Mental Calculation: its place in the development of numeracy

Introduction

The current concerns about the standards of numeracy in primary schools, as these are manifest in different official reports (HMI, 1997; DfEE, 1998), have given a revised emphasis to mental calculation. While not completely discounting the wider aspects of mathematical achievement, the topics of space and shape, data handling and measurement are being de-emphasised (Brown et al, 2000) and mental calculation is being emphasised, with there being daily opportunities for children to develop efficient and flexible mental methods of calculating (QCA, 1999; Wilson, 1999). However, the term, mental calculation is not clearly defined (Harries and Spooner, 2000) and without conceptual clarity it may be very difficult for us to recognise, let alone understand, what pedagogical practices are needed to support the objective of increased emphasis on mental calculation. What follows is some consideration of what is meant by the term mental calculation and what this meaning implies for practice.

What is meant by the term, mental?

Mental calculation is the process of carrying out arithmetical operations without the aid of external devices (Sowder, 1988). Notably mental calculation is carried out 'in the head' rather than 'on paper' though this does not preclude the need for recording symbolisation to assist mathematical reasoning (Harries and Spooner 2000). The emphasis on activity which is carried out 'in the head' rather than 'on paper' is not, however, to be understood as only the quick-fire recall of basic number facts, though effortless recall of number bonds can free up precious mental processing capacity and thus may be a necessary condition for effective mental calculation. Particularly, mental calculation necessarily uses strategies which are very different from the algorithms associated with pencil-and-paper procedures. Conventional algorithms are of a permanent and standardised form (which renders them 'correctable'), are efficient and automatic (which renders them amenable to use even if they are not understood) and are generalisable (which renders them capable of application to any domain of number but need not have any articulation with the ways in which people think about number) (Plunkett, 1979). Mental strategies, on the other hand are variable, flexible, creative and idiosyncratic. While a conventional written algorithm would treat numbers as single digits and would adopt a uniform approach to executing say subtraction operations, mental calculation would tend to work with numbers holistically and would probably not compute 83-79 in the same way as 83-51 (Sowder, 1988).

What is meant by the term, calculation?

While there is no clear definition of calculation, dictionary definitions and usage (Goulding, 1997; Haylock, 1995; Haylock and Cockburn, 1997; QCA, 1999) tend to conflate calculation with computation. Whatever the referent, the task is to manipulate numbers to achieve the desired answer. The desired answer may be a precise and exact one or it may be that an approximate value rather than a precise one will suffice. When the purpose is to obtain an exact answer to the arithmetical problem at hand then computation is required, although the extent to which the computation can be mental is constrained by the magnitude of the numbers involved (Sowder, 1988). If, on the other hand, the purpose is to achieve an approximate answer or indeed if the numbers are of large magnitude (Sowder, 1988) then estimation is appropriate. But this estimation involves computation. According to Sowder (1988), estimation is the process of converting from exact to approximate numbers and mentally computing with those numbers to obtain an answer which is reasonably close to the result of an exact computation. In other words, estimation includes, but is more than, computation. Mental calculation is then the process of carrying out arithmetical operations to achieve either an exact answer (in which case mental computation is required) or an approximate answer (in which case computational estimation is required).

Why is mental calculation important?

As was mentioned earlier, for numeracy to be useful in everyday life, an approximate answer may be perfectly adequate. For example, Mary's car gets 6.5miles to the litre of fuel. The tank holds 63 litres. When the tank is full of fuel, about how far can Mary travel before she needs to refill the empty tank? The need for a precise answer to this question is not necessary since the purpose of the approximation is that Mary makes the prudent judgement of where and when to have the tank filled up again rather than risk running out of fuel in the middle of nowhere. Being able to make a reasonable estimate is fundamentally dependent on being able to compare numbers (Sowder, 1988). This means being able to order real numbers on the basis of their size (as in selecting the larger or smaller of two numbers) or being able to compare numbers of different magnitudes (as in identifying which of two numbers is closer to a third). Sowder (1988) points to research which shows that children have great difficulty with the concepts of number size. For example, ten-year olds generally have little difficulty in comparing whole numbers of four or five digits (while eight-year olds do) but when the numbers become larger the children are not so successful in making the correct comparisons. When comparison, further, involves decimal numbers or common fractions children appear to generalise from their knowledge of whole numbers but fail to extend their knowledge to take account of the fractional elements of the numbers. While the concept of number size may be slow to develop and difficult to achieve, it is nevertheless a powerful concept. The ability to make accurate judgements about relative size enables estimation to be meaningful to the estimator, thereby enabling intuitions about number to develop. Paradoxically, perhaps, the power of estimation comes from there being many possible answers to a problem and some or all of these answers may adequately meet the contextual requirements of which the operation is a part. Sowder (1988), however, points out that teachers and children consider mental computation to be superior to estimation because estimation is just guessing! This lack of regard for the power and the value of estimation diminishes the role of mental calculation and may well encourage us to conceptualise mental calculation as drill-and-practice routines, operated according to standard rules.

