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THE AVERAGING METHOD FOR MULTIVALUED SDES WITH JUMPS AND NON-LIPSCHITZ COEFFICIENTS

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Abstract. In this paper, we study the averaging principle for multivalued SDEs with jumps and non-Lipschitz coefficients. By the Bihari's inequality and the properties of the concave function, we prove that the solution of averaged multivalued SDE with jumps converges to that of the standard one in the sense of mean square and also in probability. Finally, two examples are presented to illustrate our theory.

1. **Introduction.** In this paper, we are concerned with the averaging principle of the following equation:

$$dx(t)+A(x(t))dt \ni f(t,x(t))dt+g(t,x(t))dw(t)+\int_Z h(t,x(t^-),v)N(dt,dv), \quad (1)$$

where A is a multi-valued maximal monotone operator defined on R^n and $w(t)$ is an m dimensional Brownian motion, N is the counting measure of a stationary Poisson point process with characteristic measure π on some measurable space $(Z,\mathcal{B}(Z))$.

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As a matter of fact, Eq.(1) is called the multivalued SDEs with jumps. When $h \equiv 0$, it becomes an multivalued SDEs driven by continuous Brownian motion

$$dx(t) + A(x(t))dt \ni f(t, x(t))dt + g(t, x(t))dw(t). \tag{2}$$

Such multivalued SDEs was firstly introduced by Krée [1] and the existence and uniqueness of the solution has been discussed by Cépa [2]. After that, the theory of multivalued SDEs (2) has drawn increasing attention and it was studied subsequently by many authors. For example, Cépa [3], Lépingle and Marois [4], Bernardin [5], X.Zhang [6], J.Ren [7], Y.Ren [8], H.Zhang [9]. On the other hand, some scholars also studied the multivalued SDEs with jumps which is discontinuous in time and obtained a number of interesting results [10, 11, 12, 13, 14, 15].

As we all know, the averaging principles is an important method which a more complicated time varying system can be approximated by an autonomous differential system. Since Krylov and Bogolyubov [16] put forward the averaging principles for dynamical systems, the averaging principles have received a lot of attention. For example, the averaging principles for determined differential equations can be found in [17, 18, 19, 20, 21]. For the averaging principles of stochastic differential equations (SDEs), we refer to [22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. While the theory of averaging for SDEs is well developed, the literature involving averaging principles for multivalued SDEs is scarce. Recently, Ngoran and Modeste [32] studied the averaging principle of multivalued SDEs and proved the solution of the averaged multivalued SDEs converges to that of the original multivalued SDEs. Later, Xu and Liu [33] removed the integrability condition about A in [32] and showed the convergence of the averaged multivalued SDEs and the original one. Guo and Pei [34] extended the averaging principle [33] to the case of multivalued SDEs with compensated Poisson random measures and proved that the averaged solution converges to the original solution.

However, to the best of our knowledge, there are few literature about using the averaging methods to obtain the approximate solutions to multivalued SDEs with jumps (1). In order to fill the gap, we will study the averaging principle of Eq.(1). Different from above works mentioned, the assumption given in this paper is not the classical Lipschitz condition. In fact, the global Lipschitz condition imposed on [32, 33, 34] is seemed to be considerably strong when one discusses variable applications in real world. For example, let us consider the multivalued SDEs with pure jumps

$$dx(t) + A(x(t))dt \ni f(t, x(t))dt + \int_Z vh(t, x(t))N(dt, dv), \tag{3}$$

where $f(t, x) = h(t, x) = sin(t)k(x)$ and

$$k(x) = \begin{cases} x\sqrt{\ln x^{-1}}, & if \ x \le e^{-1}, \\ \frac{1}{2}(x + e^{-1}), & if \ x > e^{-1}. \end{cases} \tag{4}$$

It is obviously that $k(x)$ is a concave nondecreasing continuous function on R^+ and the coefficients f and h do not satisfy the global Lipschitz condition. In this case, the averaging principle obtained in [32, 33, 34] can not be applied to Eq.(3). Therefore, it is very important for us to establish the averaging principle of the multivalued SDEs with jumps (1) under some weaker conditions. In this paper, we assume that the coefficients of Eq.(1) satisfy the non-Lipschitz condition and use this condition to study the averaging principle of Eq.(1). By the Bihari's inequality and our proposed conditions, we prove that the solution of the averaged equation converges to that of the standard equation in the mean square sense. On the other hand, by lemma 3.10 and the properties of the concave function, we further give the order of the convergence for Eq.(1) in finite time interval and show the convergence in probability of the standard solution and the averaged solution. It should be pointed out that our results are also hold for multivalued SDEs without jumps (2) under non-Lipschitz conditions.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and establish the existence and uniqueness of the solution to Eq.(1). In Section 3, we establish the averaging principle of Eq.(1). By the Bihari's inequality and some useful lemmas, we prove that the solution of the averaged equation will converge to that of the standard equation in the sense of the mean square and probability. Finally, two illustrative examples will be given in Section 4.

2. **Preliminaries and multivalued SDEs with jumps.**

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with some filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Here $w(t)$ is an m-dimensional $\mathcal{F}_t$-adapted Brownian motion. Let $\mathcal{D}([0,T]; R^n)$ denote the family of all right-continuous functions with left-hand limits φ equipped with the norm $||\varphi|| = \sup\limits_{0\leq t\leq T} |\varphi(t)|$. Let $(R^n, \mathcal{B}(R^n))$ be a measurable space and $\pi(du)$ a σ- finite measure on it. Let $\{\bar{p} = \bar{p}(t), t \geq 0\}$ be a stationary $\mathcal{F}_t$-adapted and R^n-valued Poisson point process. Then, for $Z \in \mathcal{B}(R^n - \{0\})$, here $0 \notin \bar{A}$, we define the Poisson counting measure N associated with $\bar{p}$ by $N((0,t] \times Z) := \#\{0 < s \leq t, \bar{p}(s) \in Z\}$, where $\#$ denotes the cardinality of set $\{.\}$. For simplicity, we denote $N(t, Z) := N((0,t] \times Z)$. Moreover, by the Doob-Meyer's decomposition theorem, we have $N(t,Z) = \tilde{N}(t,Z) + \hat{N}(t,Z)$, where $\tilde{N}(t,Z)$ is the compensated Poisson random measure and $\hat{N}(t,Z) = \pi(Z)t$ is the compensator.

Now, we will give the definition and proposition about the multivalued maximal monotone operator. For more details, we refer to Cépa [2, 3].

Denote by 2^{R^n} for the set of all subset of R^n. Let $A: R^n \to 2^{R^n}$ be a set-valued operator. Define the domain of A by

$$D(A) = \{x \in R^n : Ax \neq \phi\}.$$

and

$$Gr(A) = \{(x,y) \in R^n \times R^n : x \in R^n, y \in Ax\}.$$

Definition 2.1. [35][35] (1) A multivalued operator A on R^n is called monotone if

$$\langle x_1 - y_1, x_2 - y_2 \rangle \geq 0, \text{ for all } (x_1,y_1),(x_2,y_2) \in Gr(A).$$

(2) A monotone operator A is called maximal monotone if and only if

$$(x_1,y_1) \in Gr(A) \Leftrightarrow \langle x_1 - y_1, x_2 - y_2 \rangle \geq 0, \ \forall \ (x_2,y_2) \in Gr(A).$$

Lemma 2.2. [3] Let A be a maximal monotone operator on R^n, then, for each $x \in D(A)$, $A(x)$ is a closed and convex subset of R^n. Let $A^\circ(x) := proj_{A(x)}(0)$ be the minimal section of A, where $proj_D$ is designated as the projection on every closed and convex subset D on R^n and $proj_\phi(0) = \infty$. Then

$$x \in D(A) \ \Leftrightarrow \ |A^\circ(x)| < \infty.$$

Definition 2.3. [35]

For every $T > 0$, a pair of $\mathcal{F}_t$-adapted random processes $(x(t), K(t))$ is called a solution of Eq.(1) if

(1) $x(t) \in \mathcal{D}([0,T]; \overline{D(A)})$ and $x(0) = x_0$ a.s.;
(2) K is of finite variation on $[0,T]$ and $K(0) = 0$ a.s.;
(3) For all $t \in [0,T]$,

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(s,x(s))ds + \int_0^t g(s,x(s))dB(s) \\ &+ \int_0^t \int_Z h(s,x(s^-),v)N(ds,dv) - K(t), \quad a.s.; \end{aligned}$$

(4) For all $\alpha, \beta \in \mathcal{D}([0,T]; R^n)$ satisfying $(\alpha(t), \beta(t)) \in Gr(A)$, the measure

$$\langle x(t) - \alpha(t), dK(t) - \beta(t)dt \rangle \geq 0 \quad a.s.$$

The following lemmas are taken from [2].

