Stability equivalence between the stochastic differential delay equations driven by G-Brownian motion and the Euler-Maruyama method

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Abstract

Consider a stochastic differential delay equation driven by G-Brownian motion (G-SDDE)

$$dx(t) = f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t) + h(x(t), x(t-\tau))d\langle B\rangle(t).$$

Under the global Lipschitz condition for the G-SDDE, we show that the G-SDDE is exponentially stable in mean square if and only if for sufficiently small step size, the Euler-Maruyama (EM) method is exponentially stable in mean square. Thus, we can carry out careful numerical simulations to investigate the exponential stability of the underlying G-SDDE in practice, in the absence of an appropriate Lyapunov function. A numerical example is provided to illustrate our results.

Keywords: Mean square stability, G-SDDE, EM method, Stability equivalence, G-Simulation.

1. Introduction

Motivated by mathematical finance problems with Knightian uncertainty, Peng has developed G-expectation and G-Brownian motion theory (see e.g. [1, 2]). Since then, stochastic differential equations driven by G-Brownian motion (G-SDEs) have received a great deal of concern due to the potential applications in uncertainty problems, risk measures and the superhedging in finance and so on (see e.g. [3, 4, 5]). In the framework of G-expectation (G-framework), many efforts have been made to investigate the stochastic stability, for example, the quasi sure stability [3], the moment stability [4], etc. It is known that a powerful tool for investigating the stochastic stability of the underlying systems is to apply the G-Lyapunov function technique (see e.g. [4, 6]). A natural problem is: in the absence of an appropriate G-Lyapunov function how do we judge the stochastic stability of the systems? Of course, we may use a numerical solution to approximate the exact solution of the corresponding system and then infer the stability of the system by the properties of the numerical solution. Now, we are faced with a key question: (Q) Does the stochastic stability of the numerical solution equivalent to that of the corresponding system?

If we can obtain a positive answer to this question, then it is feasible to judge the stochastic stability of the system from the careful numerical simulations. In the case where the SDEs are driven by the classical Brownian motion and stochastic stability means exponential stability in mean square sense, papers that answer question (Q) for SDEs, SDDEs and NSDDEs (neutral stochastic differential delay equations) can be found in [7], [8] and [9], respectively. However, in the case where the SDEs are driven by the G-Brownian motion, related papers on the stability equivalence are comparatively few and [10] is the only one, so far as we know, in which the authors showed that the stochastic θ-method is pth ($p \in (0, 1)$) moment exponentially stable for sufficiently small step size if and only if the corresponding G-SDE is also pth ($p \in (0, 1)$) moment exponentially stable under the global Lipschitz assumption.

Inspired by the aforementioned works, in this paper, we aim to study the stability equivalence between the G-SDDE and the corresponding numerical method in the sense of exponential mean square. In the G-framework, this issue is more difficult to be dealt with than SDEs, due to the non-linearity of G-expectation and distribution uncertainty of G-Brownian motion. We borrow the thought proposed by Mao in [7, 8] and apply the properties of

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G-Brownian motion to address the issue, but the computations involved to cope with time delay and integral with quadratic variation process of G-Brownian motion are nontrivial. The main contributions of this work are twofold. Firstly, we develop a numerical method for solving the G-SDDE. Secondly, we prove that in the G-framework, the mean square exponential stability of the EM numerical method is equivalent to that of the underlying system. We extend Mao's work [8] to the case of nonlinear expectation as well as Yang's results [10] to the case with delay.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. If A is a vector or matrix, its transpose is denoted by A^T, its norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$. If x is a real number, its integer part is denoted by $\lfloor x \rfloor$. Let $\tau > 0$ and $BC([-\tau, 0]; \mathbb{R}^n)$ denote the family of all bounded continuous $\mathbb{R}^n$-valued functions φ defined on $[-\tau, 0]$ to $\mathbb{R}^n$ with norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. For more details on the notions of G-expectation $\hat{\mathbb{E}}$ and G-Brownian motion on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, one can refer the reference [2]. Let $\mathcal{H}_t$ be a filtration generated by G-Brownian motion $\{B(t)\}_{t\geq 0}$. If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ which is regarded as a $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process. For $p \geq 1$ and $0 \leq T \leq +\infty$, define

