unusual orbits



- Unexpected families of periodic solutions arise.
- Interestingly, this particular orbit has monodromy matrix with spectrum

$$\{1, 1, \lambda_{r1}, 1/\lambda_{r1}, \lambda_{r2}, 1/\lambda_{r2}\}$$