Lemma 2.4. Suppose (x_1, K_1) and (x_2, K_2) are two pairs satisfying (1), (2) and (4) in Definition 2.3, then

$$\langle x_1(t) - x_2(t), dK_1(t) - dK_2(t) \rangle \geq 0.$$

In this paper, we consider the multivalued SDEs with Poisson random measure

$$dx(t) + A(x(t))dt \ni f(t,x(t))dt + g(t,x(t))dw(t) + \int_Z h(t,x(t^-),v)N(dt,dv), \quad (5)$$

where $f : [0,T] \times \mathcal{D}([0,T]; R^n) \to R^n$, $g : [0,T] \times \mathcal{D}([0,T]; R^n) \to R^{n\times m}$ and $h : [0,T] \times \mathcal{D}([0,T]; R^n) \times Z \to R^n$ are both Borel-measurable functions. The initial value $x(0) = x_0$ is an $\mathcal{F}_0$-measurable R^n-valued random variable and $E|x_0|^2 < \infty$.

Let us consider the following assumptions.

Assumption 2.5. For any $x, y \in R^n$ and $t \in [0,T]$, there exist two functions $k(.)$ and $\rho(.)$ such that

$$\begin{aligned} |f(t,x) - f(t,y)| + ||g(t,x) - g(t,y)|| &\leq \lambda(t)k(|x-y|), \\ \Big[\int_Z |h(t,x,v) - h(t,y,v)|^2 \pi(dv)\Big]^{\frac{1}{2}} &\leq \lambda(t)\rho(|x-y|), \end{aligned}$$

where $\lambda(t) \in L^2([0,T;R)$, k, ρ are two concave nondecreasing functions such that $k(0) = \rho(0) = 0$ and

$$\int_{0^+} \frac{u}{k^2(u) + \rho^2(u) + u^2} du = \infty.$$

Assumption 2.6. There exist two positive constants L_1, L_2 such that

$$\begin{aligned} |f(t,x)|^2 + ||g(t,x)||^2 + \int_Z |h(t,x,v)|^2 \pi(dv) &\leq L_1(1+|x|^2), \\ \int_Z |h(t,x,v)|^4 \pi(dv) \leq L_2(1+|x|^4). \end{aligned}$$

Assumption 2.7. A is a maximal monotone operator with $D(A) = R^n$.

Remark 2.8. Under Assumption 2.5, we can deduce that

$$\int_{0^+} \frac{1}{k^2(u)} du = \infty \quad \text{and} \quad \int_{0^+} \frac{1}{\rho^2(u)} du = \infty.$$

Remark 2.9. Let $L > 0$ and $\delta \in (0, 1/e)$ be sufficiently small. Define $k_1(u) = \rho_1(u) = Lu,\ u \geq 0$,

$$k_2(u) = \rho_2(u) = \begin{cases} u\sqrt{log(\frac{1}{u})}, & u \in [0, \delta], \\ \delta\sqrt{log(\dfrac{1}{\delta})} + k_2'(\delta-)(u-\delta), & u \in [\delta, +\infty], \end{cases}$$

and

$$k_3(u) = \rho_3(u) = \begin{cases} u\sqrt{loglog(\frac{1}{u})}, & u \in [0, \delta], \\ \delta\sqrt{loglog(\dfrac{1}{\delta})} + k_3'(\delta-)(u-\delta), & u \in [\delta, +\infty], \end{cases}$$

where $k'(\rho')$ denotes the derivative of function $k(\rho)$. They are all concave nondecreasing functions satisfying $\int_{0+} \frac{u}{k_i^2(u)+\rho_i^2(u)+u^2} du = +\infty$ $(i = 1, 2, 3)$.

In particular, we see that the Lipschitz condition is a special case of our proposed condition. In other words, we obtain a more general result than that of [32, 33, 34].

Similar to the proof of [12, 36, 37], we have the following existence result.

Theorem 2.10. *If Assumptions 2.5-2.7 hold. Then, there exists a unique solution $x(t)$ to Eq.(5) with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(\Omega; R^n)$.*

3. **Stochastic averaging principle.** In this section, we shall study the averaging principle for multi-valued SDEs with jumps. Let us consider the standard form of Eq.(5)

$$\begin{aligned} x_\varepsilon(t) &= x_0 + \varepsilon \int_0^t f(s, x_\varepsilon(s))ds + \sqrt{\varepsilon}\int_0^t g(s, x_\varepsilon(s))dw(s) \\ &+ \sqrt{\varepsilon}\int_0^t \int_Z h(s, x_\varepsilon(s^-), v)N(ds, dv) - \varepsilon K(s), \end{aligned} \tag{6}$$

with the initial value $x_\varepsilon(0) = x_0$. Here the coefficients f, g and h have the same conditions as in Assumptions 2.5, 2.6 and $\varepsilon \in [0, \varepsilon_0]$ is a positive small parameter with ε_0 is a fixed number.

Let $\bar{f}(x) : \mathcal{D}([0, T]; R^n) \to R^n$, $\bar{g}(x) : \mathcal{D}([0, T]; R^n) \to R^{n\times m}$ and $\bar{h}(x, v) : \mathcal{D}([0, T]; R^n) \times Z \to R^n$ be measurable functions, satisfying Assumptions 2.5 and 2.6. We also assume that the following condition is satisfied.

Assumption 3.1. For any $x \in R^n$ and $T_1 > 0$, there exist three positive bounded functions $\psi_i(T_1)$, $i = 1, 2, 3$, such that

$$\begin{aligned} \frac{1}{T_1}\int_0^{T_1} |f(t, x) - \bar{f}(x)|^2 dt &\leq \psi_1(T_1)(1 + |x|^2), \\ \frac{1}{T_1}\int_0^{T_1} ||g(t, x) - \bar{g}(x)||^2 dt &\leq \psi_2(T_1)(1 + |x|^2), \\ \frac{1}{T_1}\int_0^{T_1}\int_Z |h(t, x, v) - \bar{h}(x, v)|^2 \pi(dv)dt &\leq \psi_3(T_1)(1 + |x|^2), \end{aligned}$$

where $\lim_{T_1\to\infty} \psi_i(T_1) = 0$.

Then we have the averaging form of the multi-valued SDEs with jumps

$$\begin{aligned} y_\varepsilon(t) &= x_0+\varepsilon\int_0^t \bar{f}(y_\varepsilon(s))ds+\sqrt{\varepsilon}\int_0^t \bar{g}(y_\varepsilon(s))dw(s) \\ &+ \sqrt{\varepsilon}\int_0^t\int_Z \bar{h}(y_\varepsilon(s),v)N(ds,dv)-\varepsilon\bar{K}(t), \end{aligned} \tag{7}$$

with the initial value $y_\varepsilon(0)=x_0$.

Obviously, under Assumptions 2.5-2.7, the standard multi-valued SDEs with jumps (6) and the averaged one (7) have a unique solution, respectively.

In order to prove our main result, we need to introduce some lemmas.

Lemma 3.2. Let $\phi: R_+\times Z\to R^n$ and assume that $E\int_0^t\int_Z|\phi(s,v)|^2\pi(dv)ds<\infty$, then

$$E\int_0^t\int_Z|\phi(s,v)|^2N(ds,dv)=E\int_0^t\int_Z|\phi(s,v)|^2\pi(dv)ds. \tag{8}$$

Moreover, there exists a positive constant C such that

$$E\sup_{0\le t\le T}\Big|\int_0^t\int_Z\phi(s,v)\tilde{N}(ds,dv)\Big| \;\le\; CE\Big[\int_0^T\int_Z|\phi(s,v)|^2N(ds,dv)\Big]^{\frac{1}{2}}. \tag{9}$$

Proof. Obviously, by Theorem 38 in [38], we can easily obtained the equality (8). By Theorem 48 in [38], we have

$$\begin{aligned} E\sup_{0\le t\le T}\Big|\int_0^t\int_Z\phi(s,v)\tilde{N}(ds,dv)\Big| &\le CE[M_t,M_t]^{\frac{1}{2}} \\ &= CE\Big[\int_0^T\int_Z|\phi(s,v)|^2N(ds,dv)\Big]^{\frac{1}{2}}, \end{aligned}$$

where $M_t=\int_0^t\int_Z\phi(s,v)\tilde{N}(ds,dv)$ is a $\mathcal{F}_t$-adapted martingale. □

Lemma 3.3. [39] Let $k: R^+\to R^+$ be a continuous, non-decreasing function satisfying $k(0)=0$ and $\int_{0+}\frac{ds}{k(s)}=+\infty$. Let $q(.)$ be a Borel measurable bounded non-negative function defined on $[0,T]$ satisfying $q(t)\le q_0+\int_0^t v(s)k(q(s))ds),\ \ t\in[0,T]$ where $q_0>0$ and $v(.)$ is a non-negative integrable function on $[0,T]$. Then we have $q(t)\le G^{-1}(G(q_0)+\int_0^t v(s)ds)$, where $G(t)=\int_{t_0}^t\frac{ds}{k(s)}$ is well defined for some $t_0>0$, and G^{-1} is the inverse function of G. In particularly, if $q_0=0$, then $q(t)=0$ for all $t\in[0,T]$.

In what follows, $C>0$ is a constant which can change its value from line to line.

Lemma 3.4. Let Assumptions 2.5-2.7 hold. Then for any $T\ge 0$,

$$E|y_\varepsilon(t)|^2\le C,\quad \forall\ t\in[0,T]. \tag{10}$$

Proof. By the Itô formula, we have

$$\begin{aligned} |y_\varepsilon(t)|^2 &= |x_0|^2+\int_0^t 2\langle y_\varepsilon(s),\varepsilon\bar{f}(y_\varepsilon(s))\rangle ds-\int_0^t 2\langle y_\varepsilon(s),\varepsilon d\bar{K}(s)\rangle \\ &+ \int_0^t 2\langle y_\varepsilon(s),\sqrt{\varepsilon}\bar{g}(y_\varepsilon(s))\rangle dw(s)+\int_0^t\varepsilon||\bar{g}(y_\varepsilon(s))||^2ds \\ &+ \int_0^t\int_Z\Big(|y_\varepsilon(s)+\sqrt{\varepsilon}\bar{h}(y_\varepsilon(s),v)|^2-|y_\varepsilon(s)|^2\Big)N(ds,dv). \end{aligned} \tag{11}$$

As we stated above, we can obtain that Eq.(7) has a unique solution $(y_\varepsilon(t), \bar{K}(t))$. Choose $\alpha = 0$ and $\beta = A^\circ(0)$, then we have

$$\langle y_\varepsilon(t) - 0, d\bar{K}(t) - A^\circ(0)dt\rangle \geq 0$$

for any $t \in [0, T]$. So

$$\begin{aligned} & -2\int_0^t \langle y_\varepsilon(s), \varepsilon d\bar{K}(s)\rangle \\ = & -2\varepsilon\int_0^t \langle y_\varepsilon(s) - 0, d\bar{K}(s) - A^\circ(0)ds\rangle - 2\varepsilon\int_0^t \langle y_\varepsilon(s), A^\circ(0)ds\rangle \\ \leq & \ 2\varepsilon|A^\circ(0)|\int_0^t |y_\varepsilon(s)|ds \leq \varepsilon\int_0^t (|y_\varepsilon(s)|^2 + |A^\circ(0)|^2)ds. \end{aligned} \tag{12}$$

For any $t_1 \in [0, T]$, inserting (12) into (11) and taking the expectation, one gets

$$\begin{aligned} & E\sup_{0\leq t\leq t_1} |y_\varepsilon(t)|^2 \\ \leq & \ |y(0)|^2 + \varepsilon_0|A^\circ(0)|^2 T + E\int_0^{t_1} 2\langle y_\varepsilon(s), \varepsilon\bar{f}(y_\varepsilon(s))\rangle ds + E\int_0^{t_1} |y_\varepsilon(s)|^2 ds \\ + & \ \varepsilon E\sup_{0\leq t\leq t_1}\int_0^t 2\langle y_\varepsilon(s), \sqrt{\varepsilon}\bar{g}(y_\varepsilon(s))\rangle dw(s) + E\int_0^{t_1} \varepsilon||\bar{g}(y_\varepsilon(s))||^2 ds \\ & +E\sup_{0\leq t\leq t_1}\int_0^t\int_Z \Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s), v)| + \varepsilon|\bar{h}(y_\varepsilon(s), v)|^2\Big)N(ds, dv). \end{aligned} \tag{13}$$

Using the basic inequality $2ab \leq a^2 + b^2$ and Assumption 2.6, we have

$$\begin{aligned} & E\int_0^{t_1} 2\langle y_\varepsilon(s), \varepsilon\bar{f}(y_\varepsilon(s))\rangle ds \\ \leq & \ 2\varepsilon E\int_0^{t_1} |y_\varepsilon(s)||\bar{f}(y_\varepsilon(s))|ds \\ \leq & \ \varepsilon E\int_0^{t_1} |y_\varepsilon(s)|^2 ds + \varepsilon E\int_0^{t_1} |\bar{f}(y_\varepsilon(s))|^2 ds \\ \leq & \ \varepsilon E\int_0^{t_1} |y_\varepsilon(s)|^2 ds + \varepsilon L_1 E\int_0^{t_1} (1 + |y_\varepsilon(s)|^2)ds. \end{aligned} \tag{14}$$

By the Burkholder-Davis-Gundy's inequality, the Young inequality and Assumption 2.6, we obtain

$$\begin{aligned} & E\sup_{0\leq t\leq t_1}\int_0^t 2\langle y_\varepsilon(s), \sqrt{\varepsilon}\bar{g}(y_\varepsilon(s))\rangle dw(s) \\ \leq & \ 8\sqrt{\varepsilon}E[\int_0^{t_1} |y_\varepsilon(s)|^2||\bar{g}(y_\varepsilon(s))||^2 ds]^{\frac{1}{2}} \\ \leq & \ 8\sqrt{\varepsilon}E[\sup_{0\leq s\leq t_1} |y_\varepsilon(s)|^2\int_0^{t_1} ||\bar{g}(y_\varepsilon(s))||^2 ds]^{\frac{1}{2}} \\ \leq & \ \frac{1}{4}E\sup_{0\leq s\leq t_1}(1 + |y_\varepsilon(s)|^2) + C\sqrt{\varepsilon}L_1 E\int_0^{t_1} (1 + |y_\varepsilon(s)|^2)ds. \end{aligned} \tag{15}$$

Now, we give the estimation for the last term of (13). Since $N(dt,dv)=\tilde{N}(dt,dv)+\pi(dv)dt$,

$$\begin{aligned}
& E\sup_{0\le t\le t_1}\int_0^t\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)N(ds,dv)\\
\le\ & E\int_0^{t_1}\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)\pi(dv)ds\\
+\ & E\sup_{0\le t\le t_1}\int_0^t\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)\tilde{N}(ds,dv).
\end{aligned}\tag{16}$$

Similar to (14), we have

$$\begin{aligned}
& E\int_0^{t_1}\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)\pi(dv)ds\\
\le\ & \pi(Z)\sqrt{\varepsilon}E\int_0^{t_1}|y_\varepsilon(s)|^2ds+(\sqrt{\varepsilon}+\varepsilon)L_1E\int_0^{t_1}(1+|y_\varepsilon(s)|^2)ds.
\end{aligned}\tag{17}$$

Let us estimate the second term of (16). By lemma 3.2 and the basic inequality, it follows that

$$\begin{aligned}
& E\sup_{0\le t\le t_1}\int_0^t\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)\tilde{N}(ds,dv)\\
\le\ & CE\Big[\int_0^{t_1}\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)^2N(ds,dv)\Big]^{\frac{1}{2}}\\
\le\ & CE\Big[\int_0^{t_1}\int_Z\Big(8\varepsilon|y_\varepsilon(s)|^2|\bar{h}(y_\varepsilon(s),v)|^2+2\varepsilon^2|\bar{h}(y_\varepsilon(s),v)|^4\Big)N(ds,dv)\Big]^{\frac{1}{2}}.
\end{aligned}\tag{18}$$

By the basic inequality $|a+b|^{\frac{1}{2}}\le|a|^{\frac{1}{2}}+|b|^{\frac{1}{2}}$, we get

$$\begin{aligned}
& E\sup_{0\le t\le t_1}\int_0^t\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)\tilde{N}(ds,dv)\\
\le\ & C\sqrt{\varepsilon}E\Big[\int_0^{t_1}\int_Z|y_\varepsilon(s)|^2|\bar{h}(y_\varepsilon(s),v)|^2N(ds,dv)\Big]^{\frac{1}{2}}\\
+\ & C\varepsilon E\Big[\int_0^{t_1}\int_Z|\bar{h}(y_\varepsilon(s),v)|^4N(ds,dv)\Big]^{\frac{1}{2}}\\
\le\ & C\sqrt{\varepsilon}E\Big[\sup_{0\le s\le t_1}|y_\varepsilon(s)|^2\int_0^{t_1}\int_Z|\bar{h}(y_\varepsilon(s),v)|^2N(ds,dv)\Big]^{\frac{1}{2}}\\
+\ & C\varepsilon E\Big[\sup_{0\le s\le t_1}(1+|y_\varepsilon(s)|)^2\int_0^{t_1}\int_Z\frac{|\bar{h}(y_\varepsilon(s),v)|^4}{(1+|y_\varepsilon(s)|)^2}N(ds,dv)\Big]^{\frac{1}{2}}.
\end{aligned}\tag{19}$$

Then, the Young inequality and Assumption 2.6 imply that

$$\begin{aligned}
& E\sup_{0\le t\le t_1}\int_0^t\int_Z\Big(2\sqrt{\varepsilon}|y_\varepsilon(s)||\bar{h}(y_\varepsilon(s),v)|+\varepsilon|\bar{h}(y_\varepsilon(s),v)|^2\Big)\tilde{N}(ds,dv)\\
\le\ & \frac{1}{4}E\sup_{0\le s\le t_1}|y_\varepsilon(s)|^2+C\sqrt{\varepsilon}E\int_0^{t_1}\int_Z|\bar{h}(y_\varepsilon(s),v)|^2N(ds,dv)\\
+\ & \frac{1}{8}E\sup_{0\le s\le t_1}(1+|y_\varepsilon(s)|)^2+C\varepsilon E\int_0^{t_1}\int_Z\frac{|\bar{h}(y_\varepsilon(s),v)|^4}{(1+|y_\varepsilon(s)|)^2}N(ds,dv)\\
\le\ & \frac{1}{2}E\sup_{0\le s\le t_1}(1+|y_\varepsilon(s)|^2)+C\sqrt{\varepsilon}L_2E\int_0^{t_1}(1+|y_\varepsilon(s)|^2)ds\\
+\ & C\varepsilon L_2\int_0^{t_1}(1+|y_\varepsilon(s)|^2)ds. \qquad (20)
\end{aligned}$$

Consequently,

$$\begin{aligned}
& E\sup_{0\le t\le t_1}(1+|y_\varepsilon(t)|^2)\\
\le\ & 4(1+|x_0|^2+\varepsilon_0|A^\circ(0)|^2T)+C(\sqrt{\varepsilon}+\varepsilon)E\int_0^{t_1}(1+|y_\varepsilon(s)|^2)ds\\
\le\ & 4(1+|x_0|^2+\varepsilon_0|A^\circ(0)|^2T)+C(\sqrt{\varepsilon}+\varepsilon)\int_0^{t_1}E\sup_{0\le s\le t}(1+|y_\varepsilon(s)|^2)dt,
\end{aligned}$$

where C is a positive constant dependent on L_1 and L_2.

Set $r(t)=4(1+|x_0|^2+\varepsilon_0|A^\circ(0)|^2T)e^{C(\sqrt{\varepsilon}+\varepsilon)t}$, then $r(.)$ is the solution of the following ordinary differential equation

$$r(t)\ =\ 4(1+|x_0|^2+\varepsilon_0|A^\circ(0)|^2T)+C(\sqrt{\varepsilon}+\varepsilon)\int_0^t r(s)ds.$$

By recurrence, it is easy to verify that

$$E\sup_{0\le t\le t_1}(1+|y_\varepsilon(t)|^2)\ \le\ r(t_1).$$

Since $r(t)$ is continuous and bounded on $[0,T]$, we have

$$E\sup_{0\le t\le T}(1+|y_\varepsilon(t)|^2)\le r(T)<+\infty,$$

where $r(T)=4(1+|x_0|^2+\varepsilon_0|A^\circ(0)|^2T)e^{C(\sqrt{\varepsilon}+\varepsilon)T}$. The proof is therefore complete. □

Remark 3.5. By lemma 3.4, we show that if the initial value x_0 are in L^2, then the solution of multivalued SDE with jumps will be in L^2.

Now, we present our main results which are used for revealing the relationship between the processes $x_\varepsilon(t)$ and $y_\varepsilon(t)$.

Theorem 3.6. *Let Assumptions 2.5-2.7 and 3.1 hold. Then, for any $T>0$,*

$$\lim_{\varepsilon\to 0}E\sup_{0\le t\le T}|x_\varepsilon(t)-y_\varepsilon(t)|^2=0. \qquad (21)$$

Proof. From (6) and (7), we have

$$\begin{aligned} x_\varepsilon(t) - y_\varepsilon(t) &= \varepsilon \int_0^t [f(s, x_\varepsilon(s)) - \bar{f}(y_\varepsilon(s))]ds - \varepsilon[K(t) - \bar{K}(t)] \\ &+ \sqrt{\varepsilon} \int_0^t [g(s, x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))]dw(s) \\ &+ \sqrt{\varepsilon} \int_0^t \int_Z [h(s, x_\varepsilon(s^-), v) - \bar{h}(y_\varepsilon(s^-), v)]N(ds, dv). \end{aligned}$$

By the Itô formula, we have

$$\begin{aligned} |x_\varepsilon(t) - y_\varepsilon(t)|^2 &= \int_0^t 2\varepsilon\langle x_\varepsilon(s) - y_\varepsilon(s), f(s, x_\varepsilon(s)) - \bar{f}(y_\varepsilon(s))\rangle ds \\ &- \int_0^t 2\varepsilon\langle x_\varepsilon(s) - y_\varepsilon(s), dK(s) - d\bar{K}(s)\rangle \\ &+ \int_0^t 2\sqrt{\varepsilon}\langle x_\varepsilon(s) - y_\varepsilon(s), g(s, x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))\rangle dw(s) \\ &+ \int_0^t \varepsilon||g(s, x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))||^2 ds \\ &+ \int_0^t \int_Z \Big(|x_\varepsilon(s) - y_\varepsilon(s) + \sqrt{\varepsilon}[h(s, x_\varepsilon(s^-), v) \\ &- \bar{h}(y_\varepsilon(s), v)]|^2 - |x_\varepsilon(s) - y_\varepsilon(s)|^2\Big) N(ds, dv). \end{aligned} \quad (22)$$

Obviously, Lemma 2.2 implies that

$$\int_0^t 2\varepsilon\langle x_\varepsilon(s) - y_\varepsilon(s), dK(s) - d\bar{K}(s)\rangle \geq 0.$$

Taking the expectation on both sides of (22), it follows that for any $u \in [0, T]$

$$\begin{aligned} & E \sup_{0\leq t\leq u} |x_\varepsilon(t) - y_\varepsilon(t)|^2 \\ \leq\ & 2\varepsilon E \int_0^u |x_\varepsilon(s) - y_\varepsilon(s)||f(s, x_\varepsilon(s)) - \bar{f}(y_\varepsilon(s))|ds \\ +\ & E \int_0^u \varepsilon||g(s, x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))||^2 ds \\ +\ & E \int_0^u \int_Z \Big(2\sqrt{\varepsilon}|x_\varepsilon(s) - y_\varepsilon(s)||h(s, x_\varepsilon(s^-), v) - \bar{h}(y_\varepsilon(s), v)|\Big)\pi(dv)ds \\ +\ & 2\sqrt{\varepsilon} E \sup_{0\leq t\leq u} \int_0^t \langle x_\varepsilon(s) - y_\varepsilon(s), g(s, x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))\rangle dw(s) \\ +\ & E \sup_{0\leq t\leq u} \int_0^t \int_Z \Big(2\sqrt{\varepsilon}|x_\varepsilon(s) - y_\varepsilon(s)||h(s, x_\varepsilon(s^-), v) - \bar{h}(y_\varepsilon(s), v)|\Big)\tilde{N}(ds, dv) \\ +\ & E \sup_{0\leq t\leq u} \int_0^t \int_Z \varepsilon|h(s, x_\varepsilon(s^-), v) - \bar{h}(y_\varepsilon(s), v)|^2 N(ds, dv) = \sum_{i=1}^{6} Q_i. \end{aligned} \quad (23)$$

By the basic inequality $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, we can obtain

$$\begin{aligned} Q_1 &\leq \varepsilon E\int_0^u |x_\varepsilon(s)-y_\varepsilon(s)|^2 ds + \varepsilon E\int_0^u |f(s,x_\varepsilon(s)) - \bar{f}(y_\varepsilon(s))|^2 ds \\ &\leq \varepsilon E\int_0^u |x_\varepsilon(s)-y_\varepsilon(s)|^2 ds + 2\varepsilon E\int_0^u |f(s,x_\varepsilon(s)) - f(s,y_\varepsilon(s))|^2 ds \\ &+ 2\varepsilon E\int_0^u |f(s,y_\varepsilon(s)) - \bar{f}(y_\varepsilon(s))|^2 ds. \end{aligned}$$

Then, by Assumptions 2.5 and 3.1, we have

$$\begin{aligned} Q_1 &\leq \varepsilon E\int_0^u |x_\varepsilon(s)-y_\varepsilon(s)|^2 ds + \varepsilon E\int_0^u |f(s,x_\varepsilon(s)) - \bar{f}(y_\varepsilon(s))|^2 ds \\ &\leq \varepsilon E\int_0^u |x_\varepsilon(s)-y_\varepsilon(s)|^2 ds + 2\varepsilon E\int_0^u \lambda^2(s)k^2(|x_\varepsilon(s)-y_\varepsilon(s)|) ds \\ &+ 2\varepsilon u\psi_1(u)\Big(1+E|y_\varepsilon(s)|^2\Big). \end{aligned} \tag{24}$$

Similarly, we can deduce that

$$Q_2 \leq 2\varepsilon E\int_0^u \lambda^2(s)k^2(|x_\varepsilon(s)-y_\varepsilon(s)|) ds + 2\varepsilon u\psi_2(u)\Big(1+E|y_\varepsilon(s)|^2\Big) \tag{25}$$

and

$$\begin{aligned} Q_3 &\leq \sqrt{\varepsilon}\pi(Z) E\int_0^u |x_\varepsilon(s)-y_\varepsilon(s)|^2 ds + 2\sqrt{\varepsilon} E\int_0^u \lambda^2(s)\rho^2(|x_\varepsilon(s)-y_\varepsilon(s)|) ds \\ &+ 2\sqrt{\varepsilon} u\psi_3(u)\Big(1+E|y_\varepsilon(s)|^2\Big). \end{aligned} \tag{26}$$

Now, we estimate the term Q_4. By Assumptions 2.5, 3.1, the Burkholder-Davis-Gundy's inequality and the Young inequality, it follows that

$$\begin{aligned} Q_4 &\leq 8\sqrt{\varepsilon} E[\int_0^u |x_\varepsilon(s)-y_\varepsilon(s)| ||g(s,x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))||^2 ds]^{\frac{1}{2}} \\ &\leq 8\sqrt{\varepsilon} E[\sup_{0\leq s\leq u} |x_\varepsilon(s)-y_\varepsilon(s)|^2 \int_0^u ||g(s,x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))||^2 ds]^{\frac{1}{2}} \\ &\leq \frac{1}{4} E\sup_{0\leq s\leq u} |x_\varepsilon(s)-y_\varepsilon(s)|^2 + C\sqrt{\varepsilon} E\int_0^u ||g(s,x_\varepsilon(s)) - \bar{g}(y_\varepsilon(s))||^2 ds \\ &\leq \frac{1}{4} E\sup_{0\leq s\leq u} |x_\varepsilon(s)-y_\varepsilon(s)|^2 + C\sqrt{\varepsilon} u\psi_2(u)\Big(1+E|y_\varepsilon(s)|^2\Big) \\ &+ C\sqrt{\varepsilon} E\int_0^u \lambda^2(s)k^2(|x_\varepsilon(s)-y_\varepsilon(s)|) ds. \end{aligned} \tag{27}$$

Next, by Assumptions 2.5, 3.1, lemma 3.2 and the Young inequality, we compute that

$$\begin{aligned}
& Q_5 \\
\le\ & C\sqrt{\varepsilon}E\Big[\int_0^u\int_Z\Big(|x_\varepsilon(s)-y_\varepsilon(s)|^2|h(s,x_\varepsilon(s^-),v)-\bar{h}(y_\varepsilon(s),v)|^2\Big)N(ds,dv)\Big]^{\frac{1}{2}} \\
\le\ & C\sqrt{\varepsilon}E\Big[\sup_{0\le s\le u}|x_\varepsilon(s)-y_\varepsilon(s)|^2\int_0^u\int_Z|h(s,x_\varepsilon(s^-),v) \\
-\ & \bar{h}(y_\varepsilon(s),v)|^2N(ds,dv)\Big]^{\frac{1}{2}} \\
\le\ & \frac{1}{4}E\sup_{0\le s\le u}|x_\varepsilon(s)-y_\varepsilon(s)|^2+C\sqrt{\varepsilon}E\int_0^u\int_Z|h(s,x_\varepsilon(s^-),v) \\
-\ & \bar{h}(y_\varepsilon(s),v)|^2\pi(dv)ds \\
\le\ & \frac{1}{4}E\sup_{0\le s\le u}|x_\varepsilon(s)-y_\varepsilon(s)|^2+C\sqrt{\varepsilon}u\psi_3(u)\Big(1+E|y_\varepsilon(s)|^2\Big) \\
+\ & C\sqrt{\varepsilon}E\int_0^u\lambda^2(s)\rho^2(|x_\varepsilon(s)-y_\varepsilon(s)|)ds. \qquad (28)
\end{aligned}$$

It is worth noting that $\varepsilon|h(s,x_\varepsilon(s^-),v)-\bar{h}(y_\varepsilon(s),v)|^2\ge 0$, therefore by Assumption 2.5 and Lemma 3.2, we derive that

$$\begin{aligned}
Q_6 \ \le\ & \varepsilon E\int_0^u\int_Z|h(s,x_\varepsilon(s^-),v)-\bar{h}(y_\varepsilon(s),v)|^2N(ds,dv) \\
=\ & \varepsilon E\int_0^u\int_Z|h(s,x_\varepsilon(s^-),v)-\bar{h}(y_\varepsilon(s),v)|^2\pi(dv)ds \\
\le\ & 2\varepsilon E\int_0^u\lambda^2(s)\rho^2(|x_\varepsilon(s)-y_\varepsilon(s)|)ds+2\varepsilon u\psi_3(u)\Big(1+E|y_\varepsilon(s)|^2\Big). \qquad (29)
\end{aligned}$$

Combing with (23)-(29) together, we obtain

$$\begin{aligned}
E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \ \le\ & \sqrt{\varepsilon}(\pi(Z)+\sqrt{\varepsilon})E\int_0^u|x_\varepsilon(s)-y_\varepsilon(s)|^2ds \\
+\ & (C\sqrt{\varepsilon}+4\varepsilon)E\int_0^u\lambda^2(s)k^2(|x_\varepsilon(s)-y_\varepsilon(s)|)ds \\
+\ & [(C+2)\sqrt{\varepsilon}+2\varepsilon]E\int_0^u\lambda^2(s)\rho^2(|x_\varepsilon(s)-y_\varepsilon(s)|)ds \\
+\ & C(\varepsilon+\sqrt{\varepsilon})u\sum_{i=1}^3\psi_i(u)(1+E\sup_{0\le s\le u}|y_\varepsilon(s)|^2). \qquad (30)
\end{aligned}$$

By the Jensen inequality, this implies that

$$\begin{aligned}
& E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \\
\le\ & C(\sqrt{\varepsilon}+\varepsilon)\int_0^u\Big\{[(E|x_\varepsilon(s)-y_\varepsilon(s)|^2)^{\frac{1}{2}}]^2 \\
+\ & \lambda^2(s)k^2((E|x_\varepsilon(s)-y_\varepsilon(s)|^2)^{\frac{1}{2}})+\lambda^2(s)\rho^2((E|x_\varepsilon(s)-y_\varepsilon(s)|^2)^{\frac{1}{2}})\Big\}ds \\
+\ & C(\varepsilon+\sqrt{\varepsilon})u\sum_{i=1}^3\psi_i(u)(1+E\sup_{0\le s\le u}|y_\varepsilon(s)|^2). \qquad (31)
\end{aligned}$$

Letting $\gamma(x)=k^2(x^{\frac{1}{2}})+\rho^2(x^{\frac{1}{2}})+x$,

$$\begin{aligned}
& E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \\
\le\ & C(\sqrt{\varepsilon}+\varepsilon)\int_0^u\Big\{[1+\lambda^2(s)][(E|x_\varepsilon(s)-y_\varepsilon(s)|^2)^{\frac{1}{2}}]^2 \\
+\ & [1+\lambda^2(s)]k^2((E|x_\varepsilon(s)-y_\varepsilon(s)|^2)^{\frac{1}{2}})+[1+\lambda^2(s)]\rho^2((E|x_\varepsilon(s)-y_\varepsilon(s)|^2)^{\frac{1}{2}})\Big\}ds \\
+\ & C(\varepsilon+\sqrt{\varepsilon})u\sum_{i=1}^3\psi_i(u)(1+E\sup_{0\le s\le u}|y_\varepsilon(s)|^2) \\
\le\ & C(\sqrt{\varepsilon}+\varepsilon)\int_0^u[1+\lambda^2(s)]\gamma(E|x_\varepsilon(s)-y_\varepsilon(s)|^2)ds \\
+\ & C(\varepsilon+\sqrt{\varepsilon})u\sum_{i=1}^3\psi_i(u)(1+E\sup_{0\le s\le u}|y_\varepsilon(s)|^2). \qquad (32)
\end{aligned}$$

By lemma 3.4 and the boundedness of $\psi_i(u),\ i=1,2,3$, we have

$$\begin{aligned}
E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \ \le\ & C(\sqrt{\varepsilon}+\varepsilon)\int_0^u[1+\lambda^2(s)]\gamma(E|x_\varepsilon(s)-y_\varepsilon(s)|^2)ds \\
+\ & C(1+C)M(\varepsilon+\sqrt{\varepsilon})u. \qquad (33)
\end{aligned}$$

Obviously, $\gamma(x)$ is a nondecreasing function on R_+ and $\gamma(0)=0$. Moreover, we can obtain

$$\int_{0+}\frac{ds}{\gamma(s)}=\int_{0+}\frac{1}{k^2(s^{\frac{1}{2}})+\rho^2(s^{\frac{1}{2}})+s}ds=\int_{0+}\frac{u}{k^2(u)+\rho^2(u)+u^2}du=\infty.$$

For any $t_0>0$, setting $G(t)=\int_{t_0}^t\frac{ds}{\gamma(s)}$, it follows from lemma 3.3 that

$$\begin{aligned}
E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \ \le\ & G^{-1}\Big(G[C(1+C)M(\varepsilon+\sqrt{\varepsilon})u] \\
+\ & C(\sqrt{\varepsilon}+\varepsilon)(T+\int_0^T\lambda^2(s)ds)\Big). \qquad (34)
\end{aligned}$$

Noting that $C(1+C)M(\varepsilon+\sqrt{\varepsilon})u\to 0$ as $\varepsilon\to 0$. Recalling the condition $\int_{0+}\frac{ds}{\gamma(s)}=\infty$, we can conclude that

$$G[C(1+C)M(\varepsilon+\sqrt{\varepsilon})u]+C(\sqrt{\varepsilon}+\varepsilon)(T+\int_0^T\lambda^2(s)ds)\to-\infty.$$

On the other hand, because G is a strictly increasing function, then we obtain that G has an inverse function which is strictly increasing, and $G^{-1}(-\infty)=0$. That is,

$$G^{-1}\Big(G[C(1+C)M(\varepsilon+\sqrt{\varepsilon})u]+C(\sqrt{\varepsilon}+\varepsilon)(T+\int_0^T\lambda^2(s)ds)\Big)\to 0.$$

Consequently, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \ \le\ & \lim_{\varepsilon\to 0}G^{-1}\Big(G[C(1+C)M(\varepsilon+\sqrt{\varepsilon})T] \\
+\ & C(\sqrt{\varepsilon}+\varepsilon)(T+\int_0^T\lambda^2(s)ds)\Big)=0.
\end{aligned}$$

Therefore we complete the proof. □

Remark 3.7. From Theorem 3.6, we investigate the strong convergence (in moment-sense) of $x_\varepsilon(t)$ to the averaging solution $y_\varepsilon(t)$ defined by (7) under non-Lipschitz condition. In other words, as long as ε is sufficiently small, then $y_\varepsilon(t)$ and $x_\varepsilon(t)$ are close enough.

Remark 3.8. If jump term $h = \bar{h} = 0$, then equation (6) and (7) will become multi-valued SDEs which have been investigated by [2, 3, 6, 7, 8, 9, 32, 33]. Likewise, under our assumptions, we can show that the solution of the averaged multi-valued SDEs converges to that of the standard one. Moreover, even for the Wiener noise case, our result still seems to be new.

However, the order of convergence for Eq.(5) in finite time interval has not been provided by Theorem 3.6, we need to give another convergence result.

Theorem 3.9. *Let Assumptions 2.5-2.7 and 3.1 hold. For a given arbitrary small number $\delta_1 > 0$, there exist $L > 0$, $\varepsilon_1 \in (0, \varepsilon_0]$ and $\beta \in (0, 1)$ such that*

$$E|x_\varepsilon(t) - y_\varepsilon(t)|^2 \le \delta_1, \qquad \forall \ \ t \in [0, L\varepsilon^{\frac{1}{2}-\beta}], \tag{35}$$

for all $\varepsilon \in (0, \varepsilon_1]$.

In order to prove Theorem 3.9, we need the following lemma.

Lemma 3.10. Let $k_i(x)$, $i = 1, 2, 3$ be three concave nondecreasing functions on R_+ such that $k_i(0) = 0$. Then $\gamma(x) = \sum_{i=1}^3 k_i^2(x^{\frac{1}{2}})$ is a concave nondecreasing function on R_+.

Proof. Similar to lemma 2.2 in [40], we can obtain this lemma and omit its proof. □

Proof. **Proof of Theorem 3.9.** Letting $k_1(x) = k(x), k_2(x) = \rho(x), k_3(x) = x$. Then, by lemma 3.10, we can easily obtain that $\gamma(x)$ is a concave function on R_+. Hence, by the properties of the concave function, we can find a pair of positive constants a and b such that

$$\gamma(x) \le a + bx, \quad \text{for any} \ \ x \ge 0. \tag{36}$$

Therefore, (33) will become

$$\begin{aligned} E\sup_{0\le t\le u}|x_\varepsilon(t) - y_\varepsilon(t)|^2 \ \le \ & Cb(\sqrt{\varepsilon}+\varepsilon)\int_0^u [1+\lambda^2(s)]E\sup_{0\le s\le t}|x_\varepsilon(s) - y_\varepsilon(s)|^2 dt \\ + \ & Ca(\varepsilon+\sqrt{\varepsilon})u. \end{aligned} \tag{37}$$

Let

$$q(u) = E\sup_{0\le t\le u}|x_\varepsilon(t) - y_\varepsilon(t)|^2, \ q_0 = Ca(\varepsilon+\sqrt{\varepsilon})u,$$

and

$$v(t) = 1 + \lambda^2(t), \ k(t) = Cb(\sqrt{\varepsilon}+\varepsilon)t,$$

then it follows from lemma 3.3 that

$$E\sup_{0\le t\le u}|x_\varepsilon(t) - y_\varepsilon(t)|^2 \ \le \ G^{-1}\Big(G[Ca(\varepsilon+\sqrt{\varepsilon})u] + \int_0^u [1+\lambda^2(s)]ds\Big),$$

where $G(u) = \int_{u_0}^{u} \frac{1}{Cb(\sqrt{\varepsilon}+\varepsilon)t}dt = \frac{1}{Cb(\sqrt{\varepsilon}+\varepsilon)} ln(\frac{u}{u_0})$, $u_0 > 0$. Obviously, $G^{-1}(u) = u_0 e^{Cb(\sqrt{\varepsilon}+\varepsilon)u}$, $u > 0$. Hence,

$$\begin{aligned} E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 &\le u_0 e^{Cb(\sqrt{\varepsilon}+\varepsilon)\left[\frac{1}{Cb(\sqrt{\varepsilon}+\varepsilon)}ln(\frac{Ca(\varepsilon+\sqrt{\varepsilon})u}{u_0})+\int_0^u[1+\lambda^2(s)]ds\right]} \\ &\le [Ca(\varepsilon+\sqrt{\varepsilon})u]e^{Cb(\varepsilon+\sqrt{\varepsilon})\sup_{0\le s\le T}[1+\lambda^2(s)]u}. \end{aligned} \tag{38}$$

Choose $\beta \in (0,1)$ and $L > 0$ such that for every $t \in [0, L\varepsilon^{\frac{1}{2}-\beta}] \subseteq [0,T]$,

$$E\sup_{0\le t\le L\varepsilon^{\frac{1}{2}-\beta}}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \le NL\varepsilon^{1-\beta},$$

where $N = Ca(1+\sqrt{\varepsilon})e^{Cb\sup_{0\le s\le T}[1+\lambda^2(s)]L(\varepsilon^{\frac{3}{2}-\beta}+\varepsilon^{1-\beta})}$. Consequently, given any number δ_1, we can choose $\varepsilon_1 \in [0,\varepsilon_0]$ such that for each $\varepsilon \in [0,\varepsilon_1]$ and for $t \in [0, L\varepsilon^{\frac{1}{2}-\beta}]$,

$$E\sup_{0\le t\le L\varepsilon^{\frac{1}{2}-\beta}}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \le \delta_1.$$

The proof is therefore complete. □

Remark 3.11. By Theorem 3.9, we can obtain that the order of convergence for Eq.(5) is about $\varepsilon^{\frac{1}{2}-\beta}$. Moreover, we will use this theorem to show the convergence in probability between the processes $x_\varepsilon(t)$ and $y_\varepsilon(t)$.

Corollary 3.12. Let Assumptions 2.5-2.7 and 3.1 hold. For a given arbitrary small number $\delta_2 > 0$, there exist $\varepsilon_2 \in [0,\varepsilon_0]$ such that for all $\varepsilon \in (0,\varepsilon_2]$, we have

$$\lim_{\varepsilon\to 0} P(\sup_{0<t\le L\varepsilon^{\frac{1}{2}-\beta}}|x_\varepsilon(t)-y_\varepsilon(t)| > \delta_2) = 0,$$

where L and β are defined by Theorem 3.9.

Proof. By Theorem 3.9 and the Chebyshev inequality, for any given number $\delta_2 > 0$, we can obtain that

$$P(\sup_{0<t\le L\varepsilon^{\frac{1}{2}-\beta}}|x_\varepsilon(t)-y_\varepsilon(t)| > \delta_2) \le \frac{1}{\delta_2^2}E(\sup_{0<t\le L\varepsilon^{\frac{1}{2}-\beta}}|x_\varepsilon(t)-y_\varepsilon(t)|^2) \le \frac{NL\varepsilon^{1-\beta}}{\delta_2^2}.$$

Let $\varepsilon \to 0$, and the required result follows. □

4. **Examples.**

In this section, we will discuss two examples to illustrate our theory.

Example 4.1. Consider the linear multi-valued SDEs with jumps

$$\begin{aligned} dx_\varepsilon(t)+\varepsilon A(x_\varepsilon(t))dt &\ni \frac{1}{2}\varepsilon x_\varepsilon(t)dt + 4\sqrt{\varepsilon}cos^2(t)x_\varepsilon(t)dw(t) \\ &+ c\sqrt{\varepsilon}\int_Z v^2 x_\varepsilon(t)N(dt,dv), \end{aligned} \tag{39}$$

with initial data $x_\varepsilon(0) = x_0$. Here

$$f(t,x_\varepsilon(t)) = \frac{1}{2}\varepsilon x_\varepsilon(t), \quad g(t,x_\varepsilon(t)) = 4\sqrt{\varepsilon}cos^2(t)x_\varepsilon(t), \quad h(t,x_\varepsilon(t),v) = c\sqrt{\varepsilon}v^2 x_\varepsilon(t).$$

Let

$$\bar{f}(y_\varepsilon(t)) = \frac{1}{\pi}\int_0^\pi f(t,y_\varepsilon(t))dt = \frac{1}{2}\varepsilon y_\varepsilon(t), \bar{g}(y_\varepsilon(t)) = \frac{1}{\pi}\int_0^\pi g(t,y_\varepsilon(t))dt = 2\sqrt{\varepsilon}y_\varepsilon(t)$$

and

$$\bar{h}(y_\varepsilon(t), v) = \frac{1}{\pi}\int_0^\pi h(t, y_\varepsilon(t), v)dt = c\sqrt{\varepsilon}v^2 y_\varepsilon(t),$$

we obtain the corresponding averaged equation as follows

$$dy_\varepsilon(t) + \varepsilon\bar{A}(y_\varepsilon(t))dt \ni \frac{1}{2}\varepsilon y_\varepsilon(t)dt + 2\sqrt{\varepsilon}y_\varepsilon(t)dw(t) + c\sqrt{\varepsilon}\int_Z v^2 y_\varepsilon(t)N(dt, dv). \quad (40)$$

Obviously, Eq.(40) is also a linear multi-valued SDEs with jumps and its solution can be given by

$$y_\varepsilon(t) = \Phi(t)\Big(x_0 - \varepsilon\int_0^t \Phi^{-1}(s)d\bar{K}(s)\Big), \quad (41)$$

where

$$\Phi(t) \quad = \quad exp\Big\{-\frac{3}{2}\varepsilon t + 2\sqrt{\varepsilon}w(t) + \int_0^t\int_Z ln(1 + c\sqrt{\varepsilon}v^2)N(ds, dv)\Big\}.$$

It is easy to find that the conditions of Theorems 3.6, 3.9 and Corollary 3.12 are satisfied, then the solution of averaged multi-valued SDEs with jumps (41) will converge to that of the standard one (39) in the sense of mean square and in probability.

Example 4.2. Let us return to the multivalued SDEs with pure jumps (3). Consider the standard form of Eq.(3)

$$dx_\varepsilon(t) + \varepsilon A(x_\varepsilon(t))dt \ni \varepsilon f(t, x_\varepsilon(t))dt + \sqrt{\varepsilon}\int_Z vh(t, x_\varepsilon(t))N(dt, dv), \quad (42)$$

with initial data $x_\varepsilon(0) = x_0$. Here $f(t, x) = h(t, x) = sin(t)k(x)$ and $k(x)$ is defined as (4). Let

$$\bar{f}(y_\varepsilon(t)) = \frac{1}{\pi}\int_0^\pi f(t, y_\varepsilon(t))dt = \frac{2}{\pi}k(y_\varepsilon(t))$$

and

$$\bar{h}(y_\varepsilon(t)) = \frac{1}{\pi}\int_0^\pi h(t, y_\varepsilon(t))dt = \frac{2}{\pi}k(y_\varepsilon(t)),$$

we have the corresponding averaged equation

$$dy_\varepsilon(t) + \varepsilon\bar{A}(y_\varepsilon(t))dt \ni \varepsilon\frac{2}{\pi}k(y_\varepsilon(t))dt + \sqrt{\varepsilon}\frac{2}{\pi}\int_Z vk(y_\varepsilon(t))N(dt, dv). \quad (43)$$

Clearly, the coefficient $k(.)$ do not satisfy the Lipschitz condition and it is a concave nondecreasing continuous function on $[0, \infty]$ with $k(0) = 0$. In fact, Eq.(43) is not a linear stochastic equation and we can not obtain its analytic solution as Eq.(40) did.

However, by Theorem 3.6 and 3.9, we can show that the solution of averaged equation (43) will converge to that of standard one (42) in the sense of mean square and in probability. Let us consider the function $\gamma(x)$ of the inequality (32). By (4), we have that

$$\gamma(x) = \begin{cases} 2xln\frac{1}{\sqrt{x}} + x, & if\ x \le e^{-2}, \\ \frac{3}{2}x + e^{-1}\sqrt{x} + \frac{1}{2}e^{-2}, & if\ x > e^{-2}. \end{cases} \quad (44)$$

Obviously, $\gamma(x)$ is a nondecreasing continuous function on R_+ and $\gamma(0)=0$. Moreover, for any sufficiently small positive constant $\bar{\varepsilon}$, we have

$$\int_0^{\bar{\varepsilon}} \frac{ds}{\gamma(s)} = \int_0^{\bar{\varepsilon}} \frac{1}{2sln\frac{1}{\sqrt{s}}+s}ds = \infty.$$

Now, we will give the specific form of the function $G(t)$ in lemma 3.3. Choosing $t_0=\frac{1}{2}e^{-1}$, we compute that

$$G(t) = \int_{\frac{1}{2}e^{-1}}^{t} \frac{ds}{\gamma(s)} = \int_{\frac{1}{2}e^{-1}}^{t} \frac{1}{2sln\frac{1}{\sqrt{s}}+s}ds = ln\frac{2+ln2}{1-2ln\sqrt{t}}, \quad \text{for} \quad 0<t<e^{-2}. \tag{45}$$

When $0<t<e^{-2}$, $G'(t)=\frac{1}{(1-2ln\sqrt{t})t}>0$, so we have that G is a strictly increasing function on $0<t<e^{-2}$. Hence, we conclude that G has an inverse function

$$G^{-1}(t) = e^{1-(2+ln2)e^{-t}}, \quad t<ln(1+\frac{1}{2}ln2)$$

which is strictly increasing, and $G^{-1}(-\infty)=0$. By (34), it follows that

$$\begin{aligned}
& E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 \\
\le\ & G^{-1}\Big(G[C(1+C)M(\varepsilon+\sqrt{\varepsilon})u]+C(\sqrt{\varepsilon}+\varepsilon)(\frac{3}{2}T-\frac{1}{4}sin^2T)\Big) \\
=\ & e^{1-(1-2ln\sqrt{C(1+C)M(\varepsilon+\sqrt{\varepsilon})u})e^{-C(\sqrt{\varepsilon}+\varepsilon)(\frac{3}{2}T-\frac{1}{4}sin^2T)}}.
\end{aligned}$$

Note that as $\varepsilon\to 0$, $e^{1-(1-2ln\sqrt{C(1+C)M(\varepsilon+\sqrt{\varepsilon})u})e^{-C(\sqrt{\varepsilon}+\varepsilon)(\frac{3}{2}T-\frac{1}{4}sin^2T)}}\to 0$, we have

$$\lim_{\varepsilon\to 0} E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 = 0.$$

That is to say, we show that the solution of averaged equation (43) will converge to that of standard one (42) in the mean square sense. On the other hand, we may also give the order of the convergence in finite time interval. According to (44), we compute that

$$\gamma''(x) = \begin{cases} -\frac{1}{x}, & if\ x\le e^{-2}, \\ -\frac{1}{4}e^{-1}\frac{1}{x\sqrt{x}}, & if\ x>e^{-2}. \end{cases} \tag{46}$$

It is easy to see from (46) that $\gamma''(x)<0$ as $x>0$, then $\gamma(x)$ is a concave function. Moreover, using the basic inequality $ab\le\frac{1}{2}(a^2+b^2)$ and $lnx<x$ for any $x>0$, we obtain that $\gamma(x)\le 1+2x$ for $x\le e^{-2}$ and $\gamma(x)\le e^{-2}+2x$ for $x>e^{-2}$. Then, combine with these two cases, there exist two positive constants $a=1$ and $b=2$ such that

$$\gamma(x)\le 1+2x, \quad \text{for } x>0.$$

Consequently, the inequality (38) will become

$$\begin{aligned}
E\sup_{0\le t\le u}|x_\varepsilon(t)-y_\varepsilon(t)|^2 &\le (C(\varepsilon+\sqrt{\varepsilon})u)e^{2C(\varepsilon+\sqrt{\varepsilon})\sup_{0\le t\le T}[1+sin^2t]u} \\
&\le (C(\varepsilon+\sqrt{\varepsilon})u)e^{4C(\varepsilon+\sqrt{\varepsilon})u}.
\end{aligned}$$

If we choose $\beta=\frac{1}{4}$, $L=1$, then the order of the convergence is $\varepsilon^{\frac{1}{4}}$. That is, for every $t\in[0,\varepsilon^{\frac{1}{4}}]$, we have

$$E\sup_{0\le t\le\varepsilon^{\frac{1}{4}}}|x_\varepsilon(t)-y_\varepsilon(t)|^2\le N\varepsilon^{\frac{3}{4}},$$

where $N = C(1+\sqrt{\varepsilon})e^{4C(\varepsilon^{\frac{5}{4}}+\varepsilon^{\frac{3}{4}})}$. Finally, by Corollary 3.12, we can obtain that for any given number $\delta > 0$,

$$P(\sup_{0<t\leq L\varepsilon^{\frac{1}{4}}} |x_\varepsilon(t) - y_\varepsilon(t)| > \delta) \leq \frac{N\varepsilon^{\frac{3}{4}}}{\delta^2}.$$

Let $\varepsilon \to 0$, we have that the solution $y_\varepsilon(t)$ will converge to $x_\varepsilon(t)$ in probability.

Conclusions. In this paper, we discuss the averaging principle for the multivalued SDE with jumps under non-Lipschitz condition. By using the estimate for stochastic integral with respect to a Poisson random measure, the Bihari's inequality and the properties of the concave function, we prove that the solution of averaged multivalued SDE with jumps converges to that of the standard one in the sense of mean square and probability. Meantime, we also provide the order of convergence in finite time interval.

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