<u>How is mental calculation important?</u>

By virtue of their defining characteristics, conventional written algorithms encourage children to carry out the different steps of a computation without actually thinking about them. Mental strategies, on the other hand, demand that the child be actively thoughtful to determine what the numbers mean and how these might be changed in appearance but not in value. There are various strategies available (Thomson, 1999). For example, the child might *count on* and so solves 6+2 by counting ‘six seven eight’. Or the child might *use a known fact* and so solves 2+2 through instant recall. Additionally, however, the child might *derive* that 6+8=14 because he/she knows that 6+6=12 and knows also that 8 is two more that 6. Whether children be counting on, using a known fact, using a derived fact, *partitioning* (adding 29 and 36 through recognising that 29 is 20 + 9 and that 36 is 30 + 6 and concluding that the 'tens' total 50) or *bridging through ten* (dealing with the units in the last example by saying something like 9 and 1 more makes another 'ten', which then means I need to add 'ten' and 5 to the 50 I've already got), it is their *determination* of an effective mental strategy which is important. In determining mental strategies children are (albeit unconsciously) trying to answer two fundamental questions (Sowder, 1988). The first question which the child is trying to answer is how the numbers in the operation can be structurally translated so that they can be answered by the knowledge and skills already in the child's repertoire. The second question which the child is trying to answer is what the operational sequence(s) will now be as a result of the structural changes to the original operation. So, for example, in addressing the operation 73-36, the child might recognise one, or more of the following:

> $73 + 3 – 36 – 3 =37$ (knowledge of number bonds and place value; the operational sequence is two subtractions)
>
> $36 + 4 = 40$, $40 + 30 = 70$, $70 + 3 + 73$. $4 + 30 + 3 = 37$ (knowledge of number bonds to fill in the missing addends; the operational sequence is three complementary additions followed by a final addition)
>
> $(60 + 13) – (30 + 6) = (60 – 30) + (13 – 6) = 30 + 7 = 37$ (knowledge of number bonds; the operational sequence is two subtractions followed by and addition)

It is not just one strategy which can effectively address the operation. More than one strategy may be effective but not all strategies may be equally effective. As children recognise the range of knowledge and skill (knowledge of number bonds; knowledge of place value; ability to regroup numbers; ability to operate with powers of ten) which they possess and which they can use to compute mentally so their flexibility increases and choice of mental strategy can be made on the basis of speed and/or ease. As the knowledge and skill become personally useful to children, they are enabled to develop for themselves an understanding of number. Through appreciating how numbers and their operations function and through searching for efficient and economical strategies, the requirement to engage in mental computation actually promotes the development of a sense of number.

The argument so far is that mental calculation is important then because it promotes number sense. The requirement to engage in computation encourages the search for meaningful shortcuts which make use of basic number knowledge. The requirement to engage in estimation requires computation but, further, requires appropriate judgements to be made about the relative size of numbers to determine how reasonable or useful the approximate answer is. Estimation is particularly important as it allows the child to make mental sense when computation would be unreasonable because the numbers were very large or were fractional. This sense-making approach to numeracy (as distinct from a fragmented view of numeracy characterised by slavish application of meaningless techniques) would seem to underpin the recent official guidance (QCA, 1999; Wilson, 1999) on the importance of mental calculation.

Is the emphasis on mental calculation enough?

While the current emphasis given to mental calculation is to be welcomed because of its fundamental role in promoting number sense, it is an incomplete characterisation of the teaching of numeracy to try to suggest that the progressive and hierarchical introduction of mental strategies (QCA, 1999; Wilson, 1999) will be enough to secure the development of number sense. Although it is difficult to define what number sense is (Resnick, 1989a), there is considerable agreement (Case, 1989; Greeno, 1989; Resnick, 1989a) that it is a multi-faceted construct and while number sense includes flexibility in the use of strategies for precise and approximate calculation, it also includes the network of inter-connected numerical knowledge which an individual may possess. In the struggle to capture the essence of number sense there is the implication that it is the richness of the interconnections between different pieces of numerical knowledge, rather than the numerical knowledge per se, which is of critical importance. Number sense then, in the wider meaning of the term, is not a body of knowledge to be 'taught' but rather "reasoning and thinking in the number domain" (Resnick, 1989a). The achievement of (the most mature) reasoning in the number domain is, however, a developmental accomplishment which is dependent on the child's conceptualisation of what a number is. As will be adumbrated below, the meaning of what a number is changes from something fairly intuitive and elementary to something that is formal and explicit. The reasoning associated with an intuitive understanding of number is known as additive reasoning while the reasoning associated with the more sophisticated concept of number is known as multiplicative reasoning. Multiplicative reasoning allows the individual to appreciate that the operations of multiplication and division and topics such as proportion, ratio and fractions are inter-related in a complex web of meaning (Vergnaud, 1988).

Stages in the development of the concept of number

Natural Numbers

Natural numbers are those which children experience first. From a very young age, and before the start of formal schooling, children begin to develop a concept of number, and they do this in two distinct ways (Resnick, 1989b). Firstly they develop a large store of nonnumerical quantity knowledge which lets them reason intuitively about increases and decreases in quantities. They know, for example, that adding some objects to a collection will result in a larger collection and that removing objects will result in a smaller collection. Secondly, in an apparently separate line of development, children learn to count and thereby quantify their protoquantitative knowledge. Through a range of informal, social experiences in a variety of contexts (Fuson and Hall, 1983) children's numerical understanding develops to include the knowledge that counting words need to be matched to the items being counted in a one-for-one fashion; that the order in which the counting words are used is important but that the order in which items are counted need not be (Gelman and Gallistel 1978). Through counting, children demonstrate to themselves and others that sets of items can be transformed (through items being added or removed) and that these transformations can be numerically exact. However, while children are constructing this knowledge (largely through their own efforts), it has to be remembered that the child's concept of number is, as yet, only **partial**. The young child's understanding of number is bounded by being able to enumerate sets (and all that this implies) through using the natural number system (Denvir and Brown, 1987). Herein lies the critical element which allows additive reasoning: sets of items can be combined, changed or compared (Riley et al, 1983) but always, within such transformations, the underlying protoquantitive relationship is maintained. In other words the transformations which can be achieved by counting refine (and quantify) the combinations, changes and comparisons which children are able to effect perceptually, nonlinguistically and implicitly (Resnick, 1989b). What is important to note is that children's intuitive understanding of additive reasoning **is perfectly adequate** for the demands made by the formalisms of addition and subtraction operations, which emphasise the cardinal aspect of number.

Integers

Integers extend the concept of what a number is in subtle ways. Perhaps the most important nuance to be introduced is that numbers need not represent sets of things since integers include negative as well as positive whole numbers. Thus there is no 'first' number (as there is in the set of natural numbers) since numbers can extend indefinitely on both sides of zero. The idea that numbers are (only) collections of individual entities which can be physically modelled is now unhelpful. However, in spite of the fact that the cardinal aspect of number is no longer powerful, it still continues to dominate the teacher's thinking (Haylock and Cockburn, 1997). This can be inferred from at least two practices. One is the practice of encouraging children to use concrete materials to effect addition and subtraction operations (Desforges and Cockburn, 1987) in spite of the evidence that they may well be able to correctly compute the operation/solve the arithmetical problem without concrete materials (Carpenter and Moser, 1982; Solomon, 1989). This persistent reliance on concrete materials may well inhibit children from realising that objects need not be perceptually present in order to be counted but can be represented by the counting words. By extension, reliance on concrete materials may also inhibit the mental strategy of counting-on. A second practice which points to the dominance of the cardinal aspect is in the teaching of conventional, written algorithms for computing multidigit subtraction operations. Brown and Burton (1978) found that when a larger number has to be subtracted from a smaller number in any given column, children frequently inverted the digits in order to (erroneously) effect the operation. So, for example, if the minuend were 93 and the subtrahend 48, the child would take 3 from 8 in the belief that it was not possible to take 8 from 3. The error of inversion, further, intensifies when it is necessary to borrow from zero. Brown and Van Lehn, (1982) maintain that such faulty procedures develop when children forget some step of the procedure as it was taught to them and, in the face of difficulty, invent their own parts of the procedure. It does, however, seem that in this particular instance a sound appreciation of negative numbers would obviate the mistaken belief that it is impossible to take a larger number from a smaller number. This is not to deny that a grasp of place value is an essential concept in the development of number; merely an observation that an understanding of place value might be enabled by the realisation that when numbers are understood as integers, subtraction is always possible (Haylock and Cockburn, 1997).

Rational Numbers

Having understood that numbers need not be sets of things, there is now a need to extend the concept of number to include common and decimal fractions. This means a veritable explosion of the 'number of numbers' which is available. While understanding number as an integer means there is a number between 5 and 7, it also means that there is no number between 7 and 8. However, to understand number as rational is to understand not only that there can be a number between 7 and 8 but that indeed there is an indefinite number of numbers between 7 and 8. The realisation that it is always possible to insert a number between two given numbers represents a fairly sophisticated understanding of the concept of number, which is qualitatively different from understanding numbers either as natural numbers or as integers. Achieving the more sophisticated understanding requires the individual child to reorganise completely his/her understanding of what is meant by 'a number'. The nature of the unit has now changed. No longer are all quantities represented in terms of units of 'one'. The unit can now mean composite units or, indeed, partitioned units.

Operating with composite units, rather than units of 'one', signals the beginning of multiplication. A set or collection of entities can be treated as a unit or a whole which can then be replicated as many times as is determined by the scalar factor. Thus in the example, ‘a bicycle has 2 wheels - how many wheels are there when we have 6 bicycles?’ the composite unit of 2 wheels can be treated as ‘one’ while the scalar factor (or number of replications) is 6. Because the ‘6 sets of 2’ can be written as 2+2+2+2+2+2 and the answer of 12 found by adding ‘2’ repeatedly, it would be quite typical for children to be introduced to multiplication as repeated addition (Haylock and Cockburn, 1997). However, multiplication is not simply repeated addition. Although some aspects of addition form the basis of multiplication, it is unhelpful to treat the teaching of multiplication as a rather complicated form of addition because to do so implies that only the cardinal value of number is important and, further, detracts from the significance of the composite unit. That the change in what a unit can mean presents difficulties, is not a new idea. More than thirty years ago Dienes and Golding (1966) argued that children’s poor sense of number was exacerbated by teachers who did not confront what is involved in multiplication but, rather, “teach that multiplication is nothing but repeated addition”. Repeated addition allows problems involving multiplication to be solved when the quantities are **extensive** (Schwartz, 1988), that is when the quantities can be measured or counted as in how much fuel will be needed to fill 6 cars when 1 car needs 75 litres, or how many marbles there are in 3 bags when there are 9 marbles in 1 bag. Extensive quantities behave additively by mirroring the combining and partitioning which appears to be such a fundamental part of our actions on the environment (combining two extensive quantities yields a larger extensive quantity; partitioning an extensive quantity yields smaller quantities which, when re-combined result in the original quantity). The addition operation satisfactorily solves the problem because the problem only requires to know the increased amounts of fuel /marbles.

However, repeated addition is quite inadequate for solving problems which contain **intensive** quantities. Intensive quantities represent the relationship between two extensive quantities. Rates such as £20.00 per hour, £3.00 per kilo, 12 metres per hour are examples of intensive quantities. Because the two extensive quantities which form the intensive quantity represent different sorts of things - money and time, money and weight, distance and time - children have to appreciate that as one of the quantities grows larger (or smaller) by a designated amount, the other quantity will increase or decrease correspondingly. This grasp of ratio is fundamental to understanding or solving the following: 8 pizzas will feed 5 persons comfortably so how many pizzas will be needed to feed 20 people? Thus if children are to genuinely engage in multiplicative reasoning they have to appreciate that the relationship between the elements may be constant (which allows repeated addition) but they must also, critically, appreciate that the relationship may be a covarying one.

Embedded in an appreciation of ratio is an awareness of the partitioned unit. The ratio 2:3 could represent at least three different types of situation:

there are 2 girls for every 3 boys in the class so 2/5ths of the class are girls (the numerator and denominator are quantities which can be compared)

the 2 pound pizza was shared equally amongst 3 people so each person got 2/3rds of the pizza (the numerator is a quantity and the denominator is a parameter giving the idea of partitioning)

the magician was 6 feet tall but after drinking the magic potion shrank to 2/3rds of his height (the numerator is a multiplier and the denominator is the divisor to determine the 'new' equivalence).

This range of meaning is important to grasp since some conceptual underpinning for the partitioned unit may help to make sense of algorithmic activities. Typically, children at primary school (and beyond) have considerable difficulties with fractions. For example, when required to add 1/2 and 1/3, many will respond with 2/5 (Silver, 1983). Similarly when required to estimate the sum of 12/13 and 7/8, fewer than one third of the sample of thirteen and seventeen year olds suggested the response as 2 (Carpenter et al, 1981 cited in Siegler, 1998). The remainder could either offer no estimated response or suggested that the correct answer would be 1, 19 or 21! Such responses suggest that a fraction is not readily understood as a single number but, rather, as a pair of whole numbers. Difficulties in appreciating the partitioned unit are also reflected in decimal fractions. In comparing the relative size of two numbers such as 3.79 and 3.126, ten and eleven year olds would typically say that the larger number is the one with more digits to the right of the decimal point (Resnick et al, 1989 cited in Siegler, 1998). Again the persistence of additive reasoning in the children's thinking can be inferred: whole numbers with more digits are larger than whole numbers with fewer digits; thus decimal fractions with more digits must be larger than those with fewer digits.

In summary, as children move from using the operations of addition and subtraction to using the operations of multiplication and division, they have to come to terms with increasing complexities. Firstly, the operation of multiplication on natural numbers and integers includes, but is more than, the operation of addition on these same numbers. Secondly the operations of addition (and subtraction) and multiplication (and division) on rational numbers can represent a range of interpretations which were simply not needed when number was confined to natural numbers or integers. Given that many children appear to have some difficulty in mastering the concept of natural number - as this is reflected in the performance in addition and subtraction as noted by Robertson et al (1996) and Reynolds and Farrell (1996) - it is perhaps not surprising that the requirement to restructure one's already shaky, if not misconceived, notion of number presents some challenges.

<u>What does all this mean for teaching?</u>

Additive reasoning appears to be achieved by children, largely by themselves. To the extent that schools enable children in their achievement of additive reasoning, the role of the teacher focuses on helping children to make the connections between their own intuitive knowledge and the formalisms of numerical knowledge. In helping children to appreciate the power of what they already know, and the value of their preferred ways of calculating, an emphasis on mental strategies is probably very appropriate and may indeed enhance children's achievements. For example, part of the current emphasis on mental calculation is on the retrieval of addition/subtraction 'facts' from memory. However, being able to retrieve number facts from memory is very much an adult strategy to which children move gradually (Resnick, 1989b). Some children may be able to effect retrieval as young as seven years of age while other, normally developing children may be eleven or twelve years old before they use the strategy of retrieval, and may rely almost exclusively on the strategy of counting throughout primary school. Through giving prominence to mental strategies, and identifying which strategies promote the retrieval of facts, it may be that the child's developing sense of number is being enhanced through teaching, though this sense of number is confined to additive reasoning only.

If, however, multiplicative reasoning does not 'grow out of' the intuitively acquired additive reasoning but, as has been argued here, is dependent on a fundamental restructuring of one's sense of number, the achievement of multiplicative reasoning is probably too complex for it to happen without the teacher's sensitive and focused intervention. The precise nature of such intervention is, at the moment, unclear but would seem to imply that teachers should consider the extent to which their practices contribute to, or detract from, the development and achievement of multiplicative reasoning. Firstly teachers must recognise that there are many numerical tasks (such as those involving ratio and proportion) for which informal and flexible mental strategies are simply insufficient. This means that teachers themselves must have a robust understanding of the concept of number so that they can design numerical activities which do not limit the children's experiences of number. Secondly teachers (and all those responsible for curriculum design) must recognise that conventional instruction in enabling children to move from intuitive to formal number has been largely unsuccessful (Hart, 1988). The qualitative shift required in the reconceptualisation of number to allow multiplicative reasoning to develop implies that the teaching of number must now be revised.

Conclusion

The renewed emphasis on mental calculation is to be welcomed. Mental calculation can be powerful in enabling the child's development of number sense. However, in welcoming the renewed interest in mental calculation two caveats do seem important. Firstly, mental calculation has to be understood in all its complexity. It should not be thought of in the simplistic and exclusive terms of sets of rules to be recalled. Rather, it is a richly connected web of mental computation and computational estimation for which the child needs a knowledge of number relationships, a facility with basic facts, an understanding of arithmetical operations, the ability to make comparisons between numbers and possession of base-ten place value concepts.

Secondly, for all that it is helpful to be explicitly aware of the strategies which are thought to enhance proficiency in mental calculation, ensuring that children have acquired the appropriate strategies does not in itself constitute the teaching of numeracy in the primary school. These strategies will only be of use if they are located in the child's conceptual framework for number. This framework is typically at its most complete and comprehensive when the child has achieved multiplicative reasoning because then the child can see the interconnectedness of substantive mathematical knowledge rather than see mathematics as comprising discrete units of procedural knowledge. As teachers we need to be aware of the complexity of the concept of number and we also need to be familiar with the stages through which the concept of number develops. If as teachers we understand the development of number from a mathematical perspective, we can then be clearer about the impact and effects of our own attempt to teach number. In part this means confronting the idea that achieving multiplicative reasoning is probably much more difficult than has hitherto been acknowledged. It is not easy to see what should be done to develop pedagogical practices which will support the child's construction of rational number. Whatever the practices might be, they would have to be more than ensuring that children have a repertoire of mental strategies.

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