$$
\begin{aligned}
L^p_{\mathcal{H}_0}([-\tau, 0]; \mathbb{R}^n) = \Big\{&\varphi : \varphi \text{ is } \mathcal{H}_0\text{-measurable, } BC([-\tau, 0]; \mathbb{R}^n)\text{-valued random variable,}\\
&\text{such that } \varphi \in M^p_G([-\tau, 0]; \mathbb{R}^n)\Big\},\\
L^p_{\mathcal{H}_t}([-\tau, T]; \mathbb{R}^n) = \Big\{&X : X \text{ is } \mathcal{H}_t\text{-measurable, continuous on } [-\tau, T], \text{ such that } X \in M^p_G([-\tau, T]; \mathbb{R}^n)\Big\}.
\end{aligned}
$$

For $x_t \in L^2_{\mathcal{H}_t}([-\tau, T]; \mathbb{R}^n)$, define $\|x_t\|^2_{\hat{\mathbb{E}}} = \sup_{-\tau\leq\theta\leq 0} \hat{\mathbb{E}}|x(t+\theta)|^2$.

Let $B(t)$ a one-dimensional G-Brownian motion with $G(a) := \frac{1}{2}\hat{\mathbb{E}}[aB(1)^2] = \frac{1}{2}(\underline{\sigma}^2 a^+ - \overline{\sigma}^2 a^-)$, for $a \in \mathbb{R}$, where $\overline{\sigma}^2 = \hat{\mathbb{E}}[B(1)^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B(1)^2]$, and $\langle B\rangle(t)$ be the quadratic variation process of the G-Brownian motion $B(t)$. By the properties of G-Brownian motion and the Hölder inequality, we obtain that for any $\eta \in M^2_G([-\tau, T]; \mathbb{R}^n)$

$$
\hat{\mathbb{E}}\left[\left|\int_0^T \eta_t d\langle B\rangle(t)\right|^2\right] \leq \overline{\sigma}^4 T \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^2 dt\right]. \tag{2.1}
$$

In this article, we consider the following G-SDDE

$$
dx(t) = f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t) + h(x(t), x(t-\tau))d\langle B\rangle(t), \quad t \geq 0, \tag{2.2}
$$

initial data $x_0 = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in L^2_{\mathcal{H}_0}([-\tau, 0]; \mathbb{R}^n)$, where $f, g, h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, as well as $f, g, h \in M^2_G([-\tau, T]; \mathbb{R}^n)$, $\forall T \geq 0$. We impose the following hypotheses:

Assumption 2.1. *(H1) Assume that f, g, h satisfy the global Lipschitz condition, that is, there exist positive constants L_1, L_2 and L_3 such that*

$$
\begin{aligned}
|f(x, y) - f(\bar{x}, \bar{y})|^2 &\leq L_1(|x - \bar{x}|^2 + |y - \bar{y}|^2),\\
|g(x, y) - g(\bar{x}, \bar{y})|^2 &\leq L_2(|x - \bar{x}|^2 + |y - \bar{y}|^2),\\
|h(x, y) - h(\bar{x}, \bar{y})|^2 &\leq L_3(|x - \bar{x}|^2 + |y - \bar{y}|^2),
\end{aligned}
$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$. For the purpose of stability study, we further assume that $f(0, 0) = g(0, 0) = h(0, 0) = 0$.

Under condition (H1), the G-SDDE (2.2) has a unique continuous solution on $t \geq -\tau$, see [11]. We denote this solution by $x(t; 0, \xi)$.

Definition 2.2. *The G-SDDE* (2.2) *is said to be exponentially stable in mean square if there are constants α and K such that for any initial data $\xi \in L^2_{\mathcal{H}_0}([-\tau, 0]; \mathbb{R}^n)$, $\hat{\mathbb{E}}|x(t; 0, \xi)|^2 \leq K\|\xi\|^2_{\hat{\mathbb{E}}} e^{-\alpha t}$, $\forall t \geq 0$. We refer to α as the rate constant and K as the growth constant.*

Given a step size $\Delta = \tau/m$ for a positive integer m. Let $t_k = k\Delta$ for $k \geq -m$. Then the discrete EM solution for G-SDDE (2.2) is defined by

$$
y(t_{k+1}) = y(t_k) + f(y(t_k), y(t_{k-m}))\Delta + g(y(t_k), y(t_{k-m}))\Delta B_k + h(y(t_k), y(t_{k-m}))\Delta\langle B\rangle_k, \ k \geq 0, \tag{2.3}
$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$, $\Delta\langle B\rangle_k = \langle B\rangle(t_{k+1}) - \langle B\rangle(t_k)$. Set $y(t_k) = \xi(t_k)$, for $-m \le k \le 0$. Define $z(t) = y(t_k)$, for $t \in [t_k, t_{k+1})$ with the initial value $z(t) = \xi(t)$ on $[-\tau, 0]$. We extend the discrete EM solution to the continuous one by the following

$$y(t) = y(0) + \int_0^t f(z(s), z(s-\tau))ds + \int_0^t g(z(s), z(s-\tau))dB(s) + \int_0^t h(z(s), z(s-\tau))d\langle B\rangle(s), \;\; t > 0. \tag{2.4}$$

Set $y(t) = \xi(t)$, for $-\tau \le t \le 0$. It is obvious that $y(t_k) = z(t_k)$. Let us now define exponential stability in mean square for the continuous EM method.

Definition 2.3. *Given a step size $\Delta = \tau/m$ for a positive integer m, the continuous EM method is said to be exponentially stable in mean square on the G-SDDE* (2.2)*, if there are constants β and H such that for any initial data $\xi \in L^2_{\mathcal{H}_0}([-\tau, 0]; \mathbb{R}^n)$, $\hat{\mathbb{E}}|y(t; 0, \xi)|^2 \le H\|\xi\|^2_{\hat{\mathbb{E}}} e^{-\beta t}$, $\forall t \ge 0$.*

3. Main results

In this section, we prove that the EM method shares exponential mean square stability with the G-SDDEs, and vice versa.

Theorem 3.1. *Under (H1), assume that the G-SDDE* (2.2) *is exponentially stable in mean square with rate constant α and growth constant K. Choose $\bar{\Delta}$ such that for $0 < \Delta \le \bar{\Delta}$,*

$$2(C(2T - 2\tau)\Delta + Ke^{-\alpha(T-2\tau)}) \le e^{-0.5\alpha T} \text{ and } 2(C(T-\tau)\Delta + K) \le 3K. \tag{3.1}$$

Then, for such Δ the EM method is exponentially stable in mean square with rate constant $\beta = 0.5\alpha$ and growth constant $H = 3KC_1e^{0.5\alpha T}$, both of which are independent of Δ , where $T = \tau(9 + \lfloor 4\log(2K)/(\tau\alpha)\rfloor)$, C_1 and $C(\cdot)$ were defined in Lemmas 3.4 and 3.7, respectively.

Theorem 3.2. *Under (H1), assume that the EM method on the G-SDDE* (2.2) *is exponentially stable in mean square with rate constant β and growth constant H. Choose Δ such that*

$$2(C(2T - 2\tau)\Delta + He^{-\beta(T-2\tau)}) \le e^{-0.5\beta T}. \tag{3.2}$$

Then, the G-SDDE (2.2) *is exponentially stable in mean square with rate constant $\alpha = 0.5\beta$ and growth constant $K = 2C_1e^{0.5\beta T}[C(T-\tau)\Delta + H]$, where $T = \tau(9 + \lfloor 4\log(2H)/(\tau\beta)\rfloor)$, C_1 and $C(\cdot)$ were defined in Lemmas 3.4 and 3.7, respectively.*

Based on the Theorem 3.1 and Theorem 3.2, we derive the following conclusion.

