

# PATH LAPLACIAN OPERATORS AND SUPERDIFFUSIVE PROCESSES ON GRAPHS. II. TWO-DIMENSIONAL LATTICE

ERNESTO ESTRADA, EHSAN HAMEED, MATTHIAS LANGER,  
AND ALEKSANDRA PUCHALSKA

ABSTRACT. In this paper we consider a generalized diffusion equation on a square lattice corresponding to Mellin transforms of the  $k$ -path Laplacian. In particular, we prove that superdiffusion occurs when the parameter  $s$  in the Mellin transform is in the interval  $(2, 4)$  and that normal diffusion prevails when  $s > 4$ .

## 1. INTRODUCTION

Many physical systems are best represented by graphs  $G = (V, E)$ , where the set of nodes (vertices)  $V$  represents the entities of the system and the set of edges  $E$  describes the interactions between these entities [13]. Among those systems we can mention atomic and molecular ones as well as complex networks, which include a vast range of complex systems embracing biological, social, ecological, infrastructural and technological ones. Diffusion-like processes, such as diffusion, reaction-diffusion, synchronization, epidemic spreading, etc., are ubiquitous in those previously mentioned systems [6]. Apart from the normal diffusive processes, where the mean square displacement (MSD) of the diffusive particle scales linearly with time, there are many real-world examples where anomalous diffusion takes place. In these anomalous diffusive processes, MSD scales nonlinearly with time giving rise to subdiffusive and superdiffusive processes [29].

In Part I [15] of this series we introduced a new theoretical framework to study superdiffusive processes on graphs. In that work we considered transformations of the so-called  $k$ -path Laplace operators  $L_k$ . The latter are defined in a similar way as the standard graph Laplacian, but they take only nodes into account whose distance is equal to  $k$ ; here the distance is measured as the length of the shortest path connecting two nodes. Hence  $L_k$  describes hops to nodes at distance  $k$ . The above mentioned transformations of  $L_k$  are combinations of the form  $\sum_{k=1}^{\infty} c_k L_k$  with some non-negative coefficients  $c_k$ . This combination describes interactions with all nodes where different strengths are used for nodes at different distances. In general, one uses a sequence  $c_k$  that is decreasing in  $k$ . In particular, in [15] we considered the Mellin transform of  $L_k$ , which is obtained by choosing  $c_k = k^{-s}$  with some positive parameter  $s$ . The choice of the transformation has proved to be crucial in determining the diffusive behaviour. In [15] we studied, in particular, the one-dimensional path graph. We proved that superdiffusion appears when a Mellin transform of the  $k$ -path Laplace operators is considered with  $s$  satisfying  $1 < s < 3$ , while for  $s > 3$  normal diffusion is obtained; the latter occurs also if one considers different transformations of  $L_k$  like the Laplace and factorial transforms.

This new method adds new values to the already existing ones for modelling anomalous diffusion. Among such existing methods we should mention the use of random walks with Lévy flights (RWLF) [12, 34, 38, 30] and the use of the fractional

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diffusion equation (FDE) [27, 36, 23, 16]. While the first method is easy to use for computer simulations, the second is preferred for analytical studies. However, there are different types of definitions of fractional derivatives, such as the Caputo fractional operator and the Riemann–Liouville fractional operator [32], which then have different interpretations and adapt differently to the different physical phenomena studied with them (see [20, 26]). The  $k$ -path Laplace operators allow the derivation of analytical results as the FDE but use a unique framework which is very similar to the one traditionally used in graph and network theory. It also allows an easy computational implementation in the form of a random multi-hopper on graphs [14].

The goal of the current work is to study the solutions of the generalized diffusion equation in 2D graphs. In particular, we focus our attention on the abstract Cauchy problem in an infinite square lattice. Square lattices are ubiquitous in many real-world physical systems. It is frequently used to describe the spin-1/2 antiferromagnetic Heisenberg model in a variety of materials [28, 19, 3, 11]. It is also the preferred model for two-dimensional (2D) gases and optical lattices [5, 21, 18, 1, 25]. Recently, square lattices of superconducting qubits have been used for error correcting codes in quantum computers [9]. A very interesting discovery has been the experimental finding that the native architecture of certain photosynthetic membranes have square lattice shapes [35, 4, 10]. This finding is very relevant for the current work as the existence of long-range interactions (LRI) is well documented for light-harvesting complexes [17, 7, 8]. The existence of LRI like the ones mathematically described by the  $k$ -path Laplace operators considered here are well documented for other systems previously mentioned here, such as cold atomic clouds, helium Rydberg atoms and cold Rydberg gases [2, 22, 37]. Note also that anomalous diffusion has been observed for ultracold atoms in 2D and 3D lattices [33]. Consequently, the study of a generalized diffusion model on square lattices and proving the conditions for which superdiffusive behaviour exists on them is of great theoretical importance due to the many physical processes involved.

The main result of the current paper is contained in Theorem 4.7, which describes the asymptotic behaviour of the generalized diffusion equation corresponding to the Mellin-transformed  $k$ -path Laplacian. We prove that superdiffusion occurs when  $2 < s < 4$  and that normal diffusion prevails when  $s > 4$ . More precisely, we consider the time evolution of the solution of the generalized diffusion equation with initial condition concentrated at one point. As time  $t$  tends to infinity, the spread of the solution (e.g. measured by the full width at half maximum) grows like  $t^\kappa$  with  $\kappa = \frac{1}{2}$  when  $s > 4$ , which is normal diffusion, and with  $\kappa > \frac{1}{2}$  when  $2 < s < 4$ , which is a superdiffusive behaviour.

Let us give a brief outline of the contents of the paper: in Section 2 we recall results from Part I [15]. In Section 3 we study the solution of the generalized diffusion equation and give an integral representation (Theorem 3.3). Finally, in Section 4 we investigate the asymptotic behaviour of the solution as time tends to infinity. In particular, we formulate and prove our main result (Theorem 4.7). Finally, we examine the behaviour of finite truncations,  $\sum_{k=1}^N k^{-s} L_k$  of the Mellin transforms. Although normal diffusion occurs in this case, the diffusion speed can be made arbitrarily large if  $s \in (2, 4)$  and  $N$  is large enough; see Remark 4.9.

## 2. PRELIMINARIES

At the beginning let us briefly recall some of the results given in the first part [15] of this article. Let  $\Gamma = (V, E)$  be an undirected, locally finite graph with set of vertices  $V$  and set of edges  $E$ . Moreover, let  $d$  be the distance metric on  $V$ , i.e. let  $d(v, w)$  be the length of the shortest path from  $v$  to  $w$ , and let  $\delta_k(v)$  be the  $k$ -path

degree of a vertex  $v \in V$ :

$$\delta_k(v) = \#\{w \in V : d(v, w) = k\}. \quad (2.1)$$

Let  $\ell^2(V)$  be the Hilbert space of square-summable functions on  $V$  with inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v) \overline{g(v)}, \quad f, g \in \ell^2(V).$$

For  $k \in \mathbb{N}$  we consider the  $k$ -path Laplacian, which is an operator in  $\ell^2(V)$  and defined by

$$(L_k f)(v) := \sum_{w \in V: d(v, w) = k} (f(v) - f(w)), \quad f \in \text{dom}(L_k), \quad (2.2)$$

with maximal domain  $\text{dom}(L_k)$ , i.e.

$$\text{dom}(L_k) = \left\{ f \in \ell^2(V) : \sum_{v \in V} \left| \sum_{w \in V: d(v, w) = k} (f(v) - f(w)) \right|^2 < \infty \right\}.$$

The following properties were proved in the first part of the paper.

**Theorem 2.1.** [15, Theorem 2.2] *For each  $k \in \mathbb{N}$  the  $k$ -path Laplacian  $L_k$  is a self-adjointed operator in  $\ell^2(V)$ . Furthermore, the operator  $L_k$  is bounded if and only if the function  $\delta_k : V \rightarrow \mathbb{N}$  is bounded.*

Now let us consider an Abstract Cauchy Problem of the form

$$u'(t) = -Lu(t), \quad u(0) = \hat{u}, \quad (2.3)$$

where  $L$  is some operator in  $\ell^2(V)$ . Similarly to the classical description of Brownian motion, the solution to the system (2.3) with  $L = L_k$ , when rescaled properly, converges to the normal distribution as time tends to infinity. In order to build the model in which interaction among all vertices in a graph that are joined by a path are taken into account, we use the differential equation (2.3) with an operator  $L$  given by a transformed  $k$ -path Laplacian operator:

$$L = \sum_{k=1}^{\infty} c_k L_k \quad (2.4)$$

with some coefficients  $c_k \in \mathbb{C}$ .

The main goal is to examine the existence of superdiffusion in the process described by (2.3) with an operator  $L$  as in (2.4). In [15] we considered three transforms: the Laplace, the factorial and the Mellin transforms, which differ in the rate of convergence to zero of their coefficients. It appeared that for the first two the probabilities of big jumps are too small for superdiffusion to arise and a significant result happens only for the Mellin transform. In the current paper we therefore concentrate on the Mellin transform. Let us recall the definition and some properties of the latter in the following theorem from [15].

**Theorem 2.2.** [15, Theorem 3.1] *Let us consider an infinite graph  $\Gamma$  which is locally finite and such that its  $k$ -path degree  $\delta_k$ , defined in (2.1), satisfies the condition*

$$\delta_{k, \max} := \max\{\delta_k(v) : v \in V\} \leq Ck^\alpha \quad (2.5)$$

for some  $\alpha \geq 0$  and  $C > 0$ . Then the Mellin-transformed  $k$ -path Laplacian

$$L_{M, s} := \sum_{k=1}^{\infty} \frac{1}{k^s} L_k \quad (2.6)$$

is a well-defined, bounded operator in  $\ell^2(V)$  for  $s \in \mathbb{C}$  with  $\text{Re } s > \alpha + 1$ , and the series in (2.6) converges in the operator norm.

One can easily find examples of graphs for which (2.5) is satisfied and hence the operator  $L_{M,s}$  is bounded, e.g. a path graph or a square lattice where  $\delta_{k,\max}$  equals 2 and  $4k$ , respectively. On the other hand, condition (2.5) is violated for the Cayley trees with degree of the non-pendant node equal to  $r \in \mathbb{N}$ ,  $r \geq 3$ , for which  $\delta_{k,\max} = r(r-1)^{k-1}$ .

### 3. EXISTENCE AND TIME EVOLUTION OF THE MELLIN TRANSFORM OF THE $k$ -PATH LAPLACIAN ON THE SQUARE LATTICE

Let us consider the square lattice, i.e. the graph  $\Gamma = P_\infty \times P_\infty = (V, E)$  with vertices  $V = \mathbb{Z}^2$  and edges connecting vertices  $(i, j)$  and  $(m, n)$  when  $|i - m| + |j - n| = 1$ . We usually write  $(u_{x,y})_{x,y \in \mathbb{Z}}$  for functions on  $V$ .

On  $P_\infty \times P_\infty$  the  $k$ -path Laplacian  $L_k$ , defined in (2.2), is given by

$$(L_k u)_{x,y} = 4k u_{x,y} - \sum_{j=0}^{k-1} \left[ u_{x+k-j,y+j} + u_{x-k+j,y-j} + u_{x-j,y+k-j} + u_{x+j,y-k+j} \right],$$

$$x, y \in \mathbb{Z}, u \in \ell^2(V).$$

Clearly,  $L_k$  is a bounded operator. For  $m, n \in \mathbb{Z}$  let  $\sigma_{m,n} : \ell^2(V) \rightarrow \ell^2(V)$  be the shift operator defined by

$$(\sigma_{m,n} u)_{x,y} = u_{x+m,y+n}, \quad x, y \in \mathbb{Z}.$$

Then  $L_k$  can be written as

$$L_k = 4kI - \sum_{j=0}^{k-1} [\sigma_{k-j,j} + \sigma_{-k+j,-j} + \sigma_{-j,k-j} + \sigma_{j,-k+j}]. \quad (3.1)$$

Let us consider the following Fourier transform, which is a unitary operator and which is defined by

$$\mathcal{F} : \ell^2(V) \rightarrow L^2([-\pi, \pi]^2),$$

$$(\mathcal{F}u)(p, q) = \frac{1}{2\pi} \sum_{x,y \in \mathbb{Z}} u_{x,y} e^{ipx} e^{iqy}, \quad p, q \in [-\pi, \pi], u \in \ell^2(V),$$

and whose inverse given by

$$(\mathcal{F}^{-1}f)_{x,y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(p, q) e^{-ipx} e^{-iqy} dp dq, \quad x, y \in \mathbb{Z}, f \in L^2([-\pi, \pi]^2).$$

Since

$$(\mathcal{F}\sigma_{m,n}u)(p, q) = \frac{1}{2\pi} \sum_{x,y \in \mathbb{Z}} u_{x+m,y+n} e^{ipx} e^{iqy} = \frac{1}{2\pi} \sum_{x,y \in \mathbb{Z}} u_{x,y} e^{ip(x-m)} e^{iq(y-n)}$$

$$= e^{-ipm} e^{-iqn} (\mathcal{F}u)(p, q),$$

we have

$$(\mathcal{F}\sigma_{m,n}\mathcal{F}^{-1}f)(p, q) = e^{-i(pm+qn)} f(p, q), \quad p, q \in [-\pi, \pi], f \in L^2([-\pi, \pi]^2). \quad (3.2)$$

Together with (3.1) we obtain that  $L_k$  is unitarily equivalent to a multiplication operator; more precisely, the following lemma is true.

**Lemma 3.1.** *With the notations from above we have*

$$(\mathcal{F}L_k\mathcal{F}^{-1}f)(p, q) = l_k(p, q) f(p, q), \quad p, q \in [-\pi, \pi], f \in L^2([-\pi, \pi]^2), \quad (3.3)$$

where

$$l_k(p, q) = \begin{cases} 4k - i \frac{\sin p \cdot (e^{ikp} - e^{-ikp}) - \sin q \cdot (e^{ikq} - e^{-ikq})}{\cos p - \cos q}, & |p| \neq |q|, \\ 4k + i \cot p \cdot (e^{ikp} - e^{-ikp}) - k(e^{ikp} + e^{-ikp}), & |p| = |q| \neq 0, \pi, \\ 0, & p = q = 0, \\ 4k(1 - (-1)^k), & |p| = |q| = \pi. \end{cases}$$

Moreover,  $l_k$  is continuous and even in both  $p$  and  $q$ , and the following inequalities hold:

$$0 \leq l_k(p, q) \leq 8k, \quad p, q \in [-\pi, \pi], \quad (3.4)$$

$$l_1(p, q) > 0, \quad (p, q) \in [-\pi, \pi]^2 \setminus \{(0, 0)\}. \quad (3.5)$$

*Proof.* It follows from (3.1) and (3.2) that (3.3) holds with

$$\begin{aligned} l_k(p, q) &= 4k - \sum_{j=0}^{k-1} \left[ e^{-i[(k-j)p+jq]} - e^{-i[(-k+j)p-jq]} \right. \\ &\quad \left. - e^{-i[-jp+(k-j)q]} - e^{-i[jp+(-k+j)q]} \right] \\ &= 4k - e^{-ikp} \sum_{j=0}^{k-1} e^{ij(p-q)} + e^{ikp} \sum_{j=0}^{k-1} e^{-ij(p-q)} \\ &\quad + e^{-ikq} \sum_{j=0}^{k-1} e^{ij(p+q)} + e^{ikq} \sum_{j=0}^{k-1} e^{-ij(p+q)}. \end{aligned} \quad (3.6)$$

When  $|p| \neq |q|$  we can rewrite this as follows:

$$\begin{aligned} l_k(p, q) &= 4k - \frac{e^{-ikp} - e^{-ikq}}{1 - e^{i(p-q)}} - \frac{e^{ikp} - e^{ikq}}{1 - e^{-i(p-q)}} - \frac{e^{-ikq} - e^{ikp}}{1 - e^{i(p+q)}} - \frac{e^{ikq} - e^{-ikp}}{1 - e^{-i(p+q)}} \\ &= 4k - e^{ikp} \left( \frac{1}{1 - e^{-ip+iq}} - \frac{1}{1 - e^{ip+iq}} \right) - e^{-ikp} \left( \frac{1}{1 - e^{ip-iq}} - \frac{1}{1 - e^{-ip-iq}} \right) \\ &\quad + e^{ikq} \left( \frac{1}{1 - e^{-ip+iq}} - \frac{1}{1 - e^{-ip-iq}} \right) + e^{-ikq} \left( \frac{1}{1 - e^{ip-iq}} - \frac{1}{1 - e^{ip+iq}} \right). \end{aligned}$$

The expressions within the brackets can be simplified, e.g.

$$\begin{aligned} \frac{1}{1 - e^{-ip+iq}} - \frac{1}{1 - e^{ip+iq}} &= \frac{e^{-ip+iq} - e^{ip+iq}}{1 - e^{-ip+iq} - e^{ip+iq} + e^{2iq}} \\ &= \frac{e^{-ip} - e^{ip}}{e^{-iq} - e^{-ip} - e^{ip} + e^{iq}} = \frac{i \sin p}{\cos p - \cos q}. \end{aligned}$$

Hence

$$\begin{aligned} l_k(p, q) &= 4k - e^{ikp} \frac{i \sin p}{\cos p - \cos q} + e^{-ikp} \frac{i \sin p}{\cos p - \cos q} \\ &\quad + e^{ikq} \frac{i \sin q}{\cos p - \cos q} - e^{-ikq} \frac{i \sin q}{\cos p - \cos q} \\ &= 4k - \frac{i}{\cos p - \cos q} \left[ \sin p \cdot (e^{ikp} - e^{-ikp}) - \sin q \cdot (e^{ikq} - e^{-ikq}) \right]. \end{aligned}$$

For the case when  $|p| = |q|$  note that  $l_k$  is continuous by (3.6). Write  $l_k$  as

$$l_k(p, q) = 4k - i \frac{f(p) - f(q)}{g(p) - g(q)}$$

with  $f(p) = \sin p \cdot (e^{ikp} - e^{-ikp})$  and  $g(p) = \cos p$ . The Generalized Mean Value Theorem implies that

$$l_k(p, q) = 4k - i \frac{f'(\xi)}{g'(\xi)}$$

with  $\xi$  between  $p$  and  $q$ . Hence

$$\begin{aligned} l_k(p, p) &= \lim_{q \rightarrow p} l_k(p, q) = 4k - i \frac{f'(p)}{g'(p)} \\ &= 4k - i \frac{\cos p \cdot (e^{ikp} - e^{-ikp}) + ik \sin p \cdot (e^{ikp} + e^{-ikp})}{-\sin p} \\ &= 4k + i \cot p \cdot (e^{ikp} - e^{-ikp}) - k(e^{ikp} + e^{-ikp}). \end{aligned} \quad (3.7)$$

The relation  $l_k(0, 0) = 0$  follows from (3.6), and the value for  $l_k(p, q)$  when  $|p| = |q| = \pi$  follows from (3.7) by taking the limit  $p \rightarrow \pi$ .

That  $l_k$  is even in  $p$  and  $q$  is clear. Since  $L_k$  is a non-negative operator in  $\ell^2(V)$  by [15, Section 2], the function  $l_k$  is non-negative. The upper bound for  $l_k$  in (3.4) follows from (3.6).

Finally, to show (3.5) rewrite  $l_1$ ; for  $|p| \neq |q|$  we have

$$l_1(p, q) = 4 + 2 \frac{\sin^2 p - \sin^2 q}{\cos p - \cos q} = 4 - 2(\cos p + \cos q), \quad (3.8)$$

which extends to all  $p, q \in [-\pi, \pi]$  by continuity. The right-hand side of (3.8) is strictly positive unless  $p = q = 0$ .  $\square$

Let us now consider the Mellin transformation of the  $k$ -path Laplacians  $L_k$ , i.e. the operator

$$L_{M,s} = \sum_{k=1}^{\infty} \frac{1}{k^s} L_k;$$

see (2.6). Since  $\|L_k\| \leq 8k$  by Lemma 3.1, the series converges in the operator norm when  $s > 2$ . As the next lemma shows, the operator  $L_{M,s}$  is also unitarily equivalent to a multiplication operator in  $L^2([-\pi, \pi]^2)$ . In order to formulate this lemma, we have to recall the definition of the polylogarithm. For  $s \in \mathbb{C}$  the function  $\text{Li}_s$  is defined by

$$\text{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| < 1,$$

and by analytic continuation to  $\mathbb{C} \setminus [1, \infty)$  with 1 being a branch point; see, e.g. [31, 25.12.10].

**Lemma 3.2.** *For  $s > 2$  we have*

$$(\mathcal{F}L_{M,s}\mathcal{F}^{-1}f)(p, q) = l_{M,s}(p, q)f(p, q), \quad p, q \in [-\pi, \pi], f \in L^2([-\pi, \pi]^2), \quad (3.9)$$

where

$$l_{M,s}(p, q) := \sum_{k=1}^{\infty} \frac{1}{k^s} l_k(p, q) \quad (3.10)$$

$$= \begin{cases} 4\zeta(s-1) + \frac{g_s(p) - g_s(q)}{\cos p - \cos q}, & |p| \neq |q|, \\ 4\zeta(s-1) - 2 \cot p \cdot \operatorname{Im}(\operatorname{Li}_s(e^{ip})) - 2 \operatorname{Re}(\operatorname{Li}_{s-1}(e^{ip})), & |p| = |q| \neq 0, \pi, \\ 0, & p = q = 0, \\ 4(1 - (-1)^k) \zeta(s-1), & |p| = |q| = \pi, \end{cases}$$

with

$$g_s(p) := 2 \sin p \cdot \operatorname{Im}(\operatorname{Li}_s(e^{ip})). \quad (3.11)$$

The function  $l_{M,s}$  is continuous and even in both  $p$  and  $q$ , and the following inequalities hold:

$$0 \leq l_{M,s}(p, q) \leq 8\zeta(s-1), \quad p, q \in [-\pi, \pi], \quad (3.12)$$

$$l_{M,s}(p, q) > 0, \quad (p, q) \in [-\pi, \pi]^2 \setminus \{(0, 0)\}. \quad (3.13)$$

*Proof.* It follows from Lemma 3.1 that (3.9) holds with  $l_{M,s}$  defined as in (3.10). When  $|p| \neq |q|$ , we have

$$\begin{aligned} l_{M,s}(p, q) &= \sum_{k=1}^{\infty} \frac{1}{k^s} \left[ 4k - i \frac{\sin p \cdot (e^{ikp} - e^{-ikp}) - \sin q \cdot (e^{ikq} - e^{-ikq})}{\cos p - \cos q} \right] \\ &= 4 \sum_{k=1}^{\infty} \frac{1}{k^{s-1}} - \frac{i}{\cos p - \cos q} \left[ \sin p \cdot \sum_{k=1}^{\infty} \frac{1}{k^s} \left( (e^{ip})^k - (e^{-ip})^k \right) \right. \\ &\quad \left. - \sin q \cdot \sum_{k=1}^{\infty} \frac{1}{k^s} \left( (e^{iq})^k - (e^{-iq})^k \right) \right] \\ &= 4\zeta(s-1) - \frac{i}{\cos p - \cos q} \left[ \sin p \cdot \left( \operatorname{Li}_s(e^{ip}) - \operatorname{Li}_s(e^{-ip}) \right) \right. \\ &\quad \left. - \sin q \cdot \left( \operatorname{Li}_s(e^{iq}) - \operatorname{Li}_s(e^{-iq}) \right) \right], \end{aligned}$$

which proves the formula for  $l_{M,s}$  in the first case. Now assume that  $|p| = |q| \neq 0, \pi$ . Then

$$\begin{aligned} l_{M,s}(p, q) &= \sum_{k=1}^{\infty} \frac{1}{k^s} \left[ 4k + i \cot p \cdot (e^{ikp} - e^{-ikp}) - k(e^{ikp} + e^{-ikp}) \right] \\ &= 4 \sum_{k=1}^{\infty} \frac{1}{k^{s-1}} + i \cot p \cdot \sum_{k=1}^{\infty} \frac{1}{k^s} \left( (e^{ip})^k - (e^{-ip})^k \right) - \sum_{k=1}^{\infty} \frac{1}{k^{s-1}} \left( (e^{ip})^k + (e^{-ip})^k \right) \\ &= 4\zeta(s-1) + i \cot p \cdot \left( \operatorname{Li}_s(e^{ip}) - \operatorname{Li}_s(e^{-ip}) \right) - \operatorname{Li}_{s-1}(e^{ip}) - \operatorname{Li}_{s-1}(e^{-ip}). \end{aligned}$$

The remaining cases are clear.

The continuity of  $l_{M,s}$  follows from the continuity of  $l_k$  and the fact that the series in (3.10) converges uniformly. The symmetry of  $l_{M,s}$  and the inequalities in (3.12) follows directly from the symmetry of  $l_k$  and (3.4). The inequality in (3.13) follows from (3.5) and the first inequality in (3.4).  $\square$

Since  $L_{M,s}$  is a bounded operator, the Cauchy problem

$$u'(t) = -L_{M,s}u(t), \quad t > 0, \quad (3.14)$$

$$u(0) = \dot{u} \quad (3.15)$$

has a unique solution, which is given by

$$u(t) = e^{-tL_{M,s}}\dot{u}, \quad t \geq 0.$$

It follows from Lemma 3.2 that

$$\begin{aligned} (\mathcal{F}e^{-tL_{M,s}}\mathcal{F}^{-1}f)(p, q) &= e^{-tl_{M,s}(p,q)}f(p, q), \\ t \geq 0, \quad p, q &\in [-\pi, \pi], \quad f \in L^2([-\pi, \pi]^2). \end{aligned} \quad (3.16)$$

Using this relation and the fact that  $l_{M,s}$  is even one can easily show the following theorem; cf. [15, Theorem 5.2] for the case of the infinite path graph. For the formulation of the theorem let  $\mathbf{e}_{m,n} \in \ell^2(V)$  be the vector defined by

$$(\mathbf{e}_{m,n})_{x,y} = \begin{cases} 1, & m = x, n = y, \\ 0, & \text{otherwise.} \end{cases} \quad (3.17)$$

**Theorem 3.3.** *Let  $s > 2$  and  $\dot{u} \in \ell^2(V)$ . The unique solution of (3.14), (3.15) is given by*

$$u_{x,y}(t) = \frac{1}{4\pi^2} \sum_{m,n \in \mathbb{Z}} \dot{u}_{m,n} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i[(x-m)p+(y-n)q]} e^{-tl_{M,s}(p,q)} dp dq, \quad x, y \in \mathbb{Z}.$$

In particular, for  $\dot{u} = \mathbf{e}_{0,0}$  we obtain

$$u_{x,y}(t) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(xp+yq)} e^{-tl_{M,s}(p,q)} dp dq, \quad x, y \in \mathbb{Z}. \quad (3.18)$$

#### 4. DIFFUSION AND SUPERDIFFUSION FOR MELLIN-TRANSFORMED $k$ -PATH LAPLACIAN ON A SQUARE LATTICE

In this section we examine the long-time behaviour of the solution to the Cauchy problem generated by the Mellin-transformed  $k$ -path Laplacian. The main result is contained in Theorem 4.7. To prove this theorem we first examine the asymptotic behaviour of the function  $l_{M,s}$  (see Figure 4.1) as the arguments tend to zero, which is contained in Proposition 4.5. The discussion is based on similar considerations undertaken for the path graph in [15], but the arguments are more subtle. We start with a simple lemma, which is used a couple of times below.

**Lemma 4.1.** *Let  $b > 0$ , let  $f : (0, b) \rightarrow \mathbb{R}$  be differentiable and assume that*

$$|f'(t)| \leq Ct^\alpha, \quad t \in (0, b),$$

for some  $C > 0$  and  $\alpha \geq 1$ . Then

$$\left| \frac{f(p) - f(q)}{p^2 - q^2} \right| \leq \frac{C}{2} \max\{p^{\alpha-1}, q^{\alpha-1}\}, \quad p, q \in (0, b), \quad p \neq q.$$

*Proof.* Define the function  $g(x) := f(\sqrt{x})$ ,  $x \in (0, b^2)$ . Let  $p, q \in (0, b)$  such that  $p \neq q$  and set  $x := p^2$ ,  $y := q^2$ . Then

$$\left| \frac{f(p) - f(q)}{p^2 - q^2} \right| = \left| \frac{g(x) - g(y)}{x - y} \right| = |g'(\xi)| \quad (4.1)$$

for some  $\xi$  between  $x$  and  $y$  by the Mean Value Theorem. Since  $\sqrt{\xi} \leq \max\{p, q\}$ , we obtain that

$$|g'(\xi)| = \left| \frac{f'(\sqrt{\xi})}{2\sqrt{\xi}} \right| \leq \frac{C(\sqrt{\xi})^\alpha}{2\sqrt{\xi}} \leq \frac{C}{2} \max\{p^{\alpha-1}, q^{\alpha-1}\},$$

which, together with (4.1), finishes the proof.  $\square$

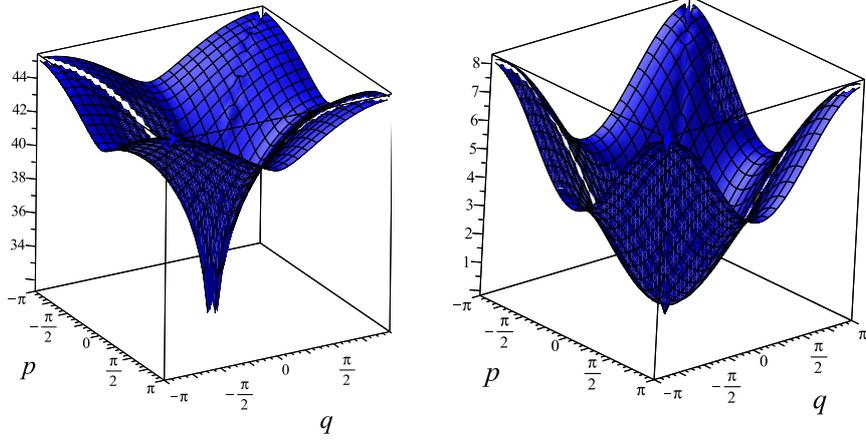
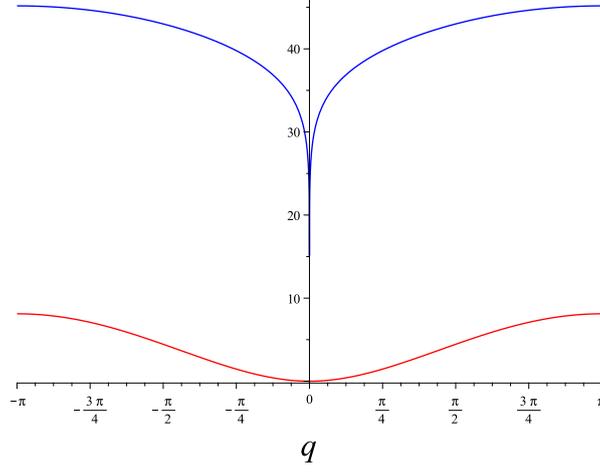
(a)  $s = 2.1$ (b)  $s = 5$ (c)  $l_{M,s}$  restricted to  $p = q$  for  $s = 2.1$  (blue) and  $s = 5$  (red), respectively

FIGURE 4.1. The graph of the function  $l_{M,s}$  on the square  $[-\pi, \pi]^2$  for the parameters  $s = 2.1$  (a) and  $s = 5$  (b), respectively. The third graph (c) shows the behaviour of both solutions  $s = 2.1$  and  $s = 5$  restricted to the line  $p = q$ .

In the next three lemmas we prove auxiliary asymptotic results, which are used to obtain the asymptotic behaviour of  $l_{M,s}$  in Proposition 4.5.

**Lemma 4.2.** *We have*

$$\frac{1}{\cos p - \cos q} = -\frac{2}{p^2 - q^2} \left[ 1 + \frac{1}{12}(p^2 + q^2) + R_1(p, q) \right], \quad p, q \in [-\pi, \pi], |p| \neq |q|.$$

where

$$R_1(p, q) = O(p^4 + q^4), \quad p, q \rightarrow 0, |p| \neq |q|.$$

*Proof.* We write

$$\cos p = 1 - \frac{p^2}{2} + \frac{p^4}{24} + f(p)$$

where  $f'(p) = O(p^5)$ ,  $p \rightarrow 0$ . For  $p, q \in (0, \pi]$  with  $p \neq q$  we have

$$\begin{aligned} \cos p - \cos q &= -\frac{p^2 - q^2}{2} + \frac{p^4 - q^4}{24} + f(p) - f(q) \\ &= -\frac{1}{2}(p^2 - q^2) \left[ 1 - \frac{1}{12}(p^2 + q^2) - 2\frac{f(p) - f(q)}{p^2 - q^2} \right] \\ &= -\frac{1}{2}(p^2 - q^2) \left[ 1 - \frac{1}{12}(p^2 + q^2) + O(p^4 + q^4) \right], \quad p, q \rightarrow 0, \end{aligned}$$

where the last relation follows from Lemma 4.1. Now the claim is obtained by taking inverses on both sides and extending the result to non-positive  $p, q$  by continuity and symmetry.  $\square$

**Lemma 4.3.** *Let  $s \in (2, \infty) \setminus \{4\}$ . Then*

$$2 \operatorname{Im}(\operatorname{Li}_s(e^{ip})) = -\frac{C_s}{2} p^{s-1} + 2\zeta(s-1)p - \frac{\zeta(s-3)}{3} p^3 + R_{2,s}(p), \quad p \in (0, 2\pi),$$

where

$$C_s := \begin{cases} -\frac{2\pi}{\Gamma(s) \sin(\frac{s\pi}{2})}, & s \notin 2\mathbb{Z}, \\ 0, & s \in 2\mathbb{Z}, \end{cases} \quad (4.2)$$

and

$$R_{2,s}(p) = O(p^5) \quad \text{and} \quad R'_{2,s}(p) = O(p^4), \quad p \searrow 0, \quad \text{if } s \neq 6,$$

and

$$R_{2,s}(p) = O(p^5 |\ln p|) \quad \text{and} \quad R'_{2,s}(p) = O(p^4 |\ln p|), \quad p \searrow 0, \quad \text{if } s = 6.$$

*Proof.* First let  $s \in (2, \infty) \setminus \mathbb{N}$ . It follows from [31, 25.12.12] that, for  $p \in (0, 2\pi)$ ,

$$\begin{aligned} 2 \operatorname{Im}(\operatorname{Li}_s(e^{ip})) &= 2 \operatorname{Im} \left[ \Gamma(1-s)(-ip)^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n) \frac{(ip)^n}{n!} \right] \\ &= 2 \operatorname{Im} \left[ \Gamma(1-s) p^{s-1} e^{-(s-1)\frac{\pi}{2}i} + \sum_{n=0}^{\infty} \zeta(s-n) \frac{i^n p^n}{n!} \right] \\ &= -2\Gamma(1-s) \sin\left((s-1)\frac{\pi}{2}\right) p^{s-1} + 2 \sum_{l=0}^{\infty} \zeta(s-2l-1) \frac{(-1)^l}{(2l+1)!} p^{2l+1} \\ &= -\frac{C_s}{2} p^{s-1} + 2\zeta(s-1)p - \frac{\zeta(s-3)}{3} p^3 + R_{2,s}(p), \end{aligned}$$

where

$$\begin{aligned} C_s &= 4\Gamma(1-s) \sin\left((s-1)\frac{\pi}{2}\right) = \frac{4\pi \sin\left((s-1)\frac{\pi}{2}\right)}{\Gamma(s) \sin(s\pi)} = -\frac{4\pi \cos\left(\frac{s\pi}{2}\right)}{\Gamma(s) \sin(s\pi)} \\ &= -\frac{2\pi}{\Gamma(s) \sin\left(\frac{s\pi}{2}\right)} \end{aligned}$$

and

$$R_{2,s}(p) = 2 \sum_{l=2}^{\infty} \zeta(s-2l-1) \frac{(-1)^l}{(2l+1)!} p^{2l+1}.$$

This relation extends to  $s$  being an odd integer with  $s \geq 3$ . Moreover,  $R_{2,s}$  satisfies

$$R_{2,s}(p) = O(p^5) \quad \text{and} \quad R'_{2,s}(p) = O(p^4), \quad p \searrow 0.$$

This proves the claim for  $s \in (2, \infty) \setminus 2\mathbb{N}$ .

Now let  $s \in \{6, 8, \dots\}$  and set

$$H_n = \sum_{j=1}^n \frac{1}{j}.$$

From [24, p. 131] we obtain, again for  $p \in (0, 2\pi)$ ,

$$\begin{aligned} 2 \operatorname{Im}(\operatorname{Li}_s(e^{ip})) &= 2 \operatorname{Im} \left[ \frac{(ip)^{s-1}}{(s-1)!} \left( H_{s-1} - \operatorname{Log}(-ip) \right) + \sum_{\substack{n=0 \\ n \neq s-1}}^{\infty} \zeta(s-n) \frac{(ip)^n}{n!} \right] \\ &= 2 \operatorname{Im} \left[ \frac{(-1)^{\frac{s}{2}-1} ip^{s-1}}{(s-1)!} \left( H_{s-1} - \ln p + i \frac{\pi}{2} \right) + \sum_{\substack{n=0 \\ n \neq s-1}}^{\infty} \zeta(s-n) \frac{i^n p^n}{n!} \right] \\ &= \frac{2(-1)^{\frac{s}{2}-1}}{(s-1)!} p^{s-1} (H_{s-1} - \ln p) + 2 \sum_{\substack{l=0 \\ l \neq \frac{s}{2}-1}}^{\infty} \zeta(s-2l-1) \frac{(-1)^l}{(2l+1)!} p^{2l+1} \\ &= 2\zeta(s-1)p - \frac{\zeta(s-3)}{3} p^3 + R_{2,s}(p), \end{aligned}$$

where

$$R_{2,s}(p) = \frac{2(-1)^{\frac{s}{2}-1}}{(s-1)!} p^{s-1} (H_{s-1} - \ln p) + 2 \sum_{\substack{l=2 \\ l \neq \frac{s}{2}-1}}^{\infty} \zeta(s-2l-1) \frac{(-1)^l}{(2l+1)!} p^{2l+1},$$

which satisfies

$$R_{2,s}(p) = O(p^5) \quad \text{and} \quad R'_{2,s}(p) = O(p^4), \quad p \searrow 0, \quad \text{if } s \geq 8,$$

and

$$R_{2,s}(p) = O(p^5 |\ln p|) \quad \text{and} \quad R'_{2,s}(p) = O(p^4 |\ln p|), \quad p \searrow 0, \quad \text{if } s = 6.$$

This finishes the proof in the case when  $s \in \{6, 8, \dots\}$ .  $\square$

**Lemma 4.4.** *Let  $s \in (2, \infty) \setminus \{4\}$  and let  $g_s$  be defined as in (3.11) and  $C_s$  as in (4.2). Then*

$$g_s(p) = -\frac{C_s}{2} p^s + 2\zeta(s-1)p^2 - \frac{\zeta(s-1) + \zeta(s-3)}{3} p^4 + R_{3,s}(p),$$

where  $R_{3,s}$  satisfies

$$R'_{3,s}(p) = \begin{cases} O(p^{s+1}), & s \in (2, 4), \\ O(p^5), & s \in (4, \infty) \setminus \{6\}, \\ O(p^5 |\ln p|), & s = 6, \end{cases} \quad (4.3)$$

as  $p \searrow 0$ .

*Proof.* Write  $\sin p = p - \frac{p^3}{6} + R_{\sin}(p)$ . From Lemma 4.3 we obtain that

$$\begin{aligned} g_s(p) &= 2 \sin p \cdot \operatorname{Im}(\operatorname{Li}_s(e^{ip})) \\ &= \left[ p - \frac{p^3}{6} + R_{\sin}(p) \right] \left[ -\frac{C_s}{2} p^{s-1} + 2\zeta(s-1)p - \frac{\zeta(s-3)}{3} p^3 + R_{2,s}(p) \right] \\ &= -\frac{C_s}{2} p^s + 2\zeta(s-1)p^2 - \frac{\zeta(s-1) + \zeta(s-3)}{3} p^4 + R_{3,s}(p) \end{aligned}$$

where

$$\begin{aligned} R_{3,s}(p) &= \frac{C_s}{12}p^{s+2} - \frac{C_s}{2}p^{s-1}R_{\sin}(p) + 2\zeta(s-1)pR_{\sin}(p) \\ &\quad + \frac{\zeta(s-3)}{18}p^6 - \frac{\zeta(s-3)}{3}p^3R_{\sin}(p) + \sin p \cdot R_{2,s}(p), \end{aligned}$$

which satisfies

$$R'_{3,s}(p) = O(p^{s+1}) + O(p^5) + O(R_{2,s}(p)) + O(pR'_{2,s}(p)).$$

The latter relation yields (4.3).  $\square$

In the next proposition we consider the asymptotic behaviour of the function  $l_{M,s}$  around the origin. In particular, we observe that the behaviour differs for the two cases  $s \in (2, 4)$  and  $s \in (4, \infty)$ . For the case when  $s = 4$  the behaviour is more complicated and involves a logarithmic term; we do not consider this case in the following.

**Proposition 4.5.** *Let  $s \in (2, \infty) \setminus \{4\}$ , let  $l_{M,s}$  be as in (3.10) and  $C_s$  as in (4.2). Moreover, define*

$$\begin{aligned} h_{1,s}(p, q) &:= \begin{cases} C_s \frac{|p|^s - |q|^s}{p^2 - q^2}, & |p| \neq |q|, \\ \frac{sC_s}{2}|p|^{s-2}, & |p| = |q|, \end{cases} \\ h_{2,s}(p, q) &:= \frac{\zeta(s-1) + 2\zeta(s-3)}{3}(p^2 + q^2). \end{aligned}$$

Then

$$l_{M,s}(p, q) = h_{1,s}(p, q) + h_{2,s}(p, q) + R_s(p, q), \quad p, q \in [-\pi, \pi],$$

where

$$R_s(p, q) = O(p^\alpha + q^\alpha), \quad p, q \rightarrow 0,$$

with

$$\alpha = \begin{cases} \min\{s, 4\}, & s \neq 6, \\ 4 - \varepsilon, & s = 6, \end{cases}$$

with an arbitrary  $\varepsilon > 0$ .

In particular, we have

$$l_{M,s}(p, q) = \begin{cases} h_{1,s}(p, q) + O(p^s + q^s), & s \in (2, 4), \\ h_{2,s}(p, q) + O(p^4 + q^4), & s \in (4, \infty) \setminus \{6\}, \\ h_{2,s}(p, q) + O(p^{4-\varepsilon} + q^{4-\varepsilon}), & s = 6, \end{cases}$$

as  $p, q \rightarrow 0$  with arbitrary  $\varepsilon > 0$  when  $s = 6$ .

*Proof.* Let  $p, q \in (0, \pi]$  such that  $p \neq q$ . From Lemmas 3.2, 4.2 and 4.4 we obtain

$$\begin{aligned} l_{M,s}(p, q) &= 4\zeta(s-1) + \frac{g_s(p) - g_s(q)}{\cos p - \cos q} \\ &= 4\zeta(s-1) - \frac{2}{p^2 - q^2} \left[ 1 + \frac{1}{12}(p^2 + q^2) + R_1(p, q) \right] \times \\ &\quad \times \left[ -\frac{C_s}{2}(p^s - q^s) + 2\zeta(s-1)(p^2 - q^2) - \frac{\zeta(s-1) + \zeta(s-3)}{3}(p^4 - q^4) \right. \\ &\quad \left. + R_{3,s}(p) - R_{3,s}(q) \right] \end{aligned}$$

$$\begin{aligned}
 &= 4\zeta(s-1) + \left[1 + \frac{1}{12}(p^2 + q^2) + R_1(p, q)\right] \cdot \left[C_s \frac{p^s - q^s}{p^2 - q^2}\right. \\
 &\quad \left. - 4\zeta(s-1) + \frac{2}{3}(\zeta(s-1) + \zeta(s-3))(p^2 + q^2) - 2\frac{R_{3,s}(p) - R_{3,s}(q)}{p^2 - q^2}\right] \\
 &= C_s \frac{p^s - q^s}{p^2 - q^2} + \left[\frac{2}{3}(\zeta(s-1) + \zeta(s-3)) - \frac{1}{3}\zeta(s-1)\right](p^2 + q^2) + R_s(p, q),
 \end{aligned}$$

where

$$\begin{aligned}
 R_s(p, q) &= C_s \left[\frac{1}{12}(p^2 + q^2) + R_1(p, q)\right] \frac{p^s - q^s}{p^2 - q^2} \\
 &\quad + R_1(p, q) \left[-4\zeta(s-1) + \frac{2}{3}(\zeta(s-1) + \zeta(s-3))(p^2 + q^2)\right] \\
 &\quad + \frac{1}{18}(\zeta(s-1) + \zeta(s-3))(p^2 + q^2)^2 \\
 &\quad - 2 \left[1 + \frac{1}{12}(p^2 + q^2) + R_1(p, q)\right] \frac{R_{3,s}(p) - R_{3,s}(q)}{p^2 - q^2}.
 \end{aligned}$$

It follows from Lemma 4.4 that

$$R'_{3,s}(p) = O(p^\beta) \quad \text{where } \beta = \begin{cases} \min\{s+1, 5\}, & s \neq 6, \\ 5 - \varepsilon, & s = 6, \end{cases} \quad (4.4)$$

for arbitrary  $\varepsilon > 0$ . Lemma 4.1 implies that

$$\frac{p^s - q^s}{p^2 - q^2} = O(p^{s-2} + q^{s-2}), \quad q, p \rightarrow 0, \quad p \neq q,$$

and

$$\frac{R_{3,s}(p) - R_{3,s}(q)}{p^2 - q^2} = O(p^{\beta-1}), \quad q, p \rightarrow 0, \quad p \neq q,$$

where  $\beta$  is as in (4.4). The error term  $R_s$  satisfies

$$R_s(p, q) = O(p^\alpha + q^\alpha), \quad p, q \rightarrow 0, \quad p \neq q,$$

where

$$\alpha = \begin{cases} \min\{s, 4\}, & s \neq 6, \\ 4 - \varepsilon, & s = 6, \end{cases}$$

with an arbitrary  $\varepsilon > 0$ . Since  $l_{M,s}$ ,  $h_{1,s}$  and  $h_{2,s}$  are continuous and even in  $p$  and  $q$ , the result extends to all  $p, q \in [-\pi, \pi]$ .  $\square$

The next lemma is the key lemma about the long-time behaviour of the solution of the Cauchy problem; it is a generalization of [15, Lemma 6.1] to the two-dimensional setting. It is more subtle than the one-dimensional case, but a further generalization to  $n$  dimensions is straightforward.

**Lemma 4.6.** *Let  $\alpha > 0$  and let  $l : [-\pi, \pi]^2 \rightarrow \mathbb{R}$  be a continuous function that satisfies*

$$l(p, q) > 0, \quad (p, q) \in [-\pi, \pi]^2 \setminus \{(0, 0)\} \quad (4.5)$$

and can be written as

$$l(p, q) = h(p, q) + R(p, q)$$

where the continuous function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies

$$h(rp, rq) = r^\alpha h(p, q), \quad r > 0, \quad p, q \in \mathbb{R},$$

and

$$R(p, q) = o(|p|^\alpha + |q|^\alpha), \quad p, q \rightarrow 0. \quad (4.6)$$

Define the function

$$f(x, y, t) := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(xp+ypq)} e^{-tl(p,q)} dp dq, \quad x, y \in \mathbb{R}.$$

Then

$$t^{\frac{2}{\alpha}} f(t^{\frac{1}{\alpha}} \xi, t^{\frac{1}{\alpha}} \eta, t) \rightarrow \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi v + \eta w)} e^{-h(v,w)} dv dw =: F(\xi, \eta), \quad (4.7)$$

$t \rightarrow \infty$ , uniformly in  $\xi, \eta \in \mathbb{R}$ .

Hence

$$f(x, y) = t^{-\frac{2}{\alpha}} F(t^{-\frac{1}{\alpha}} x, t^{-\frac{1}{\alpha}} y) + o(t^{-\frac{2}{\alpha}}), \quad (4.8)$$

$t \rightarrow \infty$ , uniformly in  $x, y \in \mathbb{R}$ .

*Proof.* Let us first show that there exists  $C > 0$  such that

$$l(p, q) \geq C(|p|^\alpha + |q|^\alpha), \quad p, q \in [-\pi, \pi]. \quad (4.9)$$

For fixed  $(p, q) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  we have

$$l(rp, rq) = r^\alpha h(p, q) + o(r^\alpha), \quad r \searrow 0,$$

which, together with (4.5) implies that  $h(p, q) > 0$  for  $(p, q) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . Set

$$C_1 := \min_{|p|^\alpha + |q|^\alpha = 1} h(p, q),$$

which is a positive number. Let  $(p, q) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  and set  $r := (|p|^\alpha + |q|^\alpha)^{\frac{1}{\alpha}}$ . Then

$$h(p, q) = h\left(r \frac{p}{r}, r \frac{q}{r}\right) = r^\alpha h\left(\frac{p}{r}, \frac{q}{r}\right) \geq C_1 r^\alpha$$

and hence

$$h(p, q) \geq C_1 (|p|^\alpha + |q|^\alpha), \quad p, q \in \mathbb{R}. \quad (4.10)$$

Together with (4.6), this implies that

$$l(p, q) \geq \frac{C_1}{2} (|p|^\alpha + |q|^\alpha), \quad p, q \in \mathbb{R} \text{ such that } |p|^\alpha + |q|^\alpha \leq r_0$$

for some  $r_0 > 0$ . Since  $l$  is continuous and satisfies (4.5), we obtain (4.9).

For  $\xi, \eta \in \mathbb{R}$  and  $t > 0$  we can use the substitution  $v = t^{\frac{1}{\alpha}} p$ ,  $w = t^{\frac{1}{\alpha}} q$  to obtain

$$\begin{aligned} t^{\frac{2}{\alpha}} f(t^{\frac{1}{\alpha}} \xi, t^{\frac{1}{\alpha}} \eta, t) &= t^{\frac{2}{\alpha}} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{it^{\frac{1}{\alpha}}(\xi p + \eta q)} e^{-tl(p,q)} dp dq \\ &= \frac{1}{4\pi^2} \int_{-t^{\frac{1}{\alpha}} \pi}^{t^{\frac{1}{\alpha}} \pi} \int_{-t^{\frac{1}{\alpha}} \pi}^{t^{\frac{1}{\alpha}} \pi} e^{i(\xi v + \eta w)} e^{-tl(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w)} dv dw. \end{aligned}$$

Hence

$$\begin{aligned}
 & \left| t^{\frac{2}{\alpha}} f\left(t^{\frac{1}{\alpha}} \xi, t^{\frac{1}{\alpha}} \eta, t\right) - F(\xi, \eta) \right| \\
 &= \left| \frac{1}{4\pi^2} \iint_{[-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2} e^{i(\xi v + \eta w)} \left( e^{-tl(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w)} - e^{-h(v, w)} \right) dv dw \right. \\
 & \quad \left. - \frac{1}{4\pi^2} \iint_{\mathbb{R}^2 \setminus [-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2} e^{i(\xi v + \eta w)} e^{-h(v, w)} dv dw \right| \\
 & \leq \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \chi_{[-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2}(v, w) \left| e^{-tl(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w)} - e^{-h(v, w)} \right| dv dw \quad (4.11)
 \end{aligned}$$

$$+ \frac{1}{4\pi^2} \iint_{\mathbb{R}^2 \setminus [-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2} e^{-h(v, w)} dv dw, \quad (4.12)$$

where  $\chi_G$  is the characteristic function of a set  $G \subseteq \mathbb{R}^2$ . The integral in (4.12) converges to 0 as  $t \rightarrow \infty$ ; note that the integral in (4.12) exists by (4.10). From (4.9) and (4.10) we obtain the following estimate for the integrand in (4.11):

$$\begin{aligned}
 & \chi_{[-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2}(v, w) \left| e^{-tl(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w)} - e^{-h(v, w)} \right| \\
 & \leq \chi_{[-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2}(v, w) \left( e^{-tl(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w)} + e^{-h(v, w)} \right) \\
 & \leq \chi_{[-t^{\frac{1}{\alpha}} \pi, t^{\frac{1}{\alpha}} \pi]^2}(v, w) \left( e^{-tC(t^{-1}|v|^\alpha + t^{-1}|w|^\alpha)} + e^{-C_1(|v|^\alpha + |w|^\alpha)} \right) \\
 & \leq e^{-C(|v|^\alpha + |w|^\alpha)} + e^{-C_1(|v|^\alpha + |w|^\alpha)},
 \end{aligned}$$

where the right-hand side is integrable on  $\mathbb{R}^2$  and independent of  $t$ . For fixed  $v, w \in \mathbb{R}^2$  and large enough  $t > 0$  we have

$$\begin{aligned}
 tl(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w) &= th(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w) + tR(t^{-\frac{1}{\alpha}} v, t^{-\frac{1}{\alpha}} w) \\
 &= h(v, w) + to(t^{-1}(|v|^\alpha + |w|^\alpha) - h(v, w)) \quad \text{as } t \rightarrow \infty.
 \end{aligned}$$

Hence the integrand in (4.11) converges to 0 pointwise as  $t \rightarrow \infty$ . Now the Dominated Convergence Theorem implies that the integral in (4.11) converges to 0 as  $t \rightarrow \infty$ . Since the integrals in (4.11) and (4.12) are independent of  $\xi$  and  $\eta$ , the convergence in (4.7) is uniform in  $\xi$  and  $\eta$ .

The relation in (4.8) follows easily from (4.7) by using the substitution  $x = t^{\frac{1}{\alpha}} \xi$ ,  $y = t^{\frac{1}{\alpha}} \eta$ .  $\square$

The next theorem is the main result of the paper. It contains the long-time behaviour of the solution of the Cauchy problem corresponding to the Mellin-transformed  $k$ -path Laplacian. It shows, in particular, that, for  $s \in (2, 4)$ , the solution exhibits superdiffusive behaviour whereas for  $s > 4$  one has normal diffusion.

**Theorem 4.7.** *Let  $\Gamma = (V, E)$  be the square lattice as described at the beginning of Section 3, let  $s > 2$ ,  $s \neq 4$ , and let  $L_{M,s}$  be the Mellin-transformed  $k$ -path Laplacian defined in (2.6). Let  $u$  be the solution in (3.18) of (3.14), (3.15) with  $\dot{u} = \mathbf{e}_{0,0}$ , where  $\mathbf{e}_{0,0}$  is defined in (3.17). Then*

$$u_{x,y}(t) = t^{-\frac{2}{\alpha}} F_s(t^{-\frac{1}{\alpha}} x, t^{-\frac{1}{\alpha}} y) + o(t^{-\frac{2}{\alpha}}), \quad t \rightarrow \infty, \text{ uniformly in } x, y \in \mathbb{Z},$$

where in the case  $s \in (2, 4)$ ,

$$\alpha = s - 2 \quad \text{and} \quad F_s(\xi, \eta) := \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi v + \eta w)} e^{-h_{1,s}(v,w)} dv dw$$

with  $h_{1,s}$  from Proposition 4.5, and in the case  $s \in (4, \infty)$ ,

$$\alpha = 2 \quad \text{and} \quad F_s(\xi, \eta) := \frac{1}{4\pi\gamma_s} e^{-\frac{\xi^2 + \eta^2}{4\gamma_s}}$$

with

$$\gamma_s = \frac{\zeta(s-1) + 2\zeta(s-3)}{3}.$$

(See Figures 4.2 and 4.3.)

*Proof.* By Proposition 4.5 and Lemma 3.2 the function  $l_{M,s}$  satisfies the assumptions of Lemma 4.6 with  $h = h_{1,s}$  and  $\alpha = s - 2$  when  $s \in (2, 4)$  and with  $h = h_{2,s}$  and  $\alpha = 2$  when  $s \in (4, \infty)$ . Hence all claims follow from Lemma 4.6.  $\square$

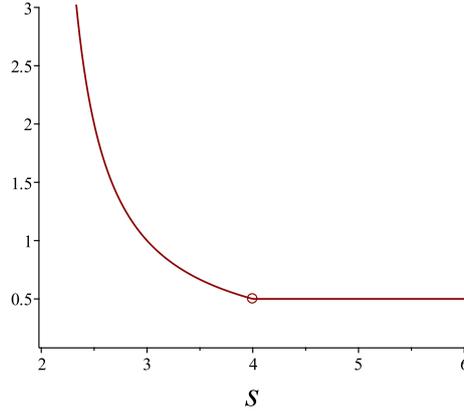


FIGURE 4.2. The  $s$ -dependence of  $\frac{1}{\alpha}$ .

*Remark 4.8.* Theorem 4.7 shows that the distribution spreads proportionally to  $t^{\frac{1}{\alpha}}$  where  $\alpha$  is as in that theorem; cf. [15, Remark 6.2]. When  $s > 4$ , one has normal diffusion since the profile spreads proportionally to  $t^{\frac{1}{2}}$  in this case. When  $2 < s < 4$ , however, we observe superdiffusion because then the spread of the profile is proportional to  $t^\kappa$  with  $\kappa = \frac{1}{s-2} > \frac{1}{2}$ . In particular, when  $s = 3$ , then the profile spreads linearly in time, which is a ballistic behaviour.

One can measure the spread, e.g. with the full width at half maximum (FWHM), which, for our purpose, we can define as

$$\text{FWHM}(t) := 2 \sup \left\{ r > 0 : u_{x,y}(t) \leq \frac{1}{2} u_{0,0}(t) \right. \\ \left. \text{for all } x, y \in \mathbb{Z} \text{ with } |x|^2 + |y|^2 \geq r^2 \right\}.$$

One can show that  $\text{FWHM}(t) \sim ct^{\frac{1}{\alpha}}$  as  $t \rightarrow \infty$  with some  $c > 0$ ; cf. [15, Remark 6.2] for the one-dimensional case.  $\diamond$

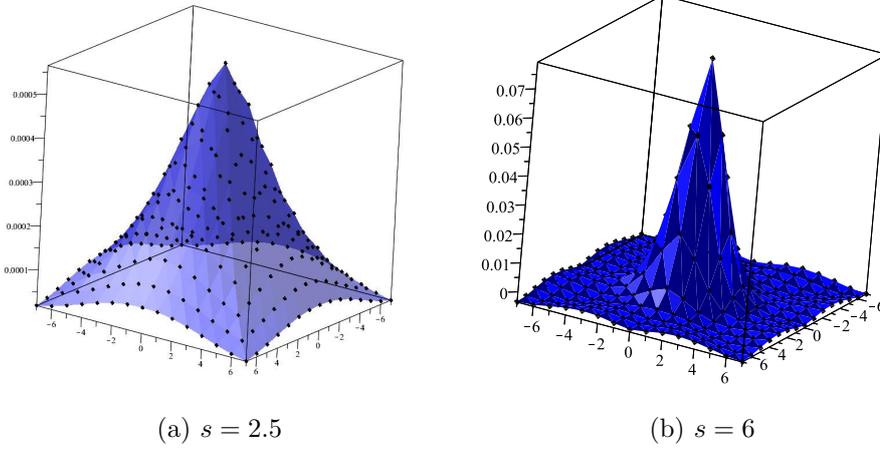


FIGURE 4.3. The graph of the limit solution  $F_s$  from Theorem 4.7 for  $s = 2.5$  (a) and  $s = 6$  (b), respectively.

*Remark 4.9.* Let us consider finite truncations of the Mellin transformation (2.6) of the  $k$ -path Laplacian, i.e. set

$$L_{M,s,N} := \sum_{k=1}^N \frac{1}{k^s} L_k$$

for  $N \in \mathbb{N}$ . By Lemma 3.1 this operator is unitarily equivalent to the operator of multiplication by the function

$$l_{M,s,N}(p, q) = \sum_{k=1}^N \frac{1}{k^s} l_k(p, q)$$

where  $l_k$  is defined in that lemma. Using Lemmas 4.1 and 4.2 one can show in a similar way as above that

$$l_k(p, q) = \frac{2k^3 + k}{3} (p^2 + q^2) + O(p^4 + q^4), \quad p, q \rightarrow 0$$

and hence

$$l_{M,s,N}(p, q) = \sum_{k=1}^N \frac{2k^3 + k}{3k^s} (p^2 + q^2) + O(p^4 + q^4), \quad p, q \rightarrow 0.$$

This leads to normal diffusion by Lemma 4.6 and not to a superdiffusive process like in the non-truncated Mellin transformation. However, the diffusion speed and the variance of the limiting normal distribution grow with  $N$ , e.g. if one measures the former with the full width at half maximum, one gets

$$\text{FWHM}(t) \sim 2 \left( (\ln 2) \sum_{k=1}^N \frac{2k^3 + k}{3k^s} \right)^{\frac{1}{2}} t^{\frac{1}{2}}, \quad t \rightarrow \infty; \quad (4.13)$$

cf. [15, Remark 6.4]. As  $N \rightarrow \infty$  one has the following behaviour,

$$\sum_{k=1}^N \frac{(2k^2 + 1)k}{3k^s} \sim \frac{2}{3(4-s)} N^{4-s}, \quad N \rightarrow \infty, \quad \text{if } s \in (2, 4),$$

$$\sum_{k=1}^N \frac{(2k^2 + 1)k}{3k^s} \rightarrow \frac{2\zeta(s-3) + \zeta(s-1)}{3}, \quad N \rightarrow \infty, \quad \text{if } s \in (4, \infty).$$

When  $s > 4$  (i.e. when  $u$  in Theorem 4.7 shows normal diffusion), the coefficient in (4.13) converges to the corresponding coefficient for  $u$  as  $N \rightarrow \infty$ , and the limiting normal distributions converge to the limiting distribution from Theorem 4.7. On the other hand, when  $s \in (2, 4)$ , the coefficient in (4.13) diverges as  $N \rightarrow \infty$ . So, although one has normal diffusion for every finite  $N$ , the speed of the diffusion — and also the variance of the limiting normal distribution — can be made arbitrarily large if  $N$  is chosen big enough.  $\diamond$

#### REFERENCES

- [1] M. Aidelsburger, M. Atala, S. Nascimbène, S. Trotzky, Y. A. Chen and I. Bloch, Experimental realization of strong effective magnetic fields in an optical lattice, *Phys. Rev. Lett.* **107** (2011), 255301
- [2] E. Akkermans, A. Gero and R. Kaiser, Photon localization and Dicke superradiance in atomic gases, *Phys. Rev. Lett.* **101** (2008), 103602
- [3] P. Babkevich, V. M. Katukuri, B. Fåk, S. Rols, T. Fennell, D. Pajić, H. Tanaka, T. Pardini, R. R. P. Singh, A. Mitrushchenkov and O. V. Yazyev, Magnetic excitations and electronic interactions in  $\text{Sr}_2\text{CuTeO}_6$ : a spin-1/2 square lattice Heisenberg antiferromagnet, *Phys. Rev. Lett.* **117** (2016), 237203
- [4] S. Bahatyrova, R. N. Frese, C. A. Siebert, J. D. Olsen, K. O. van der Werf, R. van Grondelle, R. A. Niederman, P. A. Bullough, C. Otto and C. N. Hunter, The native architecture of a photosynthetic membrane, *Nature* **430** (2004), 1058–1062
- [5] K. Binder and D. P. Landau, Square lattice gases with two- and three-body interactions: a model for the adsorption of hydrogen on Pd (100), *Surf. Sci.* **108** (1981), 503–525
- [6] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D. U. Hwang, Complex networks: structure and dynamics, *Phys. Rep.* **424** (2006), 175–308
- [7] G. L. Celardo, F. Borgonovi, M. Merkli, V. I. Tsifrinovich and G. P. Berman, Superradiance transition in photosynthetic light-harvesting complexes, *J. Phys. Chem. C* **116** (2012), 22105–22111
- [8] G. L. Celardo, G. G. Giusteri and F. Borgonovi, Cooperative robustness to static disorder: superradiance and localization in a nanoscale ring to model light-harvesting systems found in nature, *Phys. Rev. B* **90** (2014), 075113
- [9] A. D. Córcoles, E. Magesan, S. J. Srinivasan, A. W. Cross, M. Steffen, J. M. Gambetta and J. M. Chow, Demonstration of a quantum error detection code using a square lattice of four superconducting qubits, *Nature Comm.* **6** (2015), 6979
- [10] P. D. Dahlberg, P. C. Ting, S. C. Massey, M. A. Allodi, E. C. Martin, C. N. Hunter and G. S. Engel, Mapping the ultrafast flow of harvested solar energy in living photosynthetic cells, *Nature Comm.* **8** (2017), 988
- [11] B. Dalla Piazza, M. Mourigal, N. B. Christensen, G. J. Nilsen, P. Tregenna-Piggott, T. G. Per-ring, M. Enderle, D. F. McMorrow, D. A. Ivanov and H. M. Rønnow, Fractional excitations in the square-lattice quantum antiferromagnet, *Nature Phys.* **11** (2014), 3172
- [12] A. A. Dubkov, B. Spagnolo and V. V. Uchaikin, Lévy flight superdiffusion: an introduction, *Int. J. Bifur. Chaos* **18** (2008), 2649–2672
- [13] E. Estrada, Graphs and network theory, In: *Mathematical Tools for Physicists*. 2nd Edition (editor: M. Grinfeld). John Wiley & Sons, 2013
- [14] E. Estrada, J.-C. Delvenne, N. Hatano, J. L. Mateos, R. Metzler, A. P. Riascos and M. T. Schaub, Random multi-hopper model. Super-fast random walks on graphs, *J. Compl. Net.* **6** (2018), 382–403
- [15] E. Estrada, E. Hameed, N. Hatano and M. Langer, Path Laplacian operators and superdiffusive processes on graphs. I. One-dimensional case, *Linear Algebra Appl.* **523** (2017), 307–334

- [16] J. F. Gómez-Aguilar, M. Miranda-Hernández, M. G. López-López, V. M. Alvarado-Martínez and D. Baleanu, Modeling and simulation of the fractional space-time diffusion equation, *Commun. Nonlinear Sci. Numer. Simul.* **30** (2016), 115–127
- [17] J. Grad, G. Hernandez and S. Mukamel, Radiative decay and energy transfer in molecular aggregates: the role of intermolecular dephasing, *Phys. Rev. A* **37** (1998), 3835
- [18] L. C. Ha, C. L. Hung, X. Zhang, U. Eismann, S. K. Tung and C. Chin, Strongly interacting two-dimensional Bose gases, *Phys. Rev. Lett.* **110** (2013), 145302
- [19] P. R. Hammar, D. C. Dender, D. H. Reich, A. S. Albrecht and C. P. Landee, Magnetic studies of the two-dimensional,  $S = 1/2$  Heisenberg antiferromagnets  $(5\text{CAP})_2\text{CuCl}_4$  and  $(5\text{MAP})_2\text{CuCl}_4$ , *J. Appl. Phys.* **81** (1997), 4615–4617
- [20] T. T. Hartley and C. F. Lorenzo, Application of incomplete gamma functions to the initialization of fractional-order systems, *J. Comput. Nonlinear Dynam.* **3** (2008), 021103
- [21] S. Ji, C. Ates and I. Lesanovsky, Two-dimensional Rydberg gases and the quantum hard-squares model, *Phys. Rev. Lett.* **107** (2011), 060406
- [22] F. Jörder, K. Zimmermann, A. Rodriguez and A. Buchleitner, Interaction effects on dynamical localization in driven helium, *Phys. Rev. Lett.* **113** (2014), 063004
- [23] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Fractional differential equations: a emergent field in applied and mathematical sciences. In: *Factorization, Singular Operators and Related Problems*, Kluwer Acad. Publ., Dordrecht, 2003, pp. 151–173
- [24] E. Lindelöf, *Le Calcul des Résidus*, Gauthier-Villars, Paris, 1905
- [25] X. J. Liu, X. Liu, C. Wu and J. Sinova, Quantum anomalous Hall effect with cold atoms trapped in a square lattice, *Phys. Rev. A* **81** (2010), 33622
- [26] C. F. Lorenzo and T. T. Hartley, Initialization of fractional-order operators and fractional differential equations, *J. Comput. Nonlinear Dynam.* **3** (2008), 021101
- [27] J. Lu, J. Shen, J. Cao and J. Kurths, Consensus of networked multi-agent systems with delays and fractional-order dynamics. In: *Consensus and Synchronization in Complex Networks*, Springer, Berlin–Heidelberg, 2013, pp. 69–110
- [28] E. Manousakis, The spin- $\frac{1}{2}$  Heisenberg antiferromagnet on a square lattice and its application to the cuprous oxides, *Rev. Mod. Phys.* **63** (1991), 1
- [29] R. Metzler and J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339** (2000), 1–77
- [30] R. Metzler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A* **37** (2004), R161
- [31] *NIST Handbook of Mathematical Functions*, Edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. Online version: <http://dlmf.nist.gov>
- [32] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York, 1999
- [33] U. Schneider, L. Hackermüller, J. P. Ronzheimer, S. Will, S. Braun, T. Best, I. Bloch, E. Demler, S. Mandt, D. Rasch and A. Rosch, Fermionic transport and out-of-equilibrium dynamics in a homogeneous Hubbard model with ultracold atoms, *Nature Physics* **8** (2012), 213–218
- [34] M. F. Shlesinger, G. M. Zaslavsky and U. Frisch, *Lévy Flights and Related Topics in Physics*, Proceedings of the International Workshop Held at Nice. Lecture Notes in Physics, vol. 450, Springer, 1995
- [35] H. Stahlberg, J. Dubochet, H. Vogel and R. Ghosh, Are the light-harvesting I complexes from *Rhodospirillum rubrum* arranged around the reaction centre in a square geometry? *J. Mol. Biol.* **282** (1998), 819–831
- [36] H. Sun, W. Chen, C. Li and Y. Chen, Y., Fractional differential models for anomalous diffusion, *Physica A* **389** (2010), 2719–2724
- [37] M. M. Valado, C. Simonelli, M. D. Hoogerland, I. Lesanovsky, J. P. Garrahan, E. Arimondo, D. Ciampini and O. Morsch, Experimental observation of controllable kinetic constraints in a cold atomic gas, *Phys. Rev. A* **93** (2016), 040701
- [38] G. M. Viswanathan, E. P. Raposo and M. G. E. Da Luz, Lévy flights and superdiffusion in the context of biological encounters and random searches, *Phys. Life Rev.* **5** (2008), 133–150

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF STRATHCLYDE,  
26 RICHMOND STREET, GLASGOW G1 1XH, UK  
*E-mail address:* [ernesto.estrada@strath.ac.uk](mailto:ernesto.estrada@strath.ac.uk)

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF STRATHCLYDE,  
26 RICHMOND STREET, GLASGOW G1 1XH, UK  
*E-mail address:* [ehsan.hameed@strath.ac.uk](mailto:ehsan.hameed@strath.ac.uk)

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF STRATHCLYDE,  
26 RICHMOND STREET, GLASGOW G1 1XH, UK  
*E-mail address:* [m.langer@strath.ac.uk](mailto:m.langer@strath.ac.uk)

INSTITUTE OF MATHEMATICS, ŁÓDŹ UNIVERSITY OF TECHNOLOGY,  
UL. WÓLCZAŃSKA 215, 90-924ŁÓDŹ, POLAND,

INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS, UNIVERSITY OF WARSAW,  
BANACHA 2, 02-097 WARSAW, POLAND  
*E-mail address:* [aleksandrafalkiewicz@gmail.com](mailto:aleksandrafalkiewicz@gmail.com)