

## SPECTRUM OF $J$ -FRAME OPERATORS

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**Abstract.** A  $J$ -frame is a frame  $\mathcal{F}$  for a Krein space  $(\mathcal{H}, [\cdot, \cdot])$  which is compatible with the indefinite inner product  $[\cdot, \cdot]$  in the sense that it induces an indefinite reconstruction formula that resembles those produced by orthonormal bases in  $\mathcal{H}$ . With every  $J$ -frame the so-called  $J$ -frame operator is associated, which is a self-adjoint operator in the Krein space  $\mathcal{H}$ . The  $J$ -frame operator plays an essential role in the indefinite reconstruction formula. In this paper we characterize the class of  $J$ -frame operators in a Krein space by a  $2 \times 2$  block operator representation. The  $J$ -frame bounds of  $\mathcal{F}$  are then recovered as the suprema and infima of the numerical ranges of some uniformly positive operators which are build from the entries of the  $2 \times 2$  block representation. Moreover, this  $2 \times 2$  block representation is utilized to obtain enclosures for the spectrum of  $J$ -frame operators, which finally leads to the construction of a square root. This square root allows a complete description of all  $J$ -frames associated with a given  $J$ -frame operator.

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### 1. INTRODUCTION

Frame theory is a key tool in signal and image processing, data compression, and sampling theory, among other applications in engineering, applied mathematics, and computer sciences. The major advantage of a frame over an orthonormal, orthogonal, or Riesz basis is its redundancy: each vector admits several different reconstructions in terms of the frame coefficients. For instance, frames have shown to be useful in signal processing applications when noisy channels are involved, because a frame allows to reconstruct vectors (signals) even if some of the frame coefficients are missing (or corrupted); see [3, 16, 26].

A frame for a Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  is, in general, a redundant (overcomplete) family of vectors  $\mathcal{F} = \{f_i\}_{i \in I}$  in  $\mathcal{H}$  for which there exists a pair of positive constants  $0 < \alpha \leq \beta$  such that

$$\alpha \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq \beta \|f\|^2 \quad \text{for every } f \in \mathcal{H}. \tag{1.1}$$

Note that these inequalities establish an equivalence between the norm of the vector  $f \in \mathcal{H}$  and the  $\ell_2$ -norm of its frame coefficients  $\{\langle f, f_i \rangle\}_{i \in I}$ .

Every frame  $\mathcal{F}$  for  $\mathcal{H}$  has an associated frame operator  $S : \mathcal{H} \rightarrow \mathcal{H}$ , which is uniformly positive (i.e. positive definite and boundedly invertible) in  $\mathcal{H}$ , and it allows to reconstruct each  $f \in \mathcal{H}$  as follows:

$$f = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i.$$

Given a uniformly positive operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  it is possible to describe the complete family of frames whose frame operator is  $S$ . Indeed, if  $\mathcal{F} = \{f_i\}_{i \in I}$  is a frame for  $\mathcal{H}$  then the upper bound in (1.1) implies that the so-called synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$ ,

$$Tx = \sum_{i \in I} \langle x, e_i \rangle f_i,$$

is bounded, where  $\{e_i\}_{i \in I}$  is the standard orthonormal basis in  $\ell_2(I)$ . Then its frame operator is  $S$  if and only if

$$S = TT^*.$$

Therefore,  $S$  is the frame operator of  $\mathcal{F}$  if and only if the (left) polar decomposition of  $T$  can be written as

$$T = S^{1/2}V,$$

for some co-isometry  $V : \ell_2(I) \rightarrow \mathcal{H}$ ; see for instance [10, Corollary 2.15]. Hence, the family of frames whose frame operator is  $S$  is in a one-to-one correspondence with the family of co-isometries onto the Hilbert space  $\mathcal{H}$ . Also, note that this description is possible due to the existence of a (positive) square root of  $S$ .

Recently, various approaches to introduce frame theory on Krein spaces have been suggested; see [12, 13, 23]. In the current paper we further investigate the notion of  $J$ -frames that was proposed in [13]. This concept was motivated by a signal processing problem, where signals are disturbed with the same energy at high and low band frequencies; see the discussion at the beginning of Section 3 in [13].

Given a Bessel family (that is a family which satisfies the upper bound in (1.1)) of vectors  $\mathcal{F} = \{f_i\}_{i \in I}$  in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , we divide it in a family  $\mathcal{F}_+ = \{f_i\}_{i \in I_+}$  of non-negative vectors in  $\mathcal{H}$  and a family  $\mathcal{F}_- = \{f_i\}_{i \in I_-}$  of negative vectors in  $\mathcal{H}$ . Roughly speaking,  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $(\mathcal{H}, [\cdot, \cdot])$  if the ranges of the synthesis operators  $T_+$  and  $T_-$  associated with  $\mathcal{F}_+$  and  $\mathcal{F}_-$  are a maximal uniformly

positive subspace and a maximal uniformly negative subspace of  $\mathcal{H}$ , respectively (see Subsection 2.3 for further details). In particular, it is immediate that orthonormal bases in Krein spaces are  $J$ -frames as they generate maximal dual pairs [2, Section 1.10].

Each  $J$ -frame for  $(\mathcal{H}, [\cdot, \cdot])$  is also a frame (in the Hilbert space sense) for  $\mathcal{H}$ . Moreover,  $\mathcal{F}_+ = \{f_i\}_{i \in I_+}$  and  $\mathcal{F}_- = \{f_i\}_{i \in I_-}$  are frames for the Hilbert spaces  $(R(T_+), [\cdot, \cdot])$  and  $(R(T_-), -[\cdot, \cdot])$ , respectively. However, it is easy to construct frames for  $\mathcal{H}$  which are not  $J$ -frames; see [13, Example 3.3].

For each  $J$ -frame for  $\mathcal{H}$  one can define a  $J$ -frame operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  (see (1.2) below), which is a bounded and boundedly invertible self-adjoint operator in  $(\mathcal{H}, [\cdot, \cdot])$ . It can be used to describe every vector in  $\mathcal{H}$  in terms of the frame vectors  $\mathcal{F} = \{f_i\}_{i \in I}$ :

$$f = \sum_{i \in I} \sigma_i [f, S^{-1} f_i] f_i = \sum_{i \in I} \sigma_i [f, f_i] S^{-1} f_i, \quad f \in \mathcal{H},$$

where  $\sigma_i = \operatorname{sgn} [f_i, f_i]$ . This is known as the indefinite reconstruction formula associated with  $\mathcal{F}$  since it resembles the reconstruction formula provided by an orthonormal basis in a Krein space.

If  $T : \ell_2(I) \rightarrow \mathcal{H}$  is the synthesis operator of  $\mathcal{F}$ , then  $S$  is given by

$$S = TT^+, \tag{1.2}$$

where  $T^+$  stands for the adjoint of  $T$  with respect to  $[\cdot, \cdot]$ . Hence, given a  $J$ -frame operator  $S$  in  $(\mathcal{H}, [\cdot, \cdot])$ , it is natural to look for descriptions of the family of  $J$ -frames whose  $J$ -frame operator is  $S$ . This is a non-trivial problem due to several reasons.

First, the existence of a square root of a self-adjoint operator in a Krein space depends on the location of its spectrum. Some characterizations of  $J$ -frame operators can be found in [13, Subsection 5.2], but none of them guarantees the existence of a square root.

Second, in general there is no polar decomposition for linear operators acting between Krein spaces; see [5, 20–22].

The aim of this work is to obtain a deeper insight on  $J$ -frame operators, in particular, to obtain enclosures for the spectrum of a  $J$ -frame operator and a full description of the  $J$ -frames associated with a prescribed  $J$ -frame operator. The two main results of the paper are the following.

- (i) The spectrum of a  $J$ -frame operator is always contained in the open right half-plane with a positive distance to the imaginary axis. This distance can be estimated in terms of the  $J$ -frame bounds.
- (ii) Due to the location of the spectrum according to item (i), there exists a square root of a  $J$ -frame operator. This enables one to show that there is a bijection between all co-isometries and all  $J$ -frames with the same  $J$ -frame operator.

The paper is organized as follows. Section 2 contains preliminaries both on frames for Hilbert spaces and for Krein spaces. There we recall the notions of  $J$ -frames and  $J$ -frame operators, and we present some known results.

In Section 3, given a bounded self-adjoint operator  $S$  acting in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , we describe  $J$ -frame operators as block operator matrices (with respect to

a suitable fundamental decomposition) such that their entries have some particular properties. For instance, we prove that  $S$  is a  $J$ -frame operator if and only if there exists a fundamental decomposition  $\mathcal{H} = \mathcal{H}_+ [ \dot{+} ] \mathcal{H}_-$  such that

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix},$$

where  $A$  is uniformly positive in the Hilbert space  $(\mathcal{H}_+, [\cdot, \cdot])$ ,  $K : \mathcal{H}_- \rightarrow \mathcal{H}_+$  satisfies  $\|K\| < 1$ , and  $D$  is a self-adjoint operator such that  $D + K^*AK$  is uniformly positive in the Hilbert space  $(\mathcal{H}_-, -[\cdot, \cdot])$ ; see Theorem 3.1 below. Also, a dual representation is given in Theorem 3.2 below, where the roles of the operators in the diagonal of the block operator matrix are interchanged. Using these two representations we describe the inverse of  $S$  and the positive operators  $S_{\pm}$ , which are defined in (2.8) below and satisfy  $S = S_+ - S_-$ .

Given a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  for a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , the  $J$ -frame bounds of  $\mathcal{F}$  are the two pairs of frame bounds associated with  $\mathcal{F}_+$  and  $\mathcal{F}_-$  as frames for the Hilbert spaces  $(R(T_+), [\cdot, \cdot])$  and  $(R(T_-), -[\cdot, \cdot])$ , respectively. In Section 4 we recover the  $J$ -frame bounds of  $\mathcal{F}$  as spectral bounds for the  $J$ -frame operator  $S$  associated with  $\mathcal{F}$ . More precisely, we describe the  $J$ -frame bounds of  $\mathcal{F}$  in terms of suprema and infima of the numerical ranges of the uniformly positive entries in the block operator matrix representations discussed above.

In Section 5 we use the representations obtained in Section 3 to prove enclosures for the real part and the non-real part of the spectrum of a  $J$ -frame operator  $S$ . In particular, we show that the spectrum of a  $J$ -frame operator is always contained in the right half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ . Moreover, given a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  with  $J$ -frame operator  $S$ , we describe the spectral enclosures of  $S$  in terms of the  $J$ -frame bounds of  $\mathcal{F}$ .

Finally, in Section 6 we use the Riesz–Dunford functional calculus to obtain the square root of a  $J$ -frame operator. We characterize the family of all  $J$ -frames for  $\mathcal{H}$  whose  $J$ -frame operator is  $S$ . More precisely, the synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$  of a  $J$ -frame  $\mathcal{F}$  satisfies  $TT^+ = S$  if and only if

$$T = S^{1/2}U,$$

where  $S^{1/2}$  is the square root of  $S$  and  $U : \ell_2(I) \rightarrow \mathcal{H}$  is a co-isometry between Krein spaces. Hence, there is a bijection between all co-isometries and all  $J$ -frames with the same  $J$ -frame operator.

## 2. PRELIMINARIES

When  $\mathcal{H}$  and  $\mathcal{K}$  are Hilbert spaces, we denote by  $L(\mathcal{H}, \mathcal{K})$  the vector space of bounded linear operators from  $\mathcal{H}$  into  $\mathcal{K}$  and by  $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$  the algebra of bounded linear operators acting on  $\mathcal{H}$ .

For  $T \in L(\mathcal{H})$  we denote by  $\sigma(T)$  and  $\sigma_p(T)$  the *spectrum* and the *point spectrum* of  $T$ , respectively. Moreover, we denote by  $N(T)$ ,  $R(T)$  and  $W(T)$  the *kernel*, the *range* and the *numerical range* of  $T$ , respectively.

Given two closed subspaces  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{H}$ , hereafter  $\mathcal{M} \dot{+} \mathcal{N}$  denotes their direct sum. Moreover, in case that  $\mathcal{H} = \mathcal{M} \dot{+} \mathcal{N}$ , we denote the (unique) projection with range  $\mathcal{M}$  and kernel  $\mathcal{N}$  by  $P_{\mathcal{M} // \mathcal{N}}$ . In particular, if  $\mathcal{N} = \mathcal{M}^\perp$  then  $P_{\mathcal{M}} = P_{\mathcal{M} // \mathcal{M}^\perp}$  denotes the orthogonal projection onto  $\mathcal{M}$ .

## 2.1. FRAMES FOR HILBERT SPACES

We recall the standard notation for frames for Hilbert spaces and some basic results; see, e.g. [6–8, 15].

A *frame* for a Hilbert space  $\mathcal{H}$  is a family of vectors  $\mathcal{F} = \{f_i\}_{i \in I}$ ,  $f_i \in \mathcal{H}$ , for which there exist constants  $0 < \alpha \leq \beta < \infty$  such that

$$\alpha \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq \beta \|f\|^2 \quad \text{for every } f \in \mathcal{H}. \quad (2.1)$$

The optimal constants (maximal for  $\alpha$  and minimal for  $\beta$ ) are called the *lower* and the *upper frame bounds* of  $\mathcal{F}$ , respectively.

If a family of vectors  $\mathcal{F} = \{f_i\}_{i \in I}$  satisfies the upper bound condition in (2.1), then  $\mathcal{F}$  is a *Bessel family*. For a Bessel family  $\mathcal{F} = \{f_i\}_{i \in I}$  the *synthesis operator*  $T \in L(\ell_2(I), \mathcal{H})$  is defined by

$$Tx = \sum_{i \in I} \langle x, e_i \rangle f_i, \quad x \in \ell_2(I),$$

where  $\{e_i\}_{i \in I}$  is the standard orthonormal basis of  $\ell_2(I)$ . A Bessel family  $\mathcal{F}$  is a frame for  $\mathcal{H}$  if and only if  $T$  is surjective. In this case, the operator  $S := TT^* \in L(\mathcal{H})$  is uniformly positive and called *frame operator*. It can easily be verified that

$$Sf = \sum_{i \in I} \langle f, f_i \rangle f_i \quad \text{for every } f \in \mathcal{H}. \quad (2.2)$$

This implies that the frame bounds of  $\mathcal{F}$  can be computed as  $\alpha = \|S^{-1}\|^{-1}$  and  $\beta = \|S\|$ .

From (2.2) it is also easy to obtain the *frame reconstruction formula* for vectors in  $\mathcal{H}$ :

$$f = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i \quad \text{for every } f \in \mathcal{H};$$

and the frame  $\{S^{-1} f_i\}_{i \in I}$  is called the *canonical dual frame* of  $\mathcal{F}$ .

## 2.2. KREIN SPACES

We recall the standard notation and some basic results on Krein spaces. For a complete exposition on the subject (and the proofs of the results below) see the books by Azizov and Iokhvidov [2] and Bognár [4]; see also [1, 11, 17, 25].

A vector space  $\mathcal{H}$  with a Hermitian sesquilinear form  $[\cdot, \cdot]$  is called a *Krein space* if there exists a so-called *fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+ [\dot{+}] \mathcal{H}_-,$$

which is the direct (and orthogonal with respect to  $[\cdot, \cdot]$ ) sum of two Hilbert spaces  $(\mathcal{H}_+, [\cdot, \cdot])$  and  $(\mathcal{H}_-, -[\cdot, \cdot])$ . These two Hilbert spaces induce in a natural way a Hilbert space inner product  $\langle \cdot, \cdot \rangle$  and, hence, a Hilbert space topology on  $\mathcal{H}$ . Observe that the indefinite metric  $[\cdot, \cdot]$  and the Hilbert space inner product  $\langle \cdot, \cdot \rangle$  of  $\mathcal{H}$  are related by means of a *fundamental symmetry*, i.e. a unitary self-adjoint operator  $J \in L(\mathcal{H})$  that satisfies

$$\langle x, y \rangle = [Jx, y] \quad \text{for } x, y \in \mathcal{H}.$$

Although the fundamental decomposition is not unique, the norms induced by different fundamental decompositions turn out to be equivalent; see, e.g. [18, Proposition I.1.2]. Therefore, the (Hilbert space) topology in  $\mathcal{H}$  does not depend on the chosen fundamental decomposition.

If  $\mathcal{H}$  and  $\mathcal{K}$  are Krein spaces and  $T \in L(\mathcal{H}, \mathcal{K})$ , the adjoint operator of  $T$  is the unique operator  $T^+ \in L(\mathcal{K}, \mathcal{H})$  satisfying

$$[Tx, y] = [x, T^+y] \quad \text{for every } x \in \mathcal{H}, y \in \mathcal{K}.$$

An operator  $T \in L(\mathcal{H})$  is *self-adjoint in a Krein space*  $(\mathcal{H}, [\cdot, \cdot])$  if  $T = T^+$ . The spectrum of such an operator  $T$  is symmetric with respect to the real axis.

Let  $(\mathcal{H}, [\cdot, \cdot])$  be a Krein space. A vector  $x \in \mathcal{H}$  is called *positive* if  $[x, x] > 0$ , *negative* if  $[x, x] < 0$  and *neutral* if  $[x, x] = 0$ . A subspace  $\mathcal{L}$  of  $\mathcal{H}$  is *positive* if every  $x \in \mathcal{L} \setminus \{0\}$  is a positive vector. A subspace  $\mathcal{L}$  of  $\mathcal{H}$  is *uniformly positive* if there exists  $\alpha > 0$  such that  $[x, x] \geq \alpha \|x\|^2$  for every  $x \in \mathcal{L}$ , where  $\|\cdot\|$  stands for the norm of the associated Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ . *Non-negative, neutral, negative, non-positive* and *uniformly negative subspaces* are defined analogously.

Given a subspace  $\mathcal{M}$  of  $(\mathcal{H}, [\cdot, \cdot])$ , the *orthogonal companion* of  $\mathcal{M}$  is defined by

$$\mathcal{M}^{[\perp]} := \{x \in \mathcal{H} : [x, m] = 0, \text{ for every } m \in \mathcal{M}\}.$$

The *isotropic part* of  $\mathcal{M}$ ,  $\mathcal{M}^\circ := \mathcal{M} \cap \mathcal{M}^{[\perp]}$ , can be a non-trivial subspace. A subspace  $\mathcal{M}$  of  $\mathcal{H}$  is *non-degenerate* if  $\mathcal{M}^\circ = \{0\}$ . Otherwise, it is a *degenerate* subspace of  $\mathcal{H}$ .

A subspace  $\mathcal{M}$  of  $\mathcal{H}$  is *regular* if  $\mathcal{M} \dot{+} \mathcal{M}^{[\perp]} = \mathcal{H}$ . The subspace  $\mathcal{M}$  is regular if and only if there exists a (unique) self-adjoint projection  $E$  onto  $\mathcal{M}$ ; see, e.g. [2, Theorem 1.7.16].

### 2.3. FRAMES FOR KREIN SPACES

Recently, there have been various approaches to introduce frame theory on Krein spaces; see [12, 13, 23]. In the following we recall the notion of  $J$ -frames introduced in [13]. Some particular classes of  $J$ -frames were also considered in [14].

Let  $(\mathcal{H}, [\cdot, \cdot])$  be a Krein space, let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a Bessel family in  $\mathcal{H}$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$ , and set  $I_+ := \{i \in I : [f_i, f_i] \geq 0\}$  and  $I_- := \{i \in I : [f_i, f_i] < 0\}$ . Further, consider the orthogonal decomposition of  $\ell_2(I)$  induced by the partition of  $I$ ,

$$\ell_2(I) = \ell_2(I_+) \oplus \ell_2(I_-), \tag{2.3}$$

denote by  $P_{\pm}$  the orthogonal projections onto  $\ell_2(I_{\pm})$ , respectively, and set  $T_{\pm} := TP_{\pm}$ , i.e.

$$T_{\pm}x = \sum_{i \in I_{\pm}} \langle x, e_i \rangle f_i, \quad x \in \ell_2(I). \quad (2.4)$$

The following sets play an important role in the following:

$$\mathcal{M}_{\pm} := \overline{\text{span}\{f_i : i \in I_{\pm}\}}. \quad (2.5)$$

They are related to the ranges of  $T_+$  and  $T_-$  as follows:

$$\text{span}\{f_i : i \in I_{\pm}\} \subseteq R(T_{\pm}) \subseteq \mathcal{M}_{\pm}.$$

The ranges of  $T_+$  and  $T_-$  also satisfy  $R(T) = R(T_+) + R(T_-)$  and play an essential role in the definition of  $J$ -frames.

**Definition 2.1.** Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a Bessel family in a Krein space  $\mathcal{H}$  and let  $T_{\pm}$  be as in (2.4). Then  $\mathcal{F}$  is called a  $J$ -frame for  $\mathcal{H}$  if  $R(T_+)$  is a maximal uniformly positive subspace and  $R(T_-)$  is a maximal uniformly negative subspace of  $\mathcal{H}$ .

If  $\mathcal{F}$  is a  $J$ -frame for  $\mathcal{H}$ , then,  $R(T_{\pm}) = \mathcal{M}_{\pm}$  and

$$R(T) = R(T_+) \dot{+} R(T_-) = \mathcal{M}_+ \dot{+} \mathcal{M}_- = \mathcal{H}, \quad (2.6)$$

where the last equality follows from [2, Corollary 1.5.2]. Thus,  $\mathcal{F}$  is also a frame for the Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ . Moreover, it is easy to see that  $\mathcal{F}_{\pm} := \{f_i\}_{i \in I_{\pm}}$  is a frame for the Hilbert space  $(\mathcal{M}_{\pm}, \pm[\cdot, \cdot])$ .

The following is a characterization of  $J$ -frames in terms of frame inequalities; see [13, Theorem 3.9].

**Theorem 2.2.** Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a frame for  $\mathcal{H}$ . Then  $\mathcal{F}$  is a  $J$ -frame if and only if  $\mathcal{M}_{\pm}$  (defined as in (2.5)) are non-degenerate subspaces of  $\mathcal{H}$  and there exist constants  $0 < \alpha_{\pm} \leq \beta_{\pm}$  such that

$$\alpha_{\pm}(\pm[f, f]) \leq \sum_{i \in I_{\pm}} |[f, f_i]|^2 \leq \beta_{\pm}(\pm[f, f]) \quad \text{for every } f \in \mathcal{M}_{\pm}. \quad (2.7)$$

When  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $\mathcal{H}$ , we endow the coefficient space  $\ell_2(I)$  with the following indefinite inner product:

$$[x, y]_2 := \sum_{i \in I_+} x_i \bar{y}_i - \sum_{i \in I_-} x_i \bar{y}_i, \quad x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in \ell_2(I).$$

Then,  $(\ell_2(I), [\cdot, \cdot]_2)$  is a Krein space and (2.3) is a fundamental decomposition of  $\ell_2(I)$ . Now, if  $T : \ell_2(I) \rightarrow \mathcal{H}$  is the synthesis operator of  $\mathcal{F}$ , its adjoint (in the sense of Krein spaces) is given by

$$T^+f = \sum_{i \in I_+} [f, f_i] e_i - \sum_{i \in I_-} [f, f_i] e_i, \quad f \in \mathcal{H}.$$

**Definition 2.3.** Given a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  for  $\mathcal{H}$ , the  $J$ -frame operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  associated with  $\mathcal{F}$  is defined by

$$Sf := TT^+f = \sum_{i \in I_+} [f, f_i] f_i - \sum_{i \in I_-} [f, f_i] f_i, \quad f \in \mathcal{H}.$$

It is easy to see that  $S$  is a bounded and boundedly invertible self-adjoint operator  $S$  in the Krein space  $\mathcal{H}$ . We also introduce the positive operators  $S_{\pm}$  by

$$S_{\pm}f := \sum_{i \in I_{\pm}} [f, f_i] f_i, \quad f \in \mathcal{H}, \tag{2.8}$$

which are non-negative operators in  $\mathcal{H}$  since

$$[S_{\pm}f, f] = \sum_{i \in I_{\pm}} |[f, f_i]|^2 \quad \text{for every } f \in \mathcal{H}. \tag{2.9}$$

Clearly, the  $J$ -frame operator can be written as  $S = S_+ - S_-$ ; hence it is the difference of two positive operators. The operators  $S$  and  $S_{\pm}$  are related via the projection  $Q := P_{\mathcal{M}_+ // \mathcal{M}_-}$  and its adjoint  $Q^+ = P_{\mathcal{M}_-^{[\perp]} // \mathcal{M}_+^{[\perp]}}$  as follows:

$$QS = S_+ = SQ^+ \quad \text{and} \quad (I - Q)S = -S_- = S(I - Q)^+. \tag{2.10}$$

Therefore, we have that  $R(S_{\pm}) = \mathcal{M}_{\pm}$  and

$$S(\mathcal{M}_-^{[\perp]}) = \mathcal{M}_+, \quad S(\mathcal{M}_+^{[\perp]}) = \mathcal{M}_-. \tag{2.11}$$

Finally, the class of  $J$ -frame operators can be characterized in the following way; see [13, Proposition 5.7].

**Theorem 2.4.** *A bounded and boundedly invertible self-adjoint operator  $S$  in a Krein space  $\mathcal{H}$  is a  $J$ -frame operator if and only if the following conditions are satisfied:*

- (i) *there exists a maximal uniformly positive subspace  $\mathcal{L}_+$  of  $\mathcal{H}$  such that  $S(\mathcal{L}_+)$  is also maximal uniformly positive;*
- (ii)  *$[Sf, f] \geq 0$  for every  $f \in \mathcal{L}_+$ ;*
- (iii)  *$[Sg, g] \leq 0$  for every  $g \in (S(\mathcal{L}_+))^{[\perp]}$ .*

**Remark 2.5.** If  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $\mathcal{H}$  with  $J$ -frame operator  $S$  and  $\mathcal{M}_{\pm}$  are given by (2.5), then  $\mathcal{M}_-^{[\perp]}$  is a maximal uniformly positive subspace satisfying conditions (i)–(iii) in Theorem 2.4. In fact, (i) follows from (2.11). Moreover, since  $N(S_-) = R(S_-)^{[\perp]} = \mathcal{M}_-^{[\perp]}$ , we have for  $f \in \mathcal{M}_-^{[\perp]}$ :

$$[Sf, f] = [S_+f, f] = \sum_{i \in I_+} |[f, f_i]|^2 \geq 0;$$

see (2.9). Analogously,  $S(\mathcal{M}_-^{[\perp]})^{[\perp]} = \mathcal{M}_+^{[\perp]} = N(S_+)$  and if  $g \in \mathcal{M}_+^{[\perp]}$  then

$$[Sg, g] = [-S_-g, g] = - \sum_{i \in I_-} |[g, f_i]|^2 \leq 0.$$

Thus, we have also shown conditions (ii) and (iii).



If  $S$  is a  $J$ -frame operator, then  $S^{-1}$  is also a  $J$ -frame operator. More precisely, the following proposition is true; see [13, Proposition 5.4].

**Proposition 2.6.** *If  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $\mathcal{H}$  with  $J$ -frame operator  $S$ , then  $\mathcal{F}' = \{S^{-1}f_i\}_{i \in I}$  is also a  $J$ -frame for  $\mathcal{H}$ . Furthermore,*

$$\operatorname{sgn}([S^{-1}f_i, S^{-1}f_i]) = \operatorname{sgn}([f_i, f_i]) \quad \text{for every } i \in I,$$

the  $J$ -frame operator of  $\mathcal{F}'$  is  $S^{-1}$  and, if  $\mathcal{M}_{\pm}$  are given by (2.5), then

$$\overline{\operatorname{span}\{S^{-1}f_i : i \in I_{\pm}\}} = \mathcal{M}_{\mp}^{[\perp]}.$$

### 3. OPERATOR MATRIX REPRESENTATIONS FOR $J$ -FRAME OPERATORS

In the following we describe  $J$ -frame operators via  $2 \times 2$  block operator matrices. Let  $(\mathcal{H}, [\cdot, \cdot])$  be a Krein space with fundamental decomposition

$$\mathcal{H} = \mathcal{H}_+ [ + ] \mathcal{H}_-.$$

Every bounded operator  $S$  in  $\mathcal{H}$  can be written as a block operator matrix

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (3.1)$$

with bounded operators

$$A \in L(\mathcal{H}_+), \quad B \in L(\mathcal{H}_-, \mathcal{H}_+), \quad C \in L(\mathcal{H}_+, \mathcal{H}_-), \quad D \in L(\mathcal{H}_-).$$

An operator of the form (3.1) is self-adjoint in the Krein space  $\mathcal{H}$  if and only if  $A$  and  $D$  are self-adjoint in the Hilbert spaces  $(\mathcal{H}_+, [\cdot, \cdot])$  and  $(\mathcal{H}_-, -[\cdot, \cdot])$ , respectively, and  $C = -B^*$ .

**Theorem 3.1.** *Let  $S$  be a bounded self-adjoint operator in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ . Then,  $S$  is a  $J$ -frame operator if and only if there exists a fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+ [ + ] \mathcal{H}_-, \quad (3.2)$$

such that  $S$  admits a representation with respect to (3.2) of the form

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \quad (3.3)$$

where  $A$  is a uniformly positive operator in the Hilbert space  $(\mathcal{H}_+, [\cdot, \cdot])$ ,  $K : \mathcal{H}_- \rightarrow \mathcal{H}_+$  is a uniform contraction (i.e.  $\|K\| < 1$ ), and  $D$  is a self-adjoint operator such that  $D + K^*AK$  is uniformly positive in the Hilbert space  $(\mathcal{H}_-, -[\cdot, \cdot])$ .

*Proof.* First, assume that  $S$  is a  $J$ -frame operator and consider a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  for  $\mathcal{H}$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$  such that  $S = TT^+$ . Let  $\mathcal{M}_\pm$  be as in (2.5). By [2, Corollary 1.8.14], the subspace  $\mathcal{M}_-^{[\perp]}$  is maximal uniformly positive and the pair  $\mathcal{M}_-^{[\perp]}, \mathcal{M}_-$  form a fundamental decomposition of  $\mathcal{H}$ :

$$\mathcal{H} = \mathcal{M}_-^{[\perp]} [ + ] \mathcal{M}_-. \tag{3.4}$$

Since  $S$  is a bounded self-adjoint operator in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$ , there exist bounded self-adjoint operators  $A$  and  $D$  in the Hilbert spaces  $(\mathcal{M}_-^{[\perp]}, [\cdot, \cdot])$  and  $(\mathcal{M}_-, -[\cdot, \cdot])$ , respectively, and  $B : \mathcal{M}_- \rightarrow \mathcal{M}_-^{[\perp]}$  such that

$$S = \begin{bmatrix} A & B \\ -B^* & D \end{bmatrix}$$

with respect to the fundamental decomposition (3.4).

We represent the maximal uniformly definite subspaces  $\mathcal{M}_+^{[\perp]}$  and  $\mathcal{M}_+$  with the help of an angular operator with respect to the fundamental decomposition (3.4), i.e. an operator  $K : \mathcal{M}_- \rightarrow \mathcal{M}_-^{[\perp]}$  with  $\|K\| < 1$  such that

$$\mathcal{M}_+^{[\perp]} = \{Kx_- + x_- \mid x_- \in \mathcal{M}_-\} \quad \text{and} \quad \mathcal{M}_+ = \{x_+ + K^*x_+ \mid x_+ \in \mathcal{M}_-^{[\perp]}\}; \tag{3.5}$$

see, e.g. [2, Theorem 1.8.11].

Since  $S$  maps  $\mathcal{M}_-^{[\perp]}$  onto  $\mathcal{M}_+$  (see (2.11)), for every  $x \in \mathcal{M}_-^{[\perp]}$  there exists  $x_+ \in \mathcal{M}_-^{[\perp]}$  such that

$$\begin{bmatrix} A & B \\ -B^* & D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax \\ -B^*x \end{bmatrix} = \begin{bmatrix} x_+ \\ K^*x_+ \end{bmatrix}$$

(see (3.5)), i.e.  $Ax = x_+$  and  $-B^*x = K^*Ax$ . Since this holds for all  $x \in \mathcal{M}_-^{[\perp]}$ , we obtain

$$B^* = -K^*A \quad \text{and hence} \quad B = -AK,$$

which proves (3.3).

The operator  $S$  is boundedly invertible (see, e.g. [13, Proposition 5.2]) and therefore the range of  $S$  equals  $\mathcal{H} = \mathcal{M}_-^{[\perp]} [ + ] \mathcal{M}_-$ . From the first row in (3.3) we conclude that the range of  $A$  equals  $\mathcal{M}_-^{[\perp]}$ . Since  $A$  is self-adjoint, this implies that  $A$  is boundedly invertible.

Moreover, by Remark 2.5, the subspace  $\mathcal{M}_-^{[\perp]}$  satisfies conditions (i)–(iii) in Theorem 2.4. In particular, condition (ii) in Theorem 2.4 says that, for  $x \in \mathcal{M}_-^{[\perp]}$ ,

$$0 \leq [Sx, x] = \left[ \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right] = [Ax, x],$$

which implies that  $A$  is uniformly positive.

Next, let  $x_- \in \mathcal{M}_-$ . Then  $Kx_- + x_- \in \mathcal{M}_+^{[\perp]}$  and

$$\begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \begin{bmatrix} Kx_- \\ x_- \end{bmatrix} = \begin{bmatrix} 0 \\ (K^*AK + D)x_- \end{bmatrix}.$$

Since  $S(\mathcal{M}_+^{[\perp]}) = \mathcal{M}_-$  (see (2.11)), this shows that  $K^*AK + D$  is a surjective operator and hence boundedly invertible (as it is self-adjoint). Moreover, condition (iii) in Theorem 2.4 states for  $x_- \in \mathcal{M}_-$ ,

$$0 \geq \left[ \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \begin{bmatrix} Kx_- \\ x_- \end{bmatrix}, \begin{bmatrix} Kx_- \\ x_- \end{bmatrix} \right] = [(K^*AK + D)x_-, x_-],$$

which proves that  $D + K^*AK$  is uniformly positive in  $(\mathcal{M}_-, -[\cdot, \cdot])$ .

Conversely, assume that  $S$  admits a block operator matrix representation as in (3.3), with respect to the fundamental decomposition (3.2). Note that  $S$  is a boundedly invertible operator because

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D + K^*AK \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix},$$

and the three block operator matrices on the right-hand side are boundedly invertible. The space  $\mathcal{H}_+$  is uniformly positive. For  $x_+ \in \mathcal{H}_+$  we have

$$\begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \begin{bmatrix} x_+ \\ 0 \end{bmatrix} = \begin{bmatrix} Ax_+ \\ K^*Ax_+ \end{bmatrix}.$$

Obviously,  $S$  maps  $\mathcal{H}_+$  into the uniformly positive subspace  $\{x_+ + K^*x_+ \mid x_+ \in \mathcal{H}_+\}$ . Moreover, for  $x_+ \in \mathcal{H}_+$  it follows that

$$\left[ \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \begin{bmatrix} x_+ \\ 0 \end{bmatrix}, \begin{bmatrix} x_+ \\ 0 \end{bmatrix} \right] = [Ax_+, x_+] \geq 0$$

since  $A$  is positive. In a similar manner, using the positivity of  $K^*AK + D$  we obtain for

$$x \in \{x_+ + K^*x_+ \mid x_+ \in \mathcal{H}_+\}^{[\perp]} = \{Kx_- + x_- \mid x_- \in \mathcal{H}_-\}$$

the estimate

$$\left[ \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \begin{bmatrix} Kx_- \\ x_- \end{bmatrix}, \begin{bmatrix} Kx_- \\ x_- \end{bmatrix} \right] = [(K^*AK + D)x_-, x_-] \leq 0.$$

Therefore, by Theorem 2.4 the operator  $S$  is a  $J$ -frame operator.  $\square$

Given a  $J$ -frame  $\mathcal{F}$  for  $\mathcal{H}$ , consider  $\mathcal{M}_\pm$  as in (2.5). The block operator representation in Theorem 3.1 is based on the fundamental decomposition (3.4) determined by  $\mathcal{M}_-$ . Obviously, the subspace  $\mathcal{M}_+$  leads in the same way to a fundamental decomposition:

$$\mathcal{H} = \mathcal{M}_+[\perp]\mathcal{M}_+^{[\perp]}. \quad (3.6)$$

Therefore, besides (3.3), there exists in a natural way a second block operator representation with respect to (3.6). Both representations are used in the next section to relate the  $J$ -frame bounds with the spectrum of the  $J$ -frame operator.

**Theorem 3.2.** *Let  $S$  be a bounded self-adjoint operator in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ . Then,  $S$  is a  $J$ -frame operator if and only if there exists a fundamental decomposition*

$$\mathcal{H} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-, \tag{3.7}$$

such that  $S$  admits a representation with respect to (3.7) of the form

$$S = \begin{bmatrix} A' & LD' \\ -D'L^* & D' \end{bmatrix} \tag{3.8}$$

where  $A'$  is a self-adjoint operator such that  $A' + LD'L^*$  is a uniformly positive operator in the Hilbert space  $(\mathcal{K}_+, [\cdot, \cdot])$ ,  $L : \mathcal{K}_- \rightarrow \mathcal{K}_+$  is a uniform contraction (i.e.  $\|L\| < 1$ ), and  $D'$  is uniformly positive in the Hilbert space  $(\mathcal{K}_-, -[\cdot, \cdot])$ .

*Proof.* Assume that  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $\mathcal{H}$  with  $J$ -frame operator  $S$ . Consider the maximal uniformly definite subspaces  $\mathcal{M}_\pm$  given by (2.5), the fundamental decomposition

$$\mathcal{H} = \mathcal{M}_+ [\dot{+}] \mathcal{M}_+^{[\perp]},$$

and the angular operator  $L : \mathcal{M}_+^{[\perp]} \rightarrow \mathcal{M}_+$  associated with  $\mathcal{M}_-$ . Since  $S$  maps  $\mathcal{M}_+^{[\perp]}$  onto  $\mathcal{M}_-$  (see (2.11)), the representation (3.8) follows in the same way as the representation (3.3) was proved in Theorem 3.1. The converse also follows the same ideas used in Theorem 3.1.  $\square$

As it was mentioned before, if  $\mathcal{F}$  is a  $J$ -frame for  $\mathcal{H}$ , the subspaces  $\mathcal{M}_\pm$  given in (2.5) are maximal uniformly definite subspaces with opposite signature and  $\mathcal{H}$  can be decomposed as  $\mathcal{H} = \mathcal{M}_+ \dot{+} \mathcal{M}_-$ ; see (2.6). Observe that the projection  $Q = P_{\mathcal{M}_+ // \mathcal{M}_-}$  has the following block operator matrix representation with respect to the fundamental decomposition (3.4):

$$Q = \begin{bmatrix} I & 0 \\ K^* & 0 \end{bmatrix}, \tag{3.9}$$

where  $K : \mathcal{M}_- \rightarrow \mathcal{M}_+^{[\perp]}$  is the angular operator associated with  $\mathcal{M}_+^{[\perp]}$ . On the other hand, with respect to the fundamental decomposition (3.6),  $Q$  is represented as

$$Q = \begin{bmatrix} I & -L \\ 0 & 0 \end{bmatrix}, \tag{3.10}$$

where  $L : \mathcal{M}_+^{[\perp]} \rightarrow \mathcal{M}_+$  is the angular operator associated with  $\mathcal{M}_-$ .

**Corollary 3.3.** *Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a  $J$ -frame for  $\mathcal{H}$  with  $J$ -frame operator  $S$ , let  $S_\pm$  be as in (2.8), and let  $\mathcal{M}_\pm$  be as in (2.5). Then the following statements are true.*

- (i) *With respect to the fundamental decomposition (3.4), the positive operators  $S_+$  and  $S_-$  are represented as*

$$S_+ = \begin{bmatrix} A & -AK \\ K^*A & -K^*AK \end{bmatrix} \quad \text{and} \quad S_- = \begin{bmatrix} 0 & 0 \\ 0 & -(D + K^*AK) \end{bmatrix}. \tag{3.11}$$

- (ii) With respect to the fundamental decomposition (3.6), the positive operators  $S_+$  and  $S_-$  are represented as

$$S_+ = \begin{bmatrix} A' + LD'L^* & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S_- = \begin{bmatrix} LD'L^* & -LD' \\ D'L^* & -D' \end{bmatrix}. \quad (3.12)$$

*Proof.* The positive operators  $S_+$  and  $S_-$  can be written as  $S_+ = QS$  and  $S_- = -(I - Q)S$ , where  $Q = P_{\mathcal{M}_+ // \mathcal{M}_-}$ ; see (2.10). Then the block operator matrix representations of  $S_+$  and  $S_-$  follow easily by multiplying (3.9) with (3.3), and (3.10) with (3.8), respectively.  $\square$

Finally, if  $S$  is a  $J$ -frame operator, we derive block operator matrix representations for its inverse  $S^{-1}$ .

**Theorem 3.4.** *Let  $S$  be a bounded and boundedly invertible self-adjoint operator acting on a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ . Then, the following conditions are equivalent:*

- (i)  $S$  is a  $J$ -frame operator;  
(ii) there exists a fundamental decomposition  $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$  such that  $S^{-1}$  is represented as

$$S^{-1} = \begin{bmatrix} A^{-1} - KZK^* & KZ \\ -ZK^* & Z \end{bmatrix}, \quad (3.13)$$

where  $A$  is a uniformly positive operator in the Hilbert space  $(\mathcal{H}_+, [\cdot, \cdot])$ ,  $Z$  is a uniformly positive operator in the Hilbert space  $(\mathcal{H}_-, -[\cdot, \cdot])$  and  $K : \mathcal{H}_- \rightarrow \mathcal{H}_+$  is a uniform contraction;

- (iii) there exists a fundamental decomposition  $\mathcal{H} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$  such that  $S^{-1}$  is represented as

$$S^{-1} = \begin{bmatrix} Y & -YL \\ L^*Y & (D')^{-1} - L^*YL \end{bmatrix}, \quad (3.14)$$

where  $D'$  is a uniformly positive operator in the Hilbert space  $(\mathcal{K}_-, -[\cdot, \cdot])$ ,  $Y$  is a uniformly positive operator in the Hilbert space  $(\mathcal{K}_+, [\cdot, \cdot])$  and  $L : \mathcal{K}_- \rightarrow \mathcal{K}_+$  is a uniform contraction.

*Proof.* First, we prove the equivalence of (i) and (ii). By Theorem 3.1,  $S$  is a  $J$ -frame operator if and only if there exists a fundamental decomposition  $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$  such that  $S$  is represented as (3.3), where  $A$  is a uniformly positive operator in the Hilbert space  $(\mathcal{H}_+, [\cdot, \cdot])$ ,  $D$  is a self-adjoint operator in the Hilbert space  $(\mathcal{H}_-, -[\cdot, \cdot])$  such that  $D + K^*AK$  is uniformly positive, and  $K : \mathcal{H}_- \rightarrow \mathcal{H}_+$  with  $\|K\| < 1$ .

Observe that the representation of  $S$  in (3.3) can be written as

$$S = \begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D + K^*AK \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}.$$

Hence, if  $Z := (D + K^*AK)^{-1}$ ,  $S$  is a  $J$ -frame operator if and only if

$$S^{-1} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & 0 \\ -K^* & I \end{bmatrix} = \begin{bmatrix} A^{-1} - KZK^* & KZ \\ -ZK^* & Z \end{bmatrix},$$

where  $A$  and  $Z$  are uniformly positive operators and  $K$  is a uniform contraction.

The proof of the equivalence of (i) and (iii) is similar, where one uses that  $S$  can be represented as in (3.8) in Theorem 3.2 and one defines  $Y := (A' + LD'L^*)^{-1}$ .  $\square$

**Remark 3.5.** It follows from Proposition 2.6 that if  $S : \mathcal{H} \rightarrow \mathcal{H}$  is a  $J$ -frame operator, then  $S^{-1}$  is also a  $J$ -frame operator. Moreover, the representation of  $S^{-1}$  in (3.13) uses the components of the representation of  $S$  in (3.3); note that, according to the proof,  $Z = (D + K^*AK)^{-1}$ . However, (3.13) is of the form as in (3.8) in Theorem 3.2 because with the notation from that theorem one has that  $D' = Z$  is uniformly positive,  $L = K$  is a uniform contraction,  $A' = A^{-1} - KZK^*$  satisfies

$$A' + LD'L^* = (A^{-1} - KZK^*) + KZK^* = A^{-1},$$

and, hence,  $A' + LD'L^*$  is uniformly positive.

Similarly, the representation of  $S^{-1}$  in (3.14) uses the components of the representation of  $S$  in (3.8) with  $Y = (A' + LD'L^*)^{-1}$ , and the representation of  $S^{-1}$  is of the form as in (3.3).

#### 4. INTERPRETING THE $J$ -FRAME BOUNDS AS SPECTRAL BOUNDS OF THE $J$ -FRAME OPERATOR

Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a  $J$ -frame for  $\mathcal{H}$  and let  $\mathcal{M}_\pm$  be as in (2.5). By Theorem 2.2,  $\mathcal{F}_\pm = \{f_i\}_{i \in I_\pm}$  is a frame for  $(\mathcal{M}_\pm, \pm[\cdot, \cdot])$ , i.e. there exist  $0 < \alpha_\pm \leq \beta_\pm$  such that

$$\alpha_\pm(\pm[f, f]) \leq \sum_{i \in I_\pm} |[f, f_i]|^2 \leq \beta_\pm(\pm[f, f]) \quad \text{for every } f \in \mathcal{M}_\pm,$$

cf. (2.7). The frame bounds of  $\mathcal{F}_\pm$  (i.e. the optimal set of constants  $0 < \alpha_\pm \leq \beta_\pm$ ) are called the  $J$ -frame bounds of  $\mathcal{F}$ .

It is our aim to recover the  $J$ -frame bounds of  $\mathcal{F}$  as spectral bounds for the  $J$ -frame operator  $S$ . Moreover, we recover also the  $J$ -frame bounds of the dual  $J$ -frame  $\mathcal{F}' = \{S^{-1}f_i\}_{i \in I}$ . This is achieved in a straightforward approach by using the  $2 \times 2$  block operator matrix representations for  $S$ ,  $S_\pm$  and  $S^{-1}$  from Section 3.

**Proposition 4.1.** *Let  $0 < \alpha_\pm \leq \beta_\pm$  be the  $J$ -frame bounds of the  $J$ -frame  $\mathcal{F}$ . If the  $J$ -frame operator  $S$  is represented as in (3.3), then*

$$\alpha_- = \inf W(D + K^*AK) \quad \text{and} \quad \beta_- = \sup W(D + K^*AK).$$

*If the  $J$ -frame operator  $S$  is represented as in (3.8), then*

$$\alpha_+ = \inf W(A' + LD'L^*) \quad \text{and} \quad \beta_+ = \sup W(A' + LD'L^*).$$

*Proof.* If  $f \in \mathcal{M}_-$  then, by (2.9) and (3.11),

$$\begin{aligned} \sum_{i \in I_-} |[f, f_i]|^2 &= [S_-f, f] = - \left[ \begin{bmatrix} 0 & 0 \\ 0 & D + K^*AK \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix}, \begin{bmatrix} 0 \\ f \end{bmatrix} \right] \\ &= -[(D + K^*AK)f, f]. \end{aligned}$$

As  $\mathcal{F}_- = \{f_i\}_{i \in I_-}$  is a frame for  $(\mathcal{M}_-, -[\cdot, \cdot])$  with frame bounds  $0 < \alpha_- \leq \beta_-$ , it is immediate that  $\inf W(D + K^*AK) = \alpha_-$  and  $\sup W(D + K^*AK) = \beta_-$ .

Similarly, if  $f \in \mathcal{M}_+$  then, by (2.9) and (3.12),

$$\begin{aligned} \sum_{i \in I_+} |[f, f_i]|^2 &= [S_+f, f] = \begin{bmatrix} A' + LD'L^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix} \\ &= [(A' + LD'L^*)f, f]. \end{aligned}$$

Since  $\mathcal{F}_+ = \{f_i\}_{i \in I_+}$  is a frame for  $(\mathcal{M}_+, [\cdot, \cdot])$  with frame bounds  $0 < \alpha_+ \leq \beta_+$ , it follows that  $\inf W(A' + LD'L^*) = \alpha_+$  and  $\sup W(A' + LD'L^*) = \beta_+$ .  $\square$

In what follows, we derive formulae for the  $J$ -frame bounds of the canonical dual  $J$ -frame  $\mathcal{F}' = \{S^{-1}f_i\}_{i \in I}$ . Recall that it is also a  $J$ -frame for  $\mathcal{H}$ , and its  $J$ -frame operator is  $S^{-1}$ ; see Proposition 2.6. In addition,  $\mathcal{F}'_{\pm} = \{S^{-1}f_i\}_{i \in I_{\pm}}$  is a frame for  $(\mathcal{M}_{\mp}^{[\perp]}, \pm[\cdot, \cdot])$ , i.e. there exist constants  $0 < \gamma_{\pm} \leq \delta_{\pm}$  such that

$$\gamma_{\pm}(\pm[f, f]) \leq \sum_{i \in I_{\pm}} |[f, S^{-1}f_i]|^2 \leq \delta_{\pm}(\pm[f, f]) \quad \text{for every } f \in \mathcal{M}_{\mp}^{[\perp]}.$$

The optimal set of constants  $0 < \gamma_{\pm} \leq \delta_{\pm}$  are the  $J$ -frame bounds of  $\mathcal{F}'$ .

**Proposition 4.2.** *Let  $\mathcal{F}$  be a  $J$ -frame for  $\mathcal{H}$  with  $J$ -frame operator  $S$ , and let  $0 < \gamma_{\pm} \leq \delta_{\pm}$  be the  $J$ -frame bounds of the canonical dual  $J$ -frame  $\mathcal{F}'$ . If the  $J$ -frame operator  $S$  is represented as in (3.3), then*

$$\gamma_+ = \inf W(A^{-1}) \quad \text{and} \quad \delta_+ = \sup W(A^{-1}).$$

If the  $J$ -frame operator  $S$  is represented as in (3.8), then

$$\gamma_- = \inf W((D')^{-1}) \quad \text{and} \quad \delta_- = \sup W((D')^{-1}).$$

*Proof.* The  $J$ -frame operator  $S^{-1}$  can be written as  $S^{-1} = (S^{-1})_+ - (S^{-1})_-$ , where  $(S^{-1})_+ = Q^+S^{-1}$  and  $(S^{-1})_- = -(I - Q)^+S^{-1}$  are positive operators in the Krein space  $\mathcal{H}$ ; see (2.10).

Also, if  $S$  is represented as in (3.3) then, by Remark 3.5,  $S^{-1}$  is represented as in (3.13). Moreover, following the same arguments as in Corollary 3.3 it can be proved that

$$(S^{-1})_+ = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad (S^{-1})_- = \begin{bmatrix} KZK^* & -KZ \\ ZK^* & -Z \end{bmatrix}.$$

If  $f \in \mathcal{M}_-^{[\perp]}$  then, by (2.9) and the above representation for  $(S^{-1})_+$ ,

$$\sum_{i \in I_+} |[f, S^{-1}f_i]|^2 = [(S^{-1})_+f, f] = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix} = [A^{-1}f, f].$$

As  $\{S^{-1}f_i\}_{i \in I_+}$  is a frame for  $(\mathcal{M}_-^{[\perp]}, [\cdot, \cdot])$  with frame bounds  $0 < \gamma_+ \leq \delta_+$ , it follows that  $\inf W(A^{-1}) = \gamma_+$  and  $\sup W(A^{-1}) = \delta_+$ .

On the other hand, if the  $J$ -frame operator  $S$  is represented as in (3.8) then, by Remark 3.5,  $S^{-1}$  is represented as in (3.14). Then,

$$(S^{-1})_+ = \begin{bmatrix} Y & -YL \\ L^*Y & -L^*YL \end{bmatrix} \quad \text{and} \quad (S^{-1})_- = \begin{bmatrix} 0 & 0 \\ 0 & -(D')^{-1} \end{bmatrix},$$

and, since  $\{S^{-1}f_i\}_{i \in I_-}$  is a frame for  $(\mathcal{M}_+^{[\perp]}, -[\cdot, \cdot])$  with frame bounds  $0 < \gamma_- \leq \delta_-$ , the characterization of  $\gamma_-$  and  $\delta_-$  follows.  $\square$

### 5. SPECTRUM OF THE $J$ -FRAME OPERATOR

In Theorem 5.1 below we describe the location of the spectrum of  $S$ . In its proof we use the first and second Schur complements of  $S$  according to the representations of  $S$  given in Theorems 3.2 and 3.1, respectively.

Let  $S$  be a  $J$ -frame operator in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$  and assume first that  $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$  is a fundamental decomposition such that  $S$  is decomposed as in (3.3) in Theorem 3.1. In this case we use the second Schur complement of  $S$ , which is defined as

$$S_2(\lambda) := D - \lambda + K^*A(A - \lambda)^{-1}AK, \quad \lambda \in \rho(A).$$

On  $\rho(A)$  the spectra of the operator function  $S_2(\lambda)$  and the operator  $S$  coincide, i.e.

$$0 \in \sigma(S_2(\lambda)) \iff \lambda \in \sigma(S) \quad \text{for } \lambda \in \rho(A); \tag{5.1}$$

see, e.g. [27, Proposition 1.6.2].

Since  $A$  and  $D + K^*AK$  are uniformly positive in the Hilbert spaces  $(\mathcal{H}_+, [\cdot, \cdot])$  and  $(\mathcal{H}_-, -[\cdot, \cdot])$ , respectively, their numerical ranges  $W(A)$  and  $W(D + K^*AK)$  are intervals in  $(0, +\infty)$ . Also  $D$  is self-adjoint in the Hilbert space  $(\mathcal{H}_-, -[\cdot, \cdot])$ ; so its numerical range  $W(D)$  is a real interval.

On the other hand, if  $\mathcal{H} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$  is a fundamental decomposition of  $\mathcal{H}$  such that  $S$  is decomposed as in (3.8) in Theorem 3.2, we use the first Schur complement of  $S$ , which is defined as

$$S_1(\lambda) := A' - \lambda + LD'(D' - \lambda)^{-1}D'L^*, \quad \lambda \in \rho(D').$$

On  $\rho(D')$  the spectra of the operator function  $S_1(\lambda)$  and the operator  $S$  coincide, i.e.

$$0 \in \sigma(S_1(\lambda)) \iff \lambda \in \sigma(S) \quad \text{for } \lambda \in \rho(D');$$

see, e.g. [27, Proposition 1.6.2]. In this case the numerical ranges  $W(D')$  and  $W(A' + LD'L^*)$  are intervals in  $(0, +\infty)$  and  $W(A')$  is a real interval.



**Theorem 5.1.** *Let  $S$  be a  $J$ -frame operator in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$ .*

- (i) *Let  $S$  be decomposed as in (3.3) and assume that  $\overline{W(A)} = [a_-, a_+]$ ,  $\overline{W(D)} = [d_-, d_+]$ , and  $\overline{W(D + K^*AK)} = [b_-, b_+]$ . Then, the spectrum of  $S$  satisfies the following inclusions:*

$$\sigma(S) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} : |\lambda - a_+| < a_+ \text{ and } \operatorname{Re} \lambda > \frac{b_-}{2} \right\}, \quad (5.2)$$

$$\sigma(S) \cap \mathbb{R} \subseteq [\min\{a_-, b_-\}, \max\{a_+, d_+\}]. \quad (5.3)$$

- (ii) *Let  $S$  be decomposed as in (3.8) and assume that  $\overline{W(A')} = [a'_-, a'_+]$ ,  $\overline{W(D')} = [d'_-, d'_+]$ , and  $\overline{W(A' + LD'L^*)} = [b'_-, b'_+]$ . Then, the spectrum of  $S$  satisfies the following inclusions:*

$$\sigma(S) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} : |\lambda - d'_+| < d'_+ \text{ and } \operatorname{Re} \lambda > \frac{b'_-}{2} \right\}, \quad (5.4)$$

$$\sigma(S) \cap \mathbb{R} \subseteq [\min\{d'_-, b'_-\}, \max\{a'_+, d'_+\}].$$

*Proof.* To prove (i), let us rewrite  $S_2(\lambda)$  as follows:

$$\begin{aligned} S_2(\lambda) &= D - \lambda + K^*A(A - \lambda)^{-1}(A - \lambda + \lambda)K \\ &= D - \lambda + K^*AK + \lambda K^*A(A - \lambda)^{-1}K \\ &= D + K^*AK - \lambda(I - K^*K) + \lambda^2 K^*(A - \lambda)^{-1}K. \end{aligned}$$

Hence, for  $g \in \mathcal{H}_-$  with  $\|g\| = 1$  we obtain

$$\langle S_2(\lambda)g, g \rangle = \langle (D + K^*AK)g, g \rangle - \lambda(1 - \|Kg\|^2) + \lambda^2 \langle (A - \lambda)^{-1}Kg, Kg \rangle \quad (5.5)$$

$$\begin{aligned} &= \langle (D + K^*AK)g, g \rangle - \lambda(1 - \|Kg\|^2) \\ &\quad + \lambda^2 \langle A(A - \lambda)^{-1}Kg, (A - \lambda)^{-1}Kg \rangle - \lambda|\lambda|^2 \|(A - \lambda)^{-1}Kg\|^2. \end{aligned} \quad (5.6)$$

First, we show that

$$(-\infty, \min\{a_-, b_-\}) \subseteq \rho(S). \quad (5.7)$$

Let  $\lambda \in (-\infty, \min\{a_-, b_-\})$ . Then  $\lambda \in \rho(A)$  and  $(A - \lambda)^{-1}$  is positive. Therefore (5.5) yields

$$\begin{aligned} \langle S_2(\lambda)g, g \rangle &\geq \langle (D + K^*AK)g, g \rangle - \lambda(1 - \|Kg\|^2) \\ &\geq b_- - \lambda(1 - \|Kg\|^2) \geq b_- - \max\{\lambda, 0\}(1 - \|Kg\|^2) \\ &\geq b_- - \max\{\lambda, 0\} > 0 \end{aligned}$$

for every  $g \in \mathcal{H}_-$  with  $\|g\| = 1$ . This implies that 0 is not in the closure of the numerical range of  $S_2(\lambda)$  and hence  $0 \in \rho(S_2(\lambda))$ . By (5.1) this shows that  $\lambda \in \rho(S)$ . Therefore (5.7) is proved.

In [19, Theorem 2.1] it was shown that  $\sigma(S) \cap \mathbb{R} \subseteq [\min\{a_-, d_-\}, \max\{a_+, d_+\}]$ . Since  $A$  is a uniformly positive operator,

$$d_- = \min \overline{W(D)} \leq \min \overline{W(D + K^*AK)} = b_-.$$

This, together with (5.7) shows (5.3).

Next, we show that

$$\sigma(S) \setminus \mathbb{R} \subseteq \{\lambda \in \mathbb{C} \setminus \mathbb{R} : |\lambda - a_+| < a_+\}. \tag{5.8}$$

Let  $\lambda = x + iy$  be in the complement of the right-hand side of (5.8) and assume that  $\lambda$  is non-real, i.e.  $y \neq 0$  and

$$x^2 - 2a_+x + y^2 \geq 0. \tag{5.9}$$

Since the spectrum of  $S$  is symmetric with respect to the real axis, we may assume, without loss of generality, that  $y > 0$ . Consider the spectral function  $E_t$  associated with the (positive) operator  $A$ :

$$A = \int_{a_-}^{a_+} t \, dE_t.$$

Then, we can rewrite some terms that appear in (5.6) in terms of  $E_t$ :

$$\begin{aligned} \langle A(A - \lambda)^{-1}Kg, (A - \lambda)^{-1}Kg \rangle &= \int_{a_-}^{a_+} \frac{t}{|t - \lambda|^2} \, d\langle E_tKg, Kg \rangle, \\ \|(A - \lambda)^{-1}Kg\|^2 &= \int_{a_-}^{a_+} \frac{1}{|t - \lambda|^2} \, d\langle E_tKg, Kg \rangle. \end{aligned}$$

For  $g \in \mathcal{H}_-$  with  $\|g\| = 1$  we obtain from (5.6) that

$$\begin{aligned} \operatorname{Im}\langle S_2(\lambda)g, g \rangle &= -y(1 - \|Kg\|^2) + 2xy\langle A(A - \lambda)^{-1}Kg, (A - \lambda)^{-1}Kg \rangle \\ &\quad - y(x^2 + y^2)\|(A - \lambda)^{-1}Kg\|^2 \\ &= -y(1 - \|Kg\|^2) + y \int_{a_-}^{a_+} \frac{2xt - (x^2 + y^2)}{|t - \lambda|^2} \, d\langle E_tKg, Kg \rangle. \end{aligned} \tag{5.10}$$

If  $x \leq 0$ , then the numerator of the fraction in the integral in (5.10) is negative as  $t > 0$ ; if  $x > 0$ , then (5.9) yields

$$2xt - (x^2 + y^2) \leq 2xa_+ - (x^2 + y^2) \leq 0.$$

In both cases the integral in (5.10) is non-positive and hence

$$\operatorname{Im}\langle S_2(\lambda)g, g \rangle \leq -y(1 - \|Kg\|^2) \leq -y(1 - \|K\|^2) < 0.$$

This shows that 0 is not in the closure of the numerical range of  $S_2(\lambda)$  for such  $\lambda$  and hence  $0 \in \rho(S_2(\lambda))$ . Again by (5.1) this shows that  $\lambda \in \rho(S)$ . Thus (5.8) is proved.

It remains to show that

$$\sigma(S) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \operatorname{Re} \lambda > \frac{b_-}{2} \right\},$$

but we postpone this until we have completed the proof of (ii).

To prove (ii), note that

$$S_1(\lambda) = A' + LD'L^* - \lambda(I - LL^*) + \lambda^2 L(D' - \lambda)^{-1} L^*$$

for  $\lambda \in \rho(D')$ . Following the same arguments as above, it follows that

$$\sigma(S) \cap \mathbb{R} \subseteq [\min\{d'_-, b'_-\}, \max\{a'_+, d'_+\}]$$

and

$$\sigma(S) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : |\lambda - d'_+| < d'_+ \right\}. \quad (5.11)$$

It follows from Remark 3.5 that if  $S$  is represented as in (3.8) then the representation (3.14) corresponds to (3.3) for  $S^{-1}$ . Hence, applying (5.8) to the  $J$ -frame operator  $S^{-1}$  we obtain that

$$\sigma(S^{-1}) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \left| \lambda - \frac{1}{b'_-} \right| < \frac{1}{b'_-} \right\},$$

because  $Y = (A' + LD'L^*)^{-1}$  and

$$\sup W((A' + LD'L^*)^{-1}) = (\inf W(A' + LD'L^*))^{-1} = \frac{1}{b'_-}.$$

Therefore,

$$\sigma(S) \setminus \mathbb{R} = \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \frac{1}{\lambda} \in \sigma(S^{-1}) \right\} \subseteq \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \operatorname{Re} \lambda > \frac{b'_-}{2} \right\}.$$

This, together with (5.11), yields (5.4) and (ii) is shown.

Finally, to conclude the proof of (i), note that if  $S$  is represented as in (3.3) then the representation (3.13) corresponds to (3.8) for  $S^{-1}$  (see Remark 3.5). Applying (5.11) to the  $J$ -frame operator  $S^{-1}$  we obtain that

$$\sigma(S^{-1}) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \left| \lambda - \frac{1}{b_-} \right| < \frac{1}{b_-} \right\},$$

because  $Z = (D + K^*AK)^{-1}$  and

$$\sup W((D + K^*AK)^{-1}) = (\inf W(D + K^*AK))^{-1} = \frac{1}{b_-}.$$

Hence,

$$\sigma(S) \setminus \mathbb{R} = \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \frac{1}{\lambda} \in \sigma(S^{-1}) \right\} \subseteq \left\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : \operatorname{Re} \lambda > \frac{b_-}{2} \right\}.$$

This, together with (5.8), yields (5.2).  $\square$

**Remark 5.2.** Let  $S$  be a  $J$ -frame operator in  $(\mathcal{H}, [\cdot, \cdot])$  and assume that  $S$  is represented as in (3.3). By [19, Theorem 2.1], we have another enclosure for the non-real spectrum of  $S$ :

$$|\operatorname{Im} z| \leq \|AK\| \quad \text{for } z \in \sigma(S) \setminus \mathbb{R}.$$

Therefore, if  $K = 0$  then the spectrum of  $S$  is contained in  $(0, +\infty)$ . Analogously, if  $S$  is represented as in (3.8) then  $|\operatorname{Im} z| \leq \|LD'\|$  for every  $z \in \sigma(S) \setminus \mathbb{R}$ . Thus,  $L = 0$  implies that  $\sigma(S) \subseteq (0, +\infty)$ .

Assume that  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $\mathcal{H}$  with  $J$ -frame operator  $S$ , and  $\mathcal{M}_\pm$  are defined as in (2.5). Then  $K$  and  $L$  are the angular operators of  $\mathcal{M}_+^{[\perp]}$  and  $\mathcal{M}_-$  according to the decompositions (3.4) and (3.6), respectively. Hence, if  $\mathcal{M}_- = \mathcal{M}_+^{[\perp]}$  we have that  $K = L = 0$  and  $\sigma(S) \subseteq (0, +\infty)$ . Moreover, in this case  $S$  is a uniformly positive operator in  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ .

Given a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  with  $J$ -frame operator  $S$ , we can use Theorem 5.1 to give a spectral enclosure for  $\sigma(S)$  in terms of the  $J$ -frame bounds of  $\mathcal{F}$  and its canonical dual  $J$ -frame  $\mathcal{F}' = \{S^{-1}f_i\}_{i \in I}$ .

**Corollary 5.3.** *Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a  $J$ -frame for a Krein space  $(\mathcal{H}, [\cdot, \cdot])$  with  $J$ -frame operator  $S$  and  $J$ -frame bounds  $0 < \alpha_\pm \leq \beta_\pm$ . Assume also that  $0 < \gamma_\pm \leq \delta_\pm$  are the  $J$ -frame bounds of its canonical dual frame  $\mathcal{F}' = \{S^{-1}f_i\}_{i \in I}$ . Then, the spectrum of  $S$  satisfies the following inclusions:*

$$\sigma(S) \cap \mathbb{R} \subseteq [\varepsilon_-, \varepsilon_+], \tag{5.12}$$

$$\sigma(S) \setminus \mathbb{R} \subseteq \left\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda > \frac{\alpha}{2}, \quad \left| \lambda - \frac{1}{\gamma} \right| < \frac{1}{\gamma} \right\}, \tag{5.13}$$

where

$$\varepsilon_- = \max \left\{ \min \left\{ \frac{1}{\delta_+}, \alpha_- \right\}, \min \left\{ \frac{1}{\delta_-}, \alpha_+ \right\} \right\},$$

$$\varepsilon_+ = \min \left\{ \max \left\{ \frac{1}{\gamma_+}, \beta_- \right\}, \max \left\{ \frac{1}{\gamma_-}, \beta_+ \right\} \right\},$$

$$\alpha = \max\{\alpha_+, \alpha_-\} \quad \text{and} \quad \gamma = \max\{\gamma_+, \gamma_-\}.$$

*Proof.* Given a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  for  $\mathcal{H}$  with  $J$ -frame operator  $S$ , let  $\mathcal{M}_\pm$  be given by (2.5). By Theorem 3.1, according to the fundamental decomposition  $\mathcal{H} = \mathcal{M}_-^{[\perp][+]} \mathcal{M}_-$ ,  $S$  can be represented as in (3.3). Also, according to the fundamental decomposition  $\mathcal{H} = \mathcal{M}_+^{[+]} \mathcal{M}_+^{[\perp]}$ ,  $S$  can be represented as in (3.8). By Propositions 4.1 and 4.2,

$$\overline{W(A)} = \left[ \frac{1}{\delta_+}, \frac{1}{\gamma_+} \right] \quad \text{and} \quad \overline{W(D + K^*AK)} = [\alpha_-, \beta_-],$$

$$\overline{W(D')} = \left[ \frac{1}{\delta_-}, \frac{1}{\gamma_-} \right] \quad \text{and} \quad \overline{W(A' + LD'L^*)} = [\alpha_+, \beta_+].$$

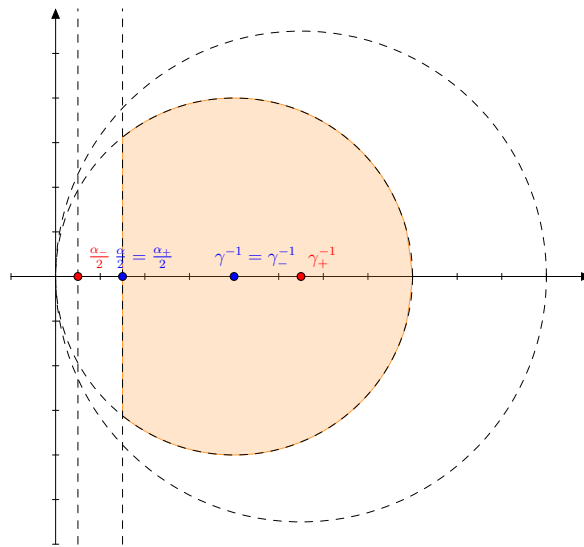
Since  $A$  is uniformly positive, we have  $\inf W(D) \leq \alpha_-$  and  $\sup W(D) \leq \beta_-$ . Analogously, the uniform positiveness of  $D'$  implies that  $\inf W(A') \leq \alpha_+$  and  $\sup W(A') \leq \beta_+$ . Hence, by Theorem 5.1, we have that

$$\begin{aligned} \sigma(S) \cap \mathbb{R} &\subseteq \left[ \min \left\{ \frac{1}{\delta_+}, \alpha_- \right\}, \max \left\{ \frac{1}{\gamma_+}, \beta_- \right\} \right], \\ \sigma(S) \cap \mathbb{R} &\subseteq \left[ \min \left\{ \frac{1}{\delta_-}, \alpha_+ \right\}, \max \left\{ \frac{1}{\gamma_-}, \beta_+ \right\} \right], \end{aligned}$$

and (5.12) follows by intersecting the above intervals. For the non-real part of the spectrum of  $S$ , Theorem 5.1 yields

$$\begin{aligned} \sigma(S) \setminus \mathbb{R} &\subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{\gamma_+} \right| < \frac{1}{\gamma_+} \text{ and } \operatorname{Re} \lambda > \frac{\alpha_-}{2} \right\}, \\ \sigma(S) \setminus \mathbb{R} &\subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{\gamma_-} \right| < \frac{1}{\gamma_-} \text{ and } \operatorname{Re} \lambda > \frac{\alpha_+}{2} \right\}, \end{aligned}$$

and (5.13) follows by intersecting the two enclosures above (see Figure 1). □



**Fig. 1.** The region containing  $\sigma(S) \setminus \mathbb{R}$  given by the set on the right-hand side of (5.13)

### 6. SQUARE ROOT OF A $J$ -FRAME OPERATOR: APPLICATIONS TO $J$ -FRAMES

It follows from Theorem 5.1 that if  $S$  is a  $J$ -frame operator acting on a Krein space  $\mathcal{H}$ , then its spectrum is located in the open right half-plane. Consider  $f(z) = z^{1/2}$  as an analytic function on the open right half-plane such that  $\operatorname{Re} f(z) > 0$  if  $\operatorname{Re} z > 0$ .

Then, we can construct a square root of  $S$  by means of the Riesz–Dunford functional calculus:

$$S^{1/2} := \frac{1}{2\pi i} \int_{\Gamma} z^{1/2}(z - S)^{-1} dz, \tag{6.1}$$

where  $\Gamma$  is a Jordan curve in the right half-plane enclosing  $\sigma(S)$ . The operator  $S^{1/2}$  is also an invertible self-adjoint operator in the Krein space  $\mathcal{H}$ .

The following theorem shows that this is (in some sense) the unique square root of  $S$ .

**Theorem 6.1.** *Let  $S$  be a  $J$ -frame operator in a Krein space  $(\mathcal{H}, [\cdot, \cdot])$ . Then there exists a unique (bounded) operator  $P$  acting on  $\mathcal{H}$  that satisfies  $P^2 = S$  and*

$$\sigma(P) \subseteq \{z \in \mathbb{C} : z = re^{it}, r > 0, t \in (-\frac{\pi}{4}, \frac{\pi}{4})\}. \tag{6.2}$$

Moreover, such an operator is self-adjoint in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$  and denoted by  $S^{1/2}$ .

*Proof.* By the discussion above and the Spectral Mapping Theorem, the operator  $S^{1/2}$  defined in (6.1) satisfies the desired conditions. In particular,  $S^{1/2}$  is invertible and we denote its inverse by  $S^{-1/2}$ .

Assume that  $P$  is another operator satisfying  $P^2 = S$  and (6.2). Then,  $PS = SP$  and also

$$PS^{-1/2} = S^{-1/2}P,$$

as  $g(z) = z^{-1/2}$  is an analytic function in an open domain containing  $\sigma(S)$ ; see [9, Proposition 4.9]. Thus, the operator  $U := PS^{-1/2}$  satisfies

$$U^2 = (S^{-1/2}P)(PS^{-1/2}) = S^{-1/2} S S^{-1/2} = I.$$

On the other hand, both  $\sigma(P)$  and  $\sigma(S^{-1/2})$  are contained in the open domain defined in the right-hand side of (6.2) and, by [24, Lemma 0.11],

$$\sigma(U) \subseteq \{\lambda \cdot \mu : \lambda \in \sigma(P), \mu \in \sigma(S^{-1/2})\} \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}.$$

So,  $(U+I)(U-I) = 0$  and  $U+I$  is invertible, or equivalently,  $U = I$  and  $P = S^{1/2}$ .  $\square$

In the following we characterize the family of  $J$ -frames for  $\mathcal{H}$  with a prescribed  $J$ -frame operator, i.e. given a  $J$ -frame operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  we describe those  $J$ -frames  $\mathcal{F}$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$  such that  $TT^+ = S$ . To do so, we show that the synthesis operator of a  $J$ -frame admits a polar decomposition (in the Krein space sense).

First, let us state some well-known facts about partial isometries in Krein spaces. Given Krein spaces  $\mathcal{H}$  and  $\mathcal{K}$ , a (bounded) operator  $U : \mathcal{H} \rightarrow \mathcal{K}$  is a *partial isometry* if  $UU^+U = U$ , an *isometry* if  $U^+U = I$ , a *co-isometry* if  $UU^+ = I$ , and a *unitary operator* if both  $U^+U = I$  and  $UU^+ = I$ .

For instance,  $U$  is a partial isometry if and only if there exist regular subspaces  $\mathcal{M}$  of  $\mathcal{H}$  and  $\mathcal{N}$  of  $\mathcal{K}$  such that

(1)  $U$  is a one-to-one mapping from  $\mathcal{M}$  onto  $\mathcal{N}$  satisfying

$$[Uf, Ug]_{\mathcal{K}} = [f, g]_{\mathcal{H}}, \quad f, g \in \mathcal{M};$$

(2)  $N(U) = \mathcal{M}^{\perp}$ .

In this case, we call  $\mathcal{M}$  and  $\mathcal{N}$  the *initial* and *final spaces* for  $U$ . Moreover,  $U^+$  is a partial isometry with initial space  $\mathcal{N}$  and final space  $\mathcal{M}$ , and

$$\begin{aligned} U^+U &= E_{\mathcal{M}}, & N(U) &= N(E_{\mathcal{M}}) = \mathcal{M}^{\perp}, \\ UU^+ &= E_{\mathcal{N}}, & N(U^+) &= N(E_{\mathcal{N}}) = \mathcal{N}^{\perp}, \end{aligned}$$

where  $E_{\mathcal{M}}$  and  $E_{\mathcal{N}}$  are the self-adjoint projections onto  $\mathcal{M}$  and  $\mathcal{N}$ , respectively; see, e.g. [11, Theorems 1.7 and 1.8].

Before stating the polar decomposition for the synthesis operator of a  $J$ -frame  $\mathcal{F}$ , recall that we are considering  $\ell_2(I)$  as a Krein space with the indefinite inner product induced by  $\mathcal{F}$ ; see (2.3).

**Proposition 6.2.** *Let  $\mathcal{F}$  be a  $J$ -frame for a Krein space  $(\mathcal{H}, [\cdot, \cdot])$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$ . Then  $T$  admits a polar decomposition in the Krein space sense: there exists a co-isometry  $U : \ell_2(I) \rightarrow \mathcal{H}$  with initial space  $N(T)^{\perp}$  such that*

$$T = (TT^+)^{1/2}U,$$

where the square root is the one defined in (6.1). Moreover, this factorization is unique in the following sense: if  $T = PW$  where  $P$  is a self-adjoint operator in  $\mathcal{H}$  satisfying (6.2) and  $W : \ell_2(I) \rightarrow \mathcal{H}$  is a co-isometry, then  $P = (TT^+)^{1/2}$  and  $W = U$ .

*Proof.* If  $S = TT^+$  is the  $J$ -frame operator of  $\mathcal{F}$  and  $S^{1/2}$  denotes the self-adjoint square root defined in (6.1), then  $U := S^{-1/2}T$  is a co-isometry from the Krein space  $(\ell_2(I), [\cdot, \cdot]_2)$  onto  $(\mathcal{H}, [\cdot, \cdot])$ . Indeed, it is immediate that

$$UU^+ = (S^{-1/2}T)(T^+S^{-1/2}) = S^{-1/2}SS^{-1/2} = I.$$

Hence  $U^+U$  is a self-adjoint projection in the Krein space  $(\ell_2(I), [\cdot, \cdot]_2)$ . Also,  $UU^+U = U$  implies that  $N(U^+U) = N(U) = N(T)$ , which is a regular subspace of  $\ell_2(I)$ , see [13, Lemma 4.1]. Hence,  $N(T)^{\perp}$  is the initial space of  $U$  and

$$T = S^{1/2}(S^{-1/2}T) = S^{1/2}U = (TT^+)^{1/2}U.$$

Finally, let us show the uniqueness of such a factorization. Assume that  $T = PW$ , where  $P$  is a self-adjoint operator in  $\mathcal{H}$  satisfying (6.2) and  $W : \ell_2(I) \rightarrow \mathcal{H}$  is a co-isometry. Then,

$$TT^+ = (PW)W^+P = P(WW^+)P = P^2.$$

By Theorem 6.1, we have that

$$P = S^{1/2} = (TT^+)^{1/2},$$

and

$$U = S^{-1/2}T = S^{-1/2}PW = W. \quad \square$$

If  $\mathcal{F} = \{f_i\}_{i \in I}$  is a  $J$ -frame for  $\mathcal{H}$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$  and  $J$ -frame operator  $S$ , then the synthesis operator of the canonical dual  $J$ -frame  $\mathcal{F}' = \{S^{-1}f_i\}_{i \in I}$  is given by  $S^{-1}T$ . Therefore the following corollary is true.

**Corollary 6.3.** *Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a  $J$ -frame for  $\mathcal{H}$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$ . If  $T$  is factorized as  $T = (TT^+)^{1/2}U$  with a co-isometry  $U : \ell_2(I) \rightarrow \mathcal{H}$ , then the canonical dual  $J$ -frame  $\mathcal{F}' = \{(TT^+)^{-1}f_i\}_{i \in I}$  has synthesis operator  $(TT^+)^{-1/2}U : \ell_2(I) \rightarrow \mathcal{H}$ .*

Let a  $J$ -frame operator  $S$  on a Krein space  $(\mathcal{H}, [\cdot, \cdot])$  be given. Then, by definition, there exists a  $J$ -frame  $\mathcal{F} = \{f_i\}_{i \in I}$  with synthesis operator  $T : \ell_2(I) \rightarrow \mathcal{H}$  such that  $TT^+ = S$ . In general, for a given  $J$ -frame operator  $S$  there exist many  $J$ -frames such that  $S$  is the associated  $J$ -frame operator. The relation between two such  $J$ -frames with the same  $J$ -frame operator  $S$  can be expressed via their corresponding synthesis operators, which is done in the following theorem.

**Theorem 6.4.** *Let  $S$  be a  $J$ -frame operator in  $(\mathcal{H}, [\cdot, \cdot])$ . Assume that  $T_k : \ell_2(I_k) \rightarrow \mathcal{H}$  is the synthesis operator of a  $J$ -frame such that  $S = T_k T_k^+$ , for  $k = 1, 2$ . Then, there exists a partial isometry  $W : \ell_2(I_2) \rightarrow \ell_2(I_1)$  (in the Krein space sense) with initial space  $N(T_2)^{[\perp]}$  and final space  $N(T_1)^{[\perp]}$  such that*

$$T_2 = T_1 W.$$

*Proof.* By Proposition 6.2, there exist co-isometries  $U_k : \ell_2(I_k) \rightarrow \mathcal{H}$  with initial spaces  $N(T_k)^{[\perp]}$  such that  $T_k = S^{1/2}U_k$ . Set  $W := U_1^+ U_2$ . Then

$$\begin{aligned} WW^+ &= (U_1^+ U_2)(U_1^+ U_2)^+ = U_1^+(U_2 U_2^+)U_1 = U_1^+ U_1 = E_1, \\ W^+ W &= (U_1^+ U_2)^+(U_1^+ U_2) = U_2^+(U_1 U_1^+)U_2 = U_2^+ U_2 = E_2, \end{aligned}$$

where  $E_k$  is the self-adjoint projection onto  $N(T_k)^{[\perp]}$ . Therefore  $W$  is a partial isometry (in the Krein space sense) with initial space  $N(T_2)^{[\perp]}$  and final space  $N(T_1)^{[\perp]}$ . Furthermore,

$$T_1 W = (S^{1/2}U_1)(U_1^+ U_2) = S^{1/2}U_2 = T_2,$$

which completes the proof. □

Finally, we obtain for a given  $J$ -frame operator  $S$  a description of all  $J$ -frames such that  $S$  is the associated  $J$ -frame operator: there is a bijection between all co-isometries and all  $J$ -frames with the same  $J$ -frame operator.

**Corollary 6.5.** *Let  $S$  be a  $J$ -frame operator in  $(\mathcal{H}, [\cdot, \cdot])$ . Then,  $T : \ell_2(I) \rightarrow \mathcal{H}$  is the synthesis operator of a  $J$ -frame such that  $S = TT^+$  if and only if there exists a co-isometry  $U : \ell_2(I) \rightarrow \mathcal{H}$  such that*

$$T = S^{1/2}U.$$



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