Theorem 3.3. *Under (H1), the G-SDDE* (2.2) *is exponentially stable in mean square if and only if for sufficiently small step size Δ, the EM method on the G-SDDE* (2.2) *is exponentially stable in mean square.*

To prove this theorem, we first need to show a number of lemmas.

Lemma 3.4. *Let (H1) hold, then*

$$\sup_{-\tau \le t \le \tau} \hat{\mathbb{E}}|y(t; 0, \xi)|^2 \le C_1\|\xi\|^2_{\hat{\mathbb{E}}}, \tag{3.3}$$

where $C_1 := 4[1 + \tau(\tau L_1 + \overline{\sigma}^2 L_2 + \overline{\sigma}^4\tau L_3)]e^{4(\tau L_1 + \overline{\sigma}^2 L_2 + \overline{\sigma}^4 \tau L_3)\tau}$.

Proof. Write $y(t; 0, \xi) = y(t)$. By the Hölder inequality, the Itô isometry, (2.1) and (H1), we have that for $0 \le t \le \tau$

$$\begin{aligned}\hat{\mathbb{E}}|y(t)|^2 &\le 4\hat{\mathbb{E}}|\xi(0)|^2 + 4\tau\hat{\mathbb{E}}\int_0^t |f(z(s), z(s-\tau))|^2 ds + 4\hat{\mathbb{E}}\int_0^t |g(z(s), z(s-\tau))|^2 d\langle B\rangle(s) + 4\hat{\mathbb{E}}\left|\int_0^t h(z(s), z(s-\tau))d\langle B\rangle(s)\right|^2 \\ &\le 4\|\xi\|^2_{\hat{\mathbb{E}}} + 4(\tau L_1 + \overline{\sigma}^2 L_2 + \overline{\sigma}^4 \tau L_3)\int_0^t (\hat{\mathbb{E}}|z(s)|^2 + \hat{\mathbb{E}}|z(s-\tau)|^2)ds.\end{aligned}$$

Now, for any $t_1 \in [0, \tau]$, we get

$$\sup_{0 \le t \le t_1} \hat{\mathbb{E}}|y(t)|^2 \le 4(1 + \tau(\tau L_1 + \overline{\sigma}^2 L_2 + \overline{\sigma}^4 \tau L_3))\|\xi\|^2_{\hat{\mathbb{E}}} + 4(\tau L_1 + \overline{\sigma}^2 L_2 + \overline{\sigma}^4 \tau L_3)\int_0^{t_1} \sup_{0 \le r \le s} \hat{\mathbb{E}}|y(r)|^2 ds.$$

Applying the Gronwall inequality and using that $\sup_{-\tau \le t \le t_1} \hat{\mathbb{E}}|y(t)|^2 \le \sup_{-\tau \le t \le 0} \hat{\mathbb{E}}|y(t)|^2 \vee \sup_{0 \le t \le t_1} \hat{\mathbb{E}}|y(t)|^2$, we obtain the desired assertion (3.3). □

Lemma 3.5. *Let (H1) hold, then*

$$\sup_{0\le t\le \tau+T} \hat{\mathbb{E}}|y(t;0,\xi)|^2 \le C_2\|\xi\|^2_{\hat{\mathbb{E}}},\ \textit{for}\ \forall T>0, \tag{3.4}$$

where $C_2 := C_2(T) = 4C_1 e^{8T(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)}$.

Proof. We write $y(t;0,\xi)=y(t)$ again. For $t\in[\tau,\tau+T]$, in the same fashion as in the proof of Lemma 3.4, we have

$$\begin{aligned}\hat{\mathbb{E}}|y(t)|^2 &\le 4\hat{\mathbb{E}}|y(\tau)|^2 + 4T\hat{\mathbb{E}}\int_\tau^t |f(z(s),z(s-\tau))|^2ds + 4\hat{\mathbb{E}}\int_\tau^t |g(z(s),z(s-\tau))|^2 d\langle B\rangle(s) + 4\hat{\mathbb{E}}\left|\int_\tau^t h(z(s),z(s-\tau))d\langle B\rangle(s)\right|^2\\ &\le 4C_1\|\xi\|^2_{\hat{\mathbb{E}}} + 4(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4\tau L_3)\int_\tau^t(\hat{\mathbb{E}}|z(s)|^2+\hat{\mathbb{E}}|z(s-\tau)|^2)ds.\end{aligned}$$

Hence, $\sup_{0\le s\le t}\hat{\mathbb{E}}|y(s)|^2 \le 4C_1\|\xi\|^2_{\hat{\mathbb{E}}} + 8(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4\tau L_3)\int_\tau^t \sup_{0\le r\le s}\hat{\mathbb{E}}|y(r)|^2ds$. The assertion (3.4) follows from the Gronwall inequality. □

Lemma 3.6. *Let (H1) hold, then for any* $T>0$,

$$\hat{\mathbb{E}}|y(t;0,\xi)-z(t;0,\xi)|^2 \le C_3\|\xi\|^2_{\hat{\mathbb{E}}}\Delta,\ \textit{for}\ \forall t\in[0,\tau+T], \tag{3.5}$$

where $C_3 := C_3(T) = 6(\tau L_1+\overline{\sigma}^2L_2+\overline{\sigma}^4\tau L_3)C_2(T)$.

Proof. Write $y(t;0,\xi)=y(t)$ and $z(t;0,\xi)=z(t)$. For any $t\in[0,\tau+T]$, there is a integer k such that $t\in[t_k,t_{k+1})$. Hence, by Lemma 3.5, we obtain

$$\begin{aligned}\hat{\mathbb{E}}|y(t)-z(t)|^2 &\le 3\Delta\hat{\mathbb{E}}\int_{t_k}^t |f(z(s),z(s-\tau))|^2ds + 3\hat{\mathbb{E}}\int_{t_k}^t |g(z(s),z(s-\tau))|^2 d\langle B\rangle(s) + 3\hat{\mathbb{E}}\left|\int_{t_k}^t h(z(s),z(s-\tau))d\langle B\rangle(s)\right|^2\\ &\le 3(\Delta L_1+\overline{\sigma}^2L_2+\overline{\sigma}^4\Delta L_3)\int_{t_k}^{t_{k+1}}(\hat{\mathbb{E}}|z(s)|^2+\hat{\mathbb{E}}|z(s-\tau)|^2)ds\\ &\le 6(\tau L_1+\overline{\sigma}^2L_2+\overline{\sigma}^4\tau L_3)C_2(T)\|\xi\|^2_{\hat{\mathbb{E}}}\Delta = C_3(T)\|\xi\|^2_{\hat{\mathbb{E}}}\Delta. \end{aligned}\tag{3.6}$$

Thus, we complete the proof. □

Lemma 3.7. *Write* $y(t)=y(t;0,\xi)$ *and define* $x(t)=x(t;\tau,y_\tau)$ *which is the solution to G-SDDE* (2.2) *with initial data* $y_\tau=\{y(\theta):0\le\theta\le\tau\}$ *at time* $t=\tau$. *Then*

$$\sup_{\tau\le t\le\tau+T}\hat{\mathbb{E}}|x(t)-y(t)|^2 \le C(T)\|\xi\|^2_{\hat{\mathbb{E}}}\Delta,\ \textit{for}\ \forall T>0, \tag{3.7}$$

where $C(T) := 3(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)(4T+\tau)C_3(T)e^{12T(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)}$.

Proof. For $\tau\le t\le\tau+T$, applying the Hölder inequality, the Itô isometry and (H1), we get that

$$\begin{aligned}\hat{\mathbb{E}}|x(t)-y(t)|^2 &\le 3T\hat{\mathbb{E}}\int_\tau^t |f(z(s),z(s-\tau))-f(x(s),x(s-\tau))|^2ds + 3\hat{\mathbb{E}}\int_\tau^t |g(z(s),z(s-\tau))-g(x(s),x(s-\tau))|^2d\langle B\rangle(s)\\ &\quad+3\hat{\mathbb{E}}\left|\int_\tau^t\Big(h(z(s),z(s-\tau))-h(x(s),x(s-\tau))\Big)d\langle B\rangle(s)\right|^2\\ &\le 3(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)\int_0^\tau \hat{\mathbb{E}}|x(s)-z(s)|^2ds + 6(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)\int_\tau^t\hat{\mathbb{E}}|x(s)-z(s)|^2ds.\end{aligned}$$

When $\tau\le s\le\tau+T$, using Lemma 3.6 gives

$$\hat{\mathbb{E}}|z(s)-x(s)|^2 \le 2\hat{\mathbb{E}}|z(s)-y(s)|^2 + 2\hat{\mathbb{E}}|x(s)-y(s)|^2 \le 2C_3(T)\|\xi\|^2_{\hat{\mathbb{E}}}\Delta + 2\hat{\mathbb{E}}|x(s)-y(s)|^2.$$

When $0\le s\le\tau$, we have $\hat{\mathbb{E}}|z(s)-x(s)|^2 = \hat{\mathbb{E}}|z(s)-y(s)|^2 \le C_3(T)\|\xi\|^2_{\hat{\mathbb{E}}}\Delta$. Hence, we obtain

$$\hat{\mathbb{E}}|x(t)-y(t)|^2 \le 3(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)C_3(T)(\tau+4T)\|\xi\|^2_{\hat{\mathbb{E}}}\Delta + 12(TL_1+\overline{\sigma}^2L_2+\overline{\sigma}^4TL_3)\int_\tau^t\hat{\mathbb{E}}|x(s)-y(s)|^2ds.$$

Using the Gronwall inequality, we obtain the desired assertion (3.7). □

Proof of Theorem 3.1 Fix any initial data ξ, write $y(t;0,\xi) = y(t)$ and define $x(t) = x(t;\tau, y_\tau)$. The exponential stability in mean square of the G-SDDE (2.2) means

$$\hat{\mathbb{E}}|x(t)|^2 \le K\|y_\tau\|_{\hat{\mathbb{E}}}^2 \mathrm{e}^{-\alpha(t-\tau)}, \text{ for } \forall t \ge \tau. \tag{3.8}$$

Letting $T = \tau(9 + \lfloor 4\log(2K)/(\tau\alpha)\rfloor)$ gives $2K\mathrm{e}^{-\alpha(T-2\tau)} \le \mathrm{e}^{-3/4\alpha T}$. Then, by the elementary inequality, we have

$$\hat{\mathbb{E}}|y(t)|^2 \le 2\hat{\mathbb{E}}|x(t) - y(t)|^2 + 2\hat{\mathbb{E}}|x(t)|^2. \tag{3.9}$$

By (3.7), we have

$$\sup_{T-\tau\le t\le 2T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le 2C(2T-2\tau)\Delta\|\xi\|_{\hat{\mathbb{E}}}^2 + 2K\|y_\tau\|_{\hat{\mathbb{E}}}^2\mathrm{e}^{-\alpha(T-2\tau)} \le R(\Delta) \sup_{-\tau\le t\le \tau} \hat{\mathbb{E}}|y(t)|^2, \tag{3.10}$$

where $R(\Delta) = 2(C(2T-2\tau)\Delta + K\mathrm{e}^{-\alpha(T-2\tau)})$. Note that $R(\Delta)$ is increasing with Δ and $R(0) = 2K\mathrm{e}^{-\alpha(T-2\tau)} \le \mathrm{e}^{-3/4\alpha T}$. We can choose a $\bar{\Delta} > 0$ satisfying $R(\Delta) \le \mathrm{e}^{-0.5\alpha T}$ for all $\Delta \le \bar{\Delta}$. Hence,

$$\sup_{T-\tau\le t\le 2T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le \mathrm{e}^{-0.5\alpha T} \sup_{-\tau\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2. \tag{3.11}$$

Similarly, using the flow property of the EM solutions, for $y(t) = y(t; jT, y_{jT})$, $j = 0, 1, 2, \cdots$, we repeat the procedure that (3.11) was obtained. Then, we also get that

$$\sup_{(j+1)T-\tau\le t\le (j+2)T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le \mathrm{e}^{-0.5\alpha T} \sup_{jT-\tau\le t\le jT+\tau} \hat{\mathbb{E}}|y(t)|^2 \le \mathrm{e}^{-0.5\alpha T} \sup_{jT-\tau\le t\le (j+1)T-\tau} \hat{\mathbb{E}}|y(t)|^2, \tag{3.12}$$

which implies

$$\sup_{(j+1)T-\tau\le t\le (j+2)T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le \mathrm{e}^{-0.5\alpha(j+1)T} \sup_{-\tau\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2. \tag{3.13}$$

By (3.7) and (3.8), we get that

$$\begin{aligned}\sup_{\tau\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2 &\le 2 \sup_{\tau\le t\le T-\tau} \hat{\mathbb{E}}|x(t) - y(t)|^2 + 2 \sup_{\tau\le t\le T-\tau} \hat{\mathbb{E}}|x(t)|^2 \\ &\le 2C(T-\tau)\Delta\|\xi\|_{\hat{\mathbb{E}}}^2 + 2K\|y_\tau\|_{\hat{\mathbb{E}}}^2 \le (2C(T-\tau)\Delta + 2K) \sup_{-\tau\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2.\end{aligned}$$

If we choose a $\bar{\Delta}$ such that $(2C(T-\tau)\Delta + 2K) \le 3K$ for all $\Delta \le \bar{\Delta}$, then $\sup_{\tau\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le 3K \sup_{-\tau\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2$. Inserting this into (3.13) and noting that $\sup_{-\tau\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2 = \sup_{-\tau\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2 \vee \sup_{\tau\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2$, we have

$$\sup_{(j+1)T-\tau\le t\le (j+2)T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le 3K\mathrm{e}^{-0.5\alpha(j+1)T} \sup_{-\tau\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2. \tag{3.14}$$

By (3.3), we get

$$\sup_{(j+1)T-\tau\le t\le (j+2)T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le 3KC_1\mathrm{e}^{-0.5\alpha(j+1)T}\|\xi\|_{\hat{\mathbb{E}}}^2, \ \forall j \ge 0.$$

Recalling that $M \ge 1$ and using Lemma 3.4, we obtain

$$\sup_{0\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2 = \sup_{0\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2 \vee \sup_{\tau\le t\le T-\tau} \hat{\mathbb{E}}|y(t)|^2 \le C_1\|\xi\|_{\hat{\mathbb{E}}}^2 \vee 3K \sup_{-\tau\le t\le\tau} \hat{\mathbb{E}}|y(t)|^2 \le 3KC_1\|\xi\|_{\hat{\mathbb{E}}}^2.$$

Hence, $\hat{\mathbb{E}}|y(t)|^2 \le 3KC_1\mathrm{e}^{0.5\alpha T}\|\xi\|_{\hat{\mathbb{E}}}^2\mathrm{e}^{-0.5\alpha t}$, which means that the EM method is exponentially stable in mean square sense with $\beta = 0.5\alpha$ and $H = 3KC_1\mathrm{e}^{0.5\alpha T}$. Thus, we complete the proof. □

In the similar way as Lemmas 3.4 and 3.7 were proved, we also have the following lemma.

Lemma 3.8. *Let (H1) hold, then*

$$\sup_{0\le t\le\tau} \hat{\mathbb{E}}|x(t;0,\xi)|^2 \le C_1\|\xi\|_{\hat{\mathbb{E}}}^2, \tag{3.15}$$

where C_1 is the same as before. Write $x(t;0,\xi) = x(t)$ and set $y(t) = y(t;\tau,x_\tau)$ which is the EM solution to the G-SDDE (2.2) *with initial data x_τ at $t = \tau$. Then,*

$$\sup_{\tau\le t\le\tau+T} \hat{\mathbb{E}}|x(t) - y(t)|^2 \le C(T)\|\xi\|_{\hat{\mathbb{E}}}^2\Delta, \ \textit{for } \forall T > 0, \tag{3.16}$$

where $C(T)$ was defined in Lemma 3.7.

Proof of Theorem 3.2 Write $x(t;0,\xi) = x(t)$ for simplicity and set $y(t) = y(t;\tau,x_\tau)$. If EM method is exponentially stable in mean square with rate constant β and growth constant H, namely,

$$\hat{\mathbb{E}}|y(t)|^2 \leq H\|y_\tau\|_{\hat{\mathbb{E}}}^2 \mathrm{e}^{-\beta(t-\tau)} \text{ for } \forall t \geq \tau.$$

Then, applying Lemma 3.8 and choosing Δ such that $2[C(2T-2\tau)\Delta + H\mathrm{e}^{-\beta(T-2\tau)}] \leq \mathrm{e}^{-0.5\beta T}$, we have

$$\sup_{T-\tau\leq t\leq 2T-\tau} \hat{\mathbb{E}}|x(t)|^2 \leq 2[C(2T-2\tau)\Delta + H\mathrm{e}^{-\beta(T-2\tau)}] \sup_{-\tau\leq t\leq\tau} \hat{\mathbb{E}}|x(t)|^2 \leq \mathrm{e}^{-0.5\beta T} \sup_{-\tau\leq t\leq\tau} \hat{\mathbb{E}}|x(t)|^2.$$

Repeating this procedure, we obtain that

$$\sup_{(j+1)T-\tau\leq t\leq(j+2)T-\tau} \hat{\mathbb{E}}|x(t)|^2 \leq \mathrm{e}^{-0.5(j+1)\beta T} \sup_{-\tau\leq t\leq T-\tau} \hat{\mathbb{E}}|x(t)|^2, \ \forall j \geq 0.$$

Applying Lemma (3.8) also gives

$$\sup_{-\tau\leq t\leq T-\tau} \hat{\mathbb{E}}|x(t)|^2 \leq 2C_1[C(T-\tau)\Delta + H]\|\xi\|_{\hat{\mathbb{E}}}^2.$$

Hence, we get $\hat{\mathbb{E}}|x(t)|^2 \leq 2C_1\mathrm{e}^{0.5\beta T}[C(T-\tau)\Delta + H]\mathrm{e}^{-0.5\beta t}\|\xi\|_{\hat{\mathbb{E}}}^2$, for $t \geq 0$, which means that the G-SDDE (2.2) is exponentially stable in mean square sense with $\alpha = 0.5\beta$ and $K = 2C_1\mathrm{e}^{0.5\beta T}[C(T-\tau)\Delta + H]$. Thus, we complete the proof. □

4. Numerical example

Example 4.1. *Let $B(t)$ be a scalar G-Brownian motion with $B(1) \sim \mathcal{N}(0,[1/5,2/5])$, $\tau = 0.1$. Define the initial data $x(t) = 1$, for $-\tau \leq t \leq 0$. Consider the scalar G-SDDE with the form:*

$$dx(t) = [-3x(t) + x(t-\tau)]dt + \frac{\sqrt{2}}{2}x(t)dB(t) + \sin x(t)d\langle B\rangle(t), \ t \geq 0. \quad (4.1)$$

One can easily verify that the G-SDDE (4.1) satisfies (H1). Setting Lyapunov function $V = |x|^2$, we compute

$$V_x f + G(2V_x h + V_{xx}|g|^2) = -6|x(t)|^2 + 2x(t)x(t-\tau) + G(4x(t)\sin x(t) + |x(t)|^2) \leq -4|x(t)|^2 + |x(t-\tau)|^2,$$

where $G(\cdot)$ was defined in Preliminaries. On one hand, according to Corollary 3.4 in [4], we have that

$$\hat{\mathbb{E}}|x(t)|^2 \leq \frac{c_2 + \tau e^{\gamma\tau}}{c_1}\|\xi\|_{\hat{\mathbb{E}}}^2 e^{-\gamma t} = 1.1309e^{-2.6912t},$$

where $c_1 = c_2 = 1, \tau = 0.1, \|\xi\|_{\hat{\mathbb{E}}} = 1, \gamma_1 = 4, \gamma_2 = