Corrections

Corrections to “On the Existence and Uniqueness of the Eigenvalue Decomposition of a Parahermitian Matrix”

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In [1], we stated that any positive semi-definite parahermitian matrix $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times M}$ that is analytic on an annulus containing at least the unit circle will admit a decomposition with analytic eigenvalues and analytic eigenvectors. In this note, we further qualify this statement, and define the class of matrices that fulfills the above properties yet does not admit an analytic EVD. We follow the notation in [1].

I. Rellich’s Analytic EVD on the Unit Circle

For a self-adjoint matrix $\boldsymbol{A}(x)$, $x \in \mathbb{R}$, Rellich [2] has shown that eigenvalues and eigenvectors exist that are analytic in x. If $\boldsymbol{R}(z)$ is evaluated on the unit circle, $z = \mathrm{e}^{\mathrm{j}\Omega}$, $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ is self-adjoint s.t. $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{R}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega})$. Further, $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ is 2π-periodic in Ω. In [1], we incorrectly assumed that the analytic eigenvalue decomposition (EVD) of $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ is also 2π-periodic, but provide the following correction below.

Theorem 1 (Analytic EVD on the unit circle): For a self-adjoint analytic $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$, Rellich’s EVD on the unit circle is given by

$$\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega/N})\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega/N})\boldsymbol{Q}^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega/N}), \tag{1}$$

where the diagonal $\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega/N})$ and unitary $\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega/N})$ can be analytic in Ω for some $N \in \mathbb{N}$.

Proof: The EVD of $\boldsymbol{R}(z)$ on the unit circle can be generally written as $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{U}(\Omega)\boldsymbol{\Gamma}(\Omega)\boldsymbol{U}^{\mathrm{H}}(\Omega)$, whereby Rellich [2] guarantees the existence of eigenvalues and eigenvectors in $\boldsymbol{\Gamma}(\Omega)$ and $\boldsymbol{U}(\Omega)$, respectively, that are analytic in $\Omega \in \mathbb{R}$ without making any assumption about their periodicity. We initially focus on the diagonal elements of $\boldsymbol{\Gamma}(\Omega)$, i.e. the analytic eigenvalues $\gamma_m(\Omega)$, $m = 1\ldots M$, only.

Let $\{\gamma_m(\Omega_0)\}$ be the set of M eigenvalues of $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_0})$ at a specific frequency Ω_0. Because of its 2π periodicity, $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega_0}) = \boldsymbol{R}(\mathrm{e}^{\mathrm{j}(\Omega_0+2\pi)})$, and the sets $\{\gamma_m(\Omega_0)|m \in \{1\ldots M\}\}$ and $\{\gamma_\mu(\Omega_0 + 2\pi)|\mu \in \{1\ldots M\}\}$ must contain the same values. Therefore $\gamma_m(\Omega_0) = \gamma_{\mu(m)}(\Omega_0 + 2\pi)$ for some $\mu(m) \in \{1\ldots M\}$, but $\mu(m) = m$ cannot be assumed. Inspecting segments of analytic eigenvalues $\gamma_m(\Omega)$ over a 2π interval, therefore we see that the end point of one eigenvalue segment, $\gamma_m(\Omega_0 + 2\pi)$, must coincide with the starting point of another eigenvalue segment, $\gamma_\mu(\Omega_0)$, for some μ.

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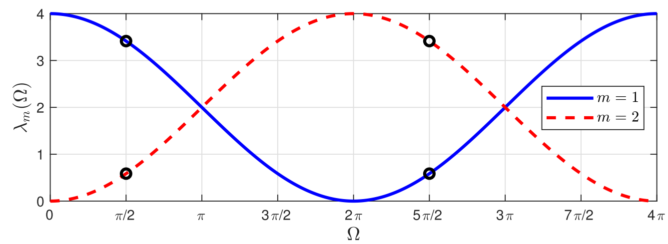

Fig. 1. Example for 4π-periodic analytic eigenvalues of $\boldsymbol{R}_1(\mathrm{e}^{\mathrm{j}\Omega})$.

Because of the above ‘chain’ rule, analytic eigenvalues must be functions that are shifted in frequency by at least $2\pi N$, for some $N \in \mathbb{N}$. Therefore, they are $2\pi N$-periodic and can be denoted as $\boldsymbol{\Lambda}(\mathrm{e}^{\mathrm{j}\Omega/N}) = \boldsymbol{\Gamma}(\Omega)$. Eigenvectors can only be analytic when eigenvalues are analytic, and due to this association have to exhibit the same periodicity, s.t. $\boldsymbol{Q}(\mathrm{e}^{\mathrm{j}\Omega/N}) = \boldsymbol{U}(\Omega)$. ■

Example 1: The matrix $\boldsymbol{R}_1(z) = [2, 1 + z^{-1}; z + 1, 2]$ from [3], [4] has the eigenvalues $\lambda_{1,2}(z) = 2 \pm (z^{1/2} + z^{-1/2})$. On the unit circle, $\lambda_{1,2}(\Omega) = 2 \pm 2\cos(\Omega/2)$ is 4π-periodic, as shown in Fig. 1. Note that values for e.g. $\Omega_0 = \frac{\pi}{2}$ and $\Omega_0 = \frac{\pi}{2} + 2\pi$ are identical but belong to different functions that are analytic in Ω.

An analytic continuation $z = \mathrm{e}^{\mathrm{j}\Omega}$ is only possible if $N = 1$; otherwise expressions in the variable $z^{1/N}$ result, which are not analytic. Therefore, in the case $N > 1$ an analytic EVD of $\boldsymbol{R}(z)$ does not exist. Nonetheless, it is possible to approximate non-analytic functions by Laurent polynomials; e.g. polynomial EVD algorithms in [8] will converge towards 2π-periodic, spectrally majorised eigenvalues for $\boldsymbol{R}_1(z)$.

II. Modulated Eigenvalues and Pseudo-Circulant Matrices

To characterise analytic matrices $\boldsymbol{R}(z)$ that do not admit an analytic EVD, we consider as a basic building block a parahermitian matrix $\boldsymbol{R}_\lambda(z) : \mathbb{C} \rightarrow \mathbb{C}^{N\times N}$, whose eigenvalues at an N-times oversampled rate are N frequency-shifted or modulated versions of a single function $\lambda(z)$,

$$\boldsymbol{\Lambda}(z) = \mathrm{diag}\Big\{\lambda(z), \lambda(z\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}}), \ldots, \lambda\big(z\mathrm{e}^{\mathrm{j}(N-1)\frac{2\pi}{N}}\big)\Big\}. \tag{2}$$

These eigenvalues will remain invariant under any paraunitary similarity transform. We are specifically interested in paraunitary matrices $\boldsymbol{W}(z)$ that yield

$$\boldsymbol{R}_\lambda(z^N) = \boldsymbol{W}(z)\boldsymbol{\Lambda}(z)\boldsymbol{W}^{\mathrm{P}}(z), \tag{3}$$

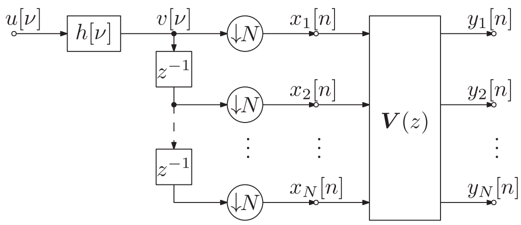

Fig. 2. Subband coding problem of finding a paraunitary $\boldsymbol{V}(z)$ to optimally compact the signals $y_i[n]$, $i = 1 \ldots N$ [6], [7].

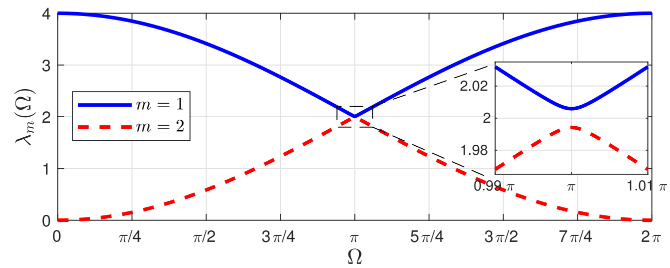

Fig. 3. Example for 2π-periodic analytic eigenvalues $\lambda_m(\Omega)$, $m = 1, 2$, algebraically calculated from the perturbed $\boldsymbol{R}_1(\mathrm{e}^{\mathrm{j}\Omega})$.

such that when undersampled by a factor N and evaluated on the unit circle, will possess a structure equivalent to (1). This is the case for $\boldsymbol{W}(z) = \boldsymbol{D}(z)\mathbf{T}$, with $\boldsymbol{D}(z) = \mathrm{diag}\{1, z^{-1}, \ldots z^{-N+1}\}$ and $\mathbf{T}$ an N-point DFT matrix scaled to be unitary [5], [6]: $\mathbf{T}$ creates a matrix $\mathbf{T}\boldsymbol{\Lambda}(z)\mathbf{T}^{\mathrm{H}}$, which is circulant and possesses elements that are N-times oversampled but time-shifted and thus offset against each other [7]; $\boldsymbol{D}(z)$ then removes these shifts such that the non-zero entries are aligned.

With the above choice for $\boldsymbol{W}(z)$, the parahermitian matrix $\boldsymbol{R}_\lambda(z)$ in (3) can be viewed as arising from the subband coding problem in Fig. 2. The signal $v[n]$ is modelled by an uncorrelated zero-mean and unit-variance noise process $u[n]$ and an innovation filter $h[n]$. For this problem, the cross-spectral density matrix $\boldsymbol{R}_\lambda(z)$ of the signals $x_i[n]$, $i = 1 \ldots N$, is pseudo-circulant [6], [7],

$$\boldsymbol{R}_\lambda(z) = \begin{bmatrix} \phi_0(z) & \phi_1(z) & \ldots & \phi_{N-1}(z) \\ z^{-1}\phi_{N-1}(z) & \phi_0(z) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z^{-1}\phi_1(z) & \ldots & z^{-1}\phi_{N-1}(z) & \phi_0(z) \end{bmatrix}. \quad (4)$$

The terms $\phi_n(z)$, $n = 0 \ldots (N-1)$, are the N polyphase components of the power spectral density (PSD) of $v[\nu]$. Based on the innovation filter $H(z) \bullet\!\!-\!\!\circ\, h[\nu]$ in Fig. 2, we therefore have $\lambda(z) = H(z)H^{\mathrm{P}}(z) = \sum_{\nu=0}^{N-1} \phi_\nu(z^N)z^{-\nu}$.

Thus any problem with modulated eigenvalues can be brought into pseudo-circulant form by a similarity transform (3). Conversely, any pseudo-circulant matrix has modulated eigenvalues. Hence (3) and (4) are equivalent. Therefore shifted eigenvalues are directly connected to pseudo-circulant matrices and the subband coding problem.

Example 2: The earlier example, $\boldsymbol{R}_1(z)$, arises from a subband coding problem for $N = 2$ with innovation filter $H(z) = 1 + z^{-1}$ and therefore PSD $\lambda(z) = z + 2 + z^{-1}$.

III. General Parahermitian Matrices

A parahermitian matrix $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M \times M}$ can potentially consist of blocks of pseudo-circulant matrices $\boldsymbol{R}_{\lambda_i}(z) : \mathbb{C} \rightarrow \mathbb{C}^{N_i \times N_i}$, $i = 1 \ldots I$, each of the type characterised in (4) and created by the structure in Fig. 2, originating from a single N_i-times oversampled eigenvalue $\lambda_i(z)$ as in (2),

$$\boldsymbol{R}(z) = \boldsymbol{V}(z)\mathrm{diag}\{\boldsymbol{R}_{\lambda_1}(z), \boldsymbol{R}_{\lambda_2}(z), \ldots, \boldsymbol{R}_{\lambda_I}(z)\}\, \boldsymbol{V}^{\mathrm{P}}(z). \quad (5)$$

Note that these eigenvalues remain invariant under a similarity transform by an arbitrary paraunitary $\boldsymbol{V}(z)$. The maximally possible fundamental period $2N_{\max}\pi$ of the analytic eigenvalues of any $\boldsymbol{R}(z)$, given its dimension M, when evaluated on the unit circle is

$$N_{\max} = \max_{\substack{N_i \in \mathbb{N} \\ i=1\ldots I}} \mathrm{lcm}\{N_1, N_2, \ldots N_I\} \quad \text{s.t.} \quad \sum_{i=1}^{I} N_i = M,$$

where $\mathrm{lmc}\{\cdot\}$ is the least common multiple of its arguments. Note that while the individual blocks $\boldsymbol{R}_{N_i}(z)$ are pseudo-circulant, the overall matrix $\boldsymbol{R}(z)$ likely loses this property due to both its block structure as well as the mixing by the paraunitary operation $\boldsymbol{V}(z)$.

Example 3: The matrix

$$\boldsymbol{R}_2(z) = \begin{bmatrix} 2 & -jz & j \\ jz^{-1} & 2 & -jz \\ -j & jz^{-1} & 2 \end{bmatrix}$$

is pseudo-circulant and can be tied to a subband coder according to Fig. 2 with $N = 3$ and $H(z) = 1 + jz^{-1}$ such that $\lambda(z) = -jz + 2 + jz^{-1}$. As a result, the eigenvalues $\lambda_m(\Omega)$ of $\boldsymbol{R}_2(\mathrm{e}^{\mathrm{j}\Omega})$, are 6π-periodic. For $\boldsymbol{R}_3(z) = \mathrm{diag}\{\boldsymbol{R}_1(z), \boldsymbol{R}_2(z)\}$, we have $N_1 = 2$, $N_2 = 3$ and therefore $N = N_{\max} = 6$, i.e. the eigenvalues will have the combined periodicity of 12π.

IV. Existence of Analytic Eigenvalues

In Section III we have established that for a parahermitian analytic matrix $\boldsymbol{R}(z)$ a paraunitary similarity transform $\boldsymbol{V}(z)$ exists such that (5) results with pseudo-circulant blocks on the main diagonal. If any of these I blocks has a dimension $N_i > 1$ (i.e. if $I < M$), then the eigenvalues $\lambda_m(z)$ associated with these blocks do not exist as analytic functions. This motivates an amendment to Theorem 3 in [1]:

Theorem 2 (Existence and Uniqueness of Eigenvalues of a Parahermitian Matrix EVD): For an analytic parahermitian matrix $\boldsymbol{R}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M \times M}$, eigenvalues $\lambda_m(z)$, $m = 1 \ldots M$, exist as absolutely convergent Laurent series if these are selected to be spectrally majorised on the unit circle. Analytic eigenvalues $\lambda_m(z)$ exist unless a paraunitary similarity transform exists that brings $\boldsymbol{R}(z)$ into the form (5) with pseudo-circulant blocks on the diagonal, with at least one of a dimension greater than one. For cases where analytic eigenvalues do not exist, a spectrally majorised solution can be found for the eigenvalues, which is absolutely convergent on the unit circle.

Proof: See [1] and above. ■

From a practical point of view, there currently is no algebraic mechanism to check condition (5). Therefore, only if $\boldsymbol{R}(z)$ arises from a subband coding-type application as depicted in Fig. 2 are we able to say that eigenvalues do not exist as analytic functions in z. Note that while analytic eigenvalues would be favoured for almost all application of a parahermitian matrix EVD, it is subband coding which requires spectrally majorised eigenvalues in order to maximise the coding gain [6], [7].

In terms of impact, in subband coding, its pseudo-circulant structure is ideally exploited when estimating $\boldsymbol{R}(z)$ [6]. If the pseudo-circulant property is not enforced for an estimate $\hat{\boldsymbol{R}}(z)$, then estimation errors will likely mask this property, such that $\hat{\boldsymbol{R}}(\mathrm{e}^{\mathrm{j}\Omega})$ will subsequently possess 2π-periodic eigenvalues on the unit circle, and can have eigenvalues that are analytic in z.

Example 4: If $\boldsymbol{R}_1(z)$ is perturbed by a parahermitian error term at -100dB, then the eigenvalues of this perturbed, no longer pseudo-circulant system, as shown in Fig. 3, are 2π-periodic on the unit circle, and now exist as analytic functions in z.

V. Conclusion

In this note, w.r.t. [1], a corrected condition for the existence of analytic eigenvalues of a parahermitian matrix $\boldsymbol{R}(z)$ has been derived. The main results in [1] on the existence of analytic eigenvalues and eigenvectors still hold; however, analytic eigenvalues do not exist if $\boldsymbol{R}(z)$ can be brought into block-diagonal form containing pseudo-circulant blocks by means of a paraunitary similarity transform.

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