Formulation of the General Momentum Equations

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Abstract: Based on the actuator disc representation of a rotor (or propeller), many simplified momentum theories have been developed and are usually coupled to blade element forces to provide a blade element momentum theory (BEM). The general momentum theory of Glauert, unlike most simplified theories, accounts for the sub-atmospheric pressure variation in the wake vortex but has lacked closure in an analytic formulation of the system of equations that would relate the flow field induction factors and wake expansion. The specific problem has been in expressing the balance in axial force (usually loosely described as the momentum equation) in an explicit way that accounts for the influence of a radial pressure gradient. Without approximation or CFD based empirical corrections, the effect of the radial variation of suction pressure in the wake as influencing the axial force balance is accounted. This enables formulation of a complete equation system relating the induction factors and wake expansion.

Notation:

- $A$: streamtube section area (circular)
- $a$: axial induction factor
- $\dot{a}$: tangential induction factor
- $C_p$: power coefficient
- $R$: streamtube radius anywhere on the streamtube that bounds the actuator disc
- $r$: radial coordinate
- $u, U$: axial velocity
- $p$: pressure
- $P$: power (as work rate or energy flow rate)
- $F_a$: axial force
- $Q$: torque
- $M$: momentum
- $\dot{m}$: mass flow rate
- $T$: rotor disc thrust
- $V$: control volume, volume flow rate
- $x$: axial coordinate

Suffixes:
- 0 - at a plane far upstream; 1 - at the actuator disc; 2 - far downstream (see Figure 1)

Integration: Where necessary for clarity, a variable of integration is indicated by a superscript “hat” as in \( \hat{\dot{p}} \) or \( d\hat{p} \).

Averages: A “bar” above a variable as in \( \bar{\dot{p}} \) represents an average quantity (streamtube section area averaged).
1. Introduction

The general momentum theory of Glauert [1] has given rise to many different simplified or approximate models of blade element momentum (BEM) theory such as are reviewed and compared in Sørensen [2]. A recurrent theme is discussion of the validity and limitations of these models especially regarding the influence of a radial pressure gradient on axial force [3], [4].

In the present formulation the general momentum equations are formulated in terms of 5 unknowns on each differential streamtube element within the bounding streamtube. These are the far wake radius, \( r_2 \), the axial and tangential induction factors, \( a_2 \) and \( \dot{a}_2 \) at the rotor plane and the corresponding factors, \( a_2 \) and \( \dot{a}_2 \) in the far wake. These 5 variables are referred to as the “key” variables and all other variables are at some stage to be expressed in terms of these.

Equations reflecting balance in power, mass flow rate and torque corresponding to conservation of energy, mass and angular momentum in a steady state flow are well established and a fourth equation arises from defining rotor loading. The remaining equation relates to axial force balance (a momentum rate balance usually loosely referred to as momentum). An integral form of this equation is well accepted but Goorjian [5] showed that the most obvious differential form obtained by removing the integral signs, is invalid. Although interesting solutions of the general momentum equations were developed for special cases or with simplifying assumptions, Joukowsky, see Sørensen [2], and Glauert [1] were unable to provide closure with an analytically determined axial force (momentum) equation and later workers Wilson [6], Sharpe [7], Sørensen [2]. developed a view that there was insufficient information.

Lack of an analytic formulation of the axial force equation prompted much work on BEM models Madsen et al [8], [9] employing empirical methods of correction typically with parameters estimated using CFD. Sørensen [2] had observed “the lateral force component is largely unknown and can only be determined from additional CFD computations solving the full Euler or Navier-Stokes equations”. The concern in this paper is not whether some empirical CFD models or others may be more accurate in modelling real flows where viscosity plays a role but rather in first establishing an analytic formulation consistent with inviscid flow. The general momentum actuator disc flow problem can be solved (see for example, Bontempo [10]) based on general methods for semi-analytic solution of potential flow as applied to the actuator disc by Conway [11] and others and so it seems hardly logical to lack analytical closure in formulation of the equations.

2. The General Momentum Theory

2.1. Streamtube source power

The expression for source power in the wind, widely disseminated as \( 0.5 \rho U_0^3 A_0 \), is incomplete being kinetic power only. This has often led to serious misunderstandings [12] especially in the context of systems that aim to exploit flow augmentation. The correct expression for source power including both kinetic and pressure power is, \( P_0 = 0.5 \rho U_0^3 A_0 + p_0 A_0 U_0 \). In the far wake of an actuator disc with rotary power extraction, excepting on the boundary, the pressure \( p_2 \) is everywhere \( < p_0 \). Hence some source pressure power as well as source kinetic power is involved in the overall process. The full recovery of atmospheric pressure in the far wake is central to the Betz analysis of the energy extracting actuator disc without wake rotation. While it seems unlikely that the formation of the wake vortex in rotary power extraction can add to total rotor performance, there is no longer any obvious reason that the local power performance coefficient on any given annular section of the actuator disc should be Betz limited.

2.2. Static pressure in the wake

Far downstream the static pressure in the external flow and also on the wake boundary must be \( p_0 \). In order to balance centrifugal forces and hence maintain the wake rotation, the pressure must then reduce within the wake. Considering \( dm \) as a mass element in the form of a differential annular ring of the far wake, \( dm = \rho 2 \pi r_2 dr_2 dx \). The centrifugal force acting on \( dm \) is \( dm(2 \dot{a}_2 \omega^2 r_2) \). The balancing pressure force is \( \{(p_2 + dp_2) - p_2\} 2 \pi dr_2 dx \). Hence \( \frac{dp_2}{dr_2} = 4 \rho \dot{a}_2^2 \omega^2 r_2 \), and relative to atmospheric pressure, \( p_0 \) as datum;
\[ p_2(r_2) = p_0 - 4\rho \omega^2 \int_{r_2}^{R_2} \dot{a}_2 r^2 dr \]  

(1)

2.3. Equation system

The equation system of Table 1 relates to necessary balances in the named quantities. Of the 5 equations listed, only the axial force balance equation is fundamentally new and its derivation is detailed in Sections 2.4. to 2.6. The other equations agree with previous derivations of Glauert [1] as later presented by Sharpe [7] differing only in explicitly introducing Equation (1). A summary discussion of the equation system follows.

**Table 1** Equation system

<table>
<thead>
<tr>
<th>Equations</th>
<th>Description</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>Power</td>
<td>( a_z^2 - 2a_z - \frac{3\lambda^2}{R_1^2} \int_{r_2}^{R_2} \dot{a}_2 r^2 dr ) + ( \frac{4\lambda^2 a_1 r_1^2}{R_1^2} \left( 1 + \frac{\dot{a}_1 r_1^2}{r_2^2} \right) = 0 )</td>
</tr>
<tr>
<td>E2</td>
<td>Mass flow</td>
<td>( r_0 dr_0 = (1 - a_1) r_1 dr_1 = (1 - a_2) r_2 dr_2 )</td>
</tr>
<tr>
<td>E3</td>
<td>Torque</td>
<td>( \dot{a}_1 r_1^2 = \dot{a}_2 r_2^2 = \dot{a} r^2 ) at any section of the same wake streamtube element</td>
</tr>
</tbody>
</table>
| E4        | Axial force | \( \dot{a}_1 (1 + \dot{a}_1) = \frac{a_2 (1 - a_1)}{2\lambda^2} \left( \frac{R_1}{r_1} \right)^2 \left( 1 + Z \right) \)  
where \( Z = \frac{4\lambda^2}{a_2 (1 - a_1) R_1^2} \int_{r_1}^{r_2} \dot{a}_2 r^2 dr \) |
| E5        | Disc loading| \( \text{Maximise: } C_P = \frac{\lambda^2}{R_1^2} \int_{r_1}^{R_1} \dot{a}_1 (1 - a_1) r_1^3 dr_1 \) |

Equations E1 to E4 are fundamental expressing conservation of energy, mass, angular momentum and linear momentum as rates in steady flow. Equation E5 is arbitrary expressing that a rotor configuration and operational state or a constraint (such as to determine values that maximise power performance) that directly or indirectly amounts to a definition of disc loading must be provided. Equation E5 in Table 1 is later derived from Equation (18) as Equation (19). Equation E5 is exactly the same in form as it would be derived from the standard BEM theories that assume atmospheric pressure everywhere in the far wake.

Regarding the wake as a vortex, the wake rotational angular velocity will tend to increase indefinitely towards the axis of symmetry. Glauert [1], Wilson and Lissaman [6] and others have assumed that it is not possible for the wake angular velocity of any stream tube element to exceed the rotor angular velocity. Sharpe [7] comments that no justification for this assumption was ever provided. Angled static vanes (or the rotor rigidly parked) with zero angular velocity can impart swirl (non-zero angular velocity) to the wake flow. Thus the wake rotational velocity in the real world and not only in the idealised world of potential flow can indeed exceed rotor angular velocity. The air, in imparting torque to the rotor, is accelerated by the equal and opposite reaction torque and the wake angular speed grows indefinitely approaching the rotor centre. In real rotors with discrete blades the flow acceleration occurs due to the pressure gradient over the blade aerofoils in the direction of resultant flow.

As rotational power remains in the wake, not all the power immediately upstream of the rotor is available for extraction. At the rotor plane the power in the air divides into power available to rotor and swirl power (kinetic power of wake rotation immediately downstream of the rotor plane). Suction pressure power associated with wake rotation is implicitly accounted in the pressure difference...
developed across the rotor plane. Immediately upstream of the rotor, the flow is not rotating and the total air power on an annulus expressed as product of axial force and axial velocity is;

\[ dP_1 = \Delta p (2\pi r_1 dr_1)u_0(1 - a_1) \]  \hspace{1cm} (2)

The rotor (and wake) torque reaction is;

\[ dQ = \rho(2\pi r_1 dr_1)u_0(1 - a_1) (2\hat{a}_1 \omega r_1) = 4\pi \rho \omega u_0(1 - a_1) \hat{a}_1 r_1^2 dr_1 \]  \hspace{1cm} (3)

Hence, considering the total air power as product of wake torque reaction and rotational speed;

\[ dP_1 = dQ \omega(1 + \hat{a}_1) \]  \hspace{1cm} (4)

The torque on the air is generated at a relative angular speed of \( \omega \) on the upstream side of the rotor plane which according to the conventional definition of \( \hat{a} \) becomes \( \omega(1 + 2\hat{a}_1) \) on the downstream side. The torque reaction may be assumed to grow linearly with change in angular speed and so an average relative speed of \( \omega(1 + \hat{a}_1) \) is appropriate in forming \( dP_1 \). Equations (2) to (4) lead to;

\[ \Delta p = 2\rho \omega^2 \hat{a}_1 (1 + \hat{a}_1) r_1^2 \]  \hspace{1cm} (5)

Equation (5) was also derived by Sharpe [7] following Glauert [1] based on there being no change of radial or axial velocity across the rotor plane and hence \( \Delta p \) must correspond to the pressure head associated with the creation of tangential velocity. The present derivation avoids the need to consider Bernoulli’s equation in a rotating reference frame. Equation (4) is the key to determining \( dP_1 \), the power available to the rotor as distinct from the total air power approaching the rotor plane. The torque reaction on the rotor must be equal to the total torque reaction on the air but relative to the ground reference frame the rotor is rotating at \( \omega \) and hence;

\[ dP_r = dQ \omega \]  \hspace{1cm} (6)

Comparing equations (4) and (6), shows that the total air power incident on the rotor plane divides into rotor available power \( dP_r \) and swirl power (kinetic power of wake rotation) \( dP_s \) in the ratio \( 1: \hat{a}_1 \). Thus from Equation (5), the part of the pressure drop to be associated with power available to the rotor is;

\[ \Delta p_r = 2\rho \omega^2 \hat{a}_1 r_1^2 \]  \hspace{1cm} (7)

Equation E2 (Table 1) simply expresses that the mass flow rate \((\rho2\pi r_0 dr_0 u_0 = \rho2\pi r_1 dr_1 u_1 = \rho2\pi r_2 dr_2 u_2)\) is constant throughout every streamtube.

The power balance equation, E1, is derived combining an analysis as in Glauert [1] or Sharpe [7] with Equation (1). The source power far upstream passing through the actuator disc divides into power available to the rotor and power remaining in the wake which is constant everywhere in the wake but most readily evaluated in the far wake. The power balance is further reduced to a pressure balance (Bernoulli type equation) using Equation E2.

\[ p_0 + 0.5\rho u_0^2 = 2\rho \hat{a}_1 \omega^2 r_1^2 + 0.5\rho u_2^2 + 2\rho \hat{a}_2^2 \omega^2 r_2^2 + p_0 - 4\rho \omega^2 \int_{r_2}^{R_2} \hat{a}_2^2 r dr \]  \hspace{1cm} (8)

Employing basic definitions for tip speed ratio, \( \lambda = \omega R_1 / u_0 \), and for induction factors such as \( u_2 = u_0 (1 - a_2) \) and also applying Equation E3, Equation (8) is put into the form given as Equation E1.

Equation E3 results from Newton’s third law. The torque reaction imparted to the wake is equal and opposite to the rotor torque and is the same everywhere in the wake. This applies to any complete section.
of the wake but also separately to every differential streamtube element comprising the wake. The torque is naturally given as rate of change of angular momentum. The angular momentum rate (torque) anywhere in the same wake differential streamtube element is 

\[ \dot{m} \omega r = (\rho \Delta v \Delta r) (\Delta \omega) r = (\Delta \dot{m} \Delta \omega) \Delta r^2. \]

Since mass flow rate, \( \dot{m} \), is constant along each wake streamtube element and rotor angular speed, \( \omega \), is constant, so is \( \Delta \dot{m} \Delta \omega \Delta r^2 \) resulting in Equation E3.

### 2.4. Axial Force Balance for the Complete Streamtube in Integral Form

In the usual derivation of the axial force balance for the complete disc and bounding streamtube, a streamtube element such as defined by the dotted lines of Figure 1 is considered. A convective force (force due to momentum changes in the fluid) at the disc element BC is evaluated from the change in fluid momentum rate between far up stream and far downstream as

\[ \dot{m}(u_0 - u_2) = \rho 2\pi r_2 dr_2 u_2 (u_0 - u_2) = \rho 2\pi r_2 dr_2 u_0^2 a_2 (1 - a_2) \]

where \( \dot{m} \) is the constant mass flow rate through the streamtube element. A direct differential force exists on the annular element of disc at BC (Figure 1) due to the pressure drop \( \Delta p_r \), as \( \Delta p_r 2\pi r_1 dr_1 \). Atmospheric pressure, \( p_0 \), acts on the upstream streamtube end and on the streamtube boundary (curved surface expanding from radius \( R_0 \) far upstream to \( R_1 \) at the disc and to \( R_2 \) far downstream) while a varying pressure, \( p_2 \), acts on the far downstream end of the wake. This leads to an overall axial force balance as;

\[ \int_0^{R_2} (p_0 - p_2) 2\pi r_2 dr_2 + \int_0^{R_2} \rho u_0^2 (1 - a_2) a_2 2\pi r_2 dr_2 = \int_0^{R_1} \Delta p 2\pi r_1 dr_1 \]

Equation (9) in this integral form appears in Glauert [5] and Sharpe [7] and is long accepted in propeller, helicopter and wind turbine rotor theory. However Goorjian [5] proved in 1971, that the obvious differential form obtained by removing the integral signs is incorrect.

### 2.5. Derivation of the axial force balance

Figure 1 shows two elements of the flow field within the streamtube bounding the actuator disc. Each is of infinite extent axially but of infinitesimal extent radially. Both include the same annular differential element of the actuator disc at radius, \( r_1 \), and of radial width, \( dr_1 \) (BC in Figure 1). The element bounded
by solid lines extends in the purely axial direction over the width $dr_1$. The other element with dotted lines is the streamtube that includes the actuator disc element $dr_1$ at BC.

The axial force on the annular ring at $r_2$ of width $dr_2$ at plane 2 is $p_2(r_2)2\pi r_2 dr_2$. This force cannot act on the actuator disc differential element (BC) at radius $r_1$ as it could only do so by shear action which is impossible in an inviscid fluid. Only a pressure force on the axial differential element which is collinear with it can affect the axial force on the actuator disc element and the relevant force acting at plane 2 is therefore $p_2(r_2)2\pi r_1 dr_1$. By a relatively sophisticated argument, Goorjian [5] showed in effect that the differential form with $p_2(r_2)2\pi r_2 dr_2$ was incorrect. The preceding argument shows not only that it is very obviously incorrect but now enables derivation of a valid differential form of axial force balance. The essential considerations in forming the new equation for axial force balance are that:

- Only convective forces within the streamtube differential element (dotted lines) can have influence on the actuator disc differential element BC as mass flow passing through BC is entirely contained within this streamtube element.
- Only pressure forces acting collinearly with BC along the axial streamtube element can influence the differential axial force on the annular ring BC of width $dr_1$.

The upstream semi-infinite segment of the axial differential element terminating at the streamtube segment AB is shown in Figure 2. Oblique lines in represent streamtube boundaries. Considering a differential streamtube segment such as HJKL of axial width $dx$, and radial width $dr \equiv dr_1$, the axial momentum rate entering along HK is transported across HK by $\nu$, the radial velocity component and is of magnitude $\bar{m}u = (\rho 2\pi r_1 \nu)u = (\rho 2\pi r_1 dr_1 u)u = \rho 2\pi r_1 dr_1 u^2$.

Any change $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial r} dr$ in axial velocity (for example, between the values at HK and JL), as a product with the infinitesimal annular area, $2\pi r_1 dr_1$, will introduce second order infinitesimal terms in the axial momentum rate that may be neglected. In view of this, changes in axial momentum rate over the whole element EFBA may be considered of second order and therefore introduce no significant convective axial force. For the equilibrium of EFBA, this implies that the end axial pressure forces are equal and that the pressure acting on AB is $p_0$. A similar analysis of the corresponding downstream axial segment concludes that the pressure on CD is $p_2$, but specifically $p_2(r_1)$ and not $p_2(r_2)$.

![Figure 2](image-url)  
**Figure 2**  
Upstream axial element terminating on streamtube segment AB

The convective forces induced by the pressure drop across the disc at BC arise from the complete upstream and downstream parts of the streamtube element of Figure 1. They are indicated in Figure 4 along with the pressures deduced from discussion of Figure 2. From this the axial force balance is determined on upstream segment ABC and on downstream segment BCD as:

\begin{align}
ABC & \quad p_0(2\pi r_1 dr_1) + \rho(2\pi r_0 dr_0)u_0^2 - \rho(2\pi r_1 dr_1)u_1^2 - p_1(2\pi r_1 dr_1) = 0 \quad (10) \\
BCD & \quad (p_1 - \Delta p)(2\pi r_1 dr_1) + \rho(2\pi r_1 dr_1)u_1^2 - \rho(2\pi r_2 dr_2)u_2^2 - p_2(r_1)(2\pi r_1 dr_1) = 0 \quad (11)
\end{align}
Adding Equations (10) and (11) leads to the complete axial force equation in differential form as:

\[ [p_0 - \Delta p - p_2(r_1)](2\pi r_1 dr_1) + \rho(2\pi r_0 dr_0)u_0^2 - \rho(2\pi r_2 dr_2)u_2^2 = 0 \]  

(12)

2.6. Induction factor relationships from the axial force equation

Equation E2 (Table 1), \( r_0 dr_0 = (1 - a_1)r_1 dr_1 = (1 - a_2)r_2 dr_2 \) enables elimination of the differentials \( dr_0, dr_1 \) and \( dr_2 \) in Equation (12). From Equation (1), \( p_2(r_1) = p_0 - 4\rho \omega^2 \int_{r_1}^{R_2} \hat{a}_2^2 r^2 dr \) and from Equation (5), \( \Delta p = 2\rho \omega^2 \hat{a}_t (1 + \hat{a}_t) r_t^2 \). Equation (12) can then be expressed in the form;

\[ \hat{a}_t (1 + \hat{a}_t) = \frac{a_2(1-a_1)}{2\lambda^2} \left( \frac{R_1}{r_1} \right)^2 \{ 1 + Z \} \text{ where } Z = \frac{4\lambda^2}{a_2(1-a_1)R_1} \int_{r_1}^{R_2} \hat{a}_2^2 r^2 dr \hat{r}_2. \]  

(13)

In simplified BEM theories that relate torque to tangential induction at the rotor plane but ignore the sub-atmospheric variation of pressure within the wake, \( Z = 0, a_2 = 2a_1 \), and Equation (13) returns to an established relationship [2], [12], [13] between the induction factors.

3. Discussion

3.1. The average pressure in the far wake

The pressure far downstream varies radially as established in Equation (1) and the area average value can be determined as;

\[ \bar{p}_2 = \frac{1}{A_2} \int_{A_2} p_2 dA_2 = p_0 - \frac{2\rho \omega^2}{R_2^2} \int_{0}^{R_2} \hat{a}_2^2 r^3 dr = p_0 - \frac{2\rho u_0^2 \lambda^2}{R_1^2 R_2} \int_{0}^{R_2} \hat{a}_2^2 r^3 dr \]  

(14)

The rotor power coefficient \( C_p \) must remain finite if the average far wake pressure is finite. Considering the wake as a vortex in inviscid flow, there will always be infinite suction at the centre of the wake but that singularity need not lead to \( \bar{p}_2 \) being infinite. If near the axis of symmetry, \( \hat{a}_2 \) varies inversely as \( r_2 \) to a power even fractionally less than 2, then \( \bar{p}_2 \) will clearly be finite. Analysis of Equation (13) in the limits \( \hat{a}_1 \rightarrow \infty, r_1 \rightarrow 0 \) and \( \lambda \rightarrow 0 \) confirms that \( \hat{a}_1 \) or \( \hat{a}_2 \) has limiting behavior such that \( \bar{p}_2 \) and hence \( C_p \) are always finite.

The most general Euler equation in cylindrical polar coordinates for inviscid flow in the radial direction is:

\[ \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_r}{\partial \theta} + u_x \frac{\partial u_r}{\partial x} - \frac{u_\theta^2}{r} = \frac{-1}{\rho} \frac{\partial p}{\partial r} \]  

(15)

There is no time variation in any steady state flow. In the far wake there is no azimuthal variation in tangential velocity, no axial variation in axial velocity and there is zero radial velocity. Equation (15)
then reduces to \( \frac{u_\theta^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r} \) which is effectively Equation (1). This means that Equations (1) and (14) are direct consequences of the Euler equation and fully compatible with inviscid flow. It follows that, \( \alpha_2 \) must vary inversely as \( r_2 \) to a power less than 2 otherwise from Equation (14), the average wake pressure and hence rotor \( C_p \) would always be infinite which is clearly absurd.

3.2. Wake rotation and suction power (14)

Expressing each term in Equation (1) as a fraction of the upstream dynamic pressure, \( 0.5 \rho u_0^2 \), a suction pressure coefficient \( c_s \) may be defined as;

\[
c_s = \frac{p_0 - p_2(r_2)}{0.5 \rho u_0^2} = \frac{8 \lambda^2}{R_2^2} \int_{r_2}^{R_2} \alpha_2^2 r^2 dr
\]

From Equation (5), the pressure component associated only with wake rotation is \( \Delta p_{wr} = 2 \rho \omega^2 \hat{a}^2 r^2 \) and in non-dimensional form as a fraction of \( 0.5 \rho u_0^2 \), \( \Delta p_{wr} \) becomes; \( c_{wr} = \frac{4 \lambda^2 \alpha_2^2 r_2^2}{R_2^4} \). Examination of the difference \( (c_{wr} - c_s) \) shows that in general for any given radius \( r \), it is not zero. This is obvious, for example at the streamtube boundary, \( R \) where \( c_s \) must be zero and \( c_{wr1} \) must be positive. However, defining \( \Psi \) as the integral of \( (c_{wr} - c_s) \) over the far downstream wake surface normal to the axis of symmetry;

\[
\Psi = \frac{2}{R_2^2} \int_0^{R_2} (c_{wr} - c_s) r dr = \frac{8 \lambda^2}{R_2^4} \int_0^{R_2} \alpha_2^2 r^3 dr - \frac{16 \lambda^2}{R_2^4} \int_0^{R_2} \left( \int_{r_2}^{R_2} \alpha_2^2 r^2 dr \right) dr
\]

Simplification of Equation (17) leads readily to \( \Psi = 0 \). This demonstrates a necessary balance in total radial force associated with centrifugal and suction effects in the far wake.

3.3. Rotor power coefficient

In the literature concerning BEM models, there have been many comments regarding a few disputing wake rotation as a “loss” [7], [14]. It certainly is a loss in the sense that some of the total air power immediately upstream of the rotor plane is always unavailable to the rotor and remains in the air as rotational kinetic power. This is inevitable in any kind of rotary energy extraction involving finite torque reaction. However this is not a complete view of the process. The suction pressure associated with wake rotation augments the rotor plane pressure drop and may induce extra mass flow through some regions of the rotor modifying the flow field and consequently affecting the air power available immediately upstream of the rotor. The power contribution \( dP \) at radial station, \( r_1 \) in the rotor plane is given in Equation 5. Using also Equation (7), the associated local power coefficient is;

\[
C_p(r_1) = \frac{dP}{0.5 \rho U_0^2 (2 \pi r_1 \hat{r}_1)} = \frac{1 - \alpha_1}{0.5 \rho U_0^2} \Delta p_r = \frac{4 \lambda^2 \alpha_1}{R_1^4} \left( \frac{r_1}{R_1} \right)^2 (1 - \alpha_1)
\]

and the whole rotor power coefficient is;

\[
C_p = \frac{2}{R_1^2} \int_0^{R_1} C_p(r_1) \hat{r}_1 d\hat{r}_1 = \frac{8 \lambda^2}{R_1^4} \int_0^{R_1} \alpha_1 (1 - \alpha_1) \hat{r}_1^3 d\hat{r}_1
\]

Using Equation E1 of Table 1, \( C_p \) may be developed in an alternative form;

\[
c_p = \frac{2}{R_1^2} \int_0^{R_1} a_2 (2 - a_2) (1 - a_1) r_1 dr_1 + \frac{8 \lambda^2}{R_1^4} \int_0^{R_1} \alpha_2 \hat{a}_2^2 r_2^2 d r_2 - \frac{16 \lambda^2}{R_1^4} \int_0^{R_2} \alpha_2 \left( \int_{r_2}^{R_2} \hat{a}_2^2 \hat{r}_2 d \hat{r}_2 \right) r_2 d r_2
\]
In the simplest actuator disc model, where energy extraction occurs without any rotary process involving torque reaction, \( a_2 = 0 \) and \( a_2 = 2a_1 = 2a \). Equation (20) then yields the usual Betz formula \( C_p = 4a(1 - a)^2 \) which is a rigorous result if an area averaged value of \( a \) is understood.

If \( a_2 \) is constant, independent of radius, \( r_2 \), the terms in Equation (20) involving \( \lambda \) will cancel similarly as in Equation (17). Assuming constant circulation and a cylindrical vortex wake with a line vortex on the axis, \( u_2 = \left( u_0^2 - \frac{\omega r^2}{\pi} - \frac{r^2}{4\pi^2R_2^2} \right)^{0.5} \) is constant [15] and so \( a_2 \) is constant. However the circulation, \( \Gamma = 2\pi a_2 \omega r_2^2 \), being constant implies \( a_2 \propto \frac{1}{r^2} \) in conflict with the limiting behavior of Equation (13) as \( r \to 0 \) and also causing the problematic singularity of the Joukowski model [16]. Consequently, while the essential balance in radial force remains, \( u_2 \) is not constant everywhere in the wake and there is not an exact balance between suction power and rotational kinetic power in the far wake implying a velocity field where the vortex core flow is extruded along the axis.

Constant circulation is evidently not compatible with accurate solution of the general momentum equations and, considering the discussion in Section 3.1. It has been determined that, \( \dot{a}_2 \) must vary inversely as \( r_2 \) to a power \((2 - \varepsilon)\) where \( \varepsilon > 0 \) with the effect that the circulation, \( \Gamma \to 0 \) as \( r \to 0 \). That any positive value of \( \varepsilon \), however small, will make the average wake pressure, \( \bar{p}_2 \) finite accords with analyses of Sorensen and van Kuik [17] where any friction or lateral force however small resolves the problem of an infinite power coefficient in the limit of low tip speed ratio.

### 3.4. The differential axial force

In Equation (12), the differential form proven by Goorjian as incorrect would replace \( \{p_0 \ 2\pi r_3 \ dr_1 - p_2(r_3) \ 2\pi r_3 \ dr_1 \} \) with \( \{p_0 \ 2\pi r_2 \ dr_3 = p_2(r_2) \ 2\pi r_2 \ dr_3 \} \). This would have the effect of changing the lower limit of integral in the definition of \( Z \) as in Equation (13) from \( r_1 \) to \( r_2 \) thereby underestimating the wake suction term. Underestimating wake suction will underestimate axial force as is consistent with much previous work [3], [8], [9] concerning corrections to axial force estimates.

### 4. Conclusions

The explicit formulation of the general momentum equations enables a number of long standing issues to be resolved analytically and without exclusion of the vortex core singularity.

1. An equation for the axial force balance in differential form is provided resolving the problem first identified by Goorjian [5].
2. The average wake pressure and whole rotor \( C_p \) is always finite including the case \( \lambda \to 0 \) resolving issues discussed at length by Sharpe [7], van Kuik [17] and others.
3. The total power in the far wake comprising axial kinetic power, rotational kinetic power and pressure power is necessarily constant. However there is not an exact separate balance in far wake pressure power and rotational kinetic power as suggested in some comments/analyses by Glaubert [1], Xiros and Xiros [4], de Vries [14], Sharpe [7] and van Kuik [16]. This is because the circulation cannot be constant near the vortex core and consequently \( u_2 \) varies as \( r_2 \to 0 \). A circulation that is radially constant is impossible although it may have been a useful approximation in practical models where the vortex core is excluded.

Differences from momentum theories which ignore far wake suction and rotational power are significant only in the region near the vortex core accounting for the practical success of many simplified theories. In reality, even without blockage of the rotor center, as in all real wind turbine designs, viscosity will limit tangential velocity in the core, will cause breakdown of the vortex core downstream and will in principle propagate flow rotation upstream.

It is already established from numerical analyses and otherwise that the general momentum theory differs little from simplified theories at the usual design tip speed ratios of large modern wind turbines. It has more significant implications for design at low tip speed ratios. As the local tip speed ratio is always low at some radial station of a rotor, the general theory may also be significant for design of the
inboard rotor. It is cautioned however that more dominant effects can easily arise from outward displacement of flow from the hub region assuming blockage of the rotor center with a hub cover [12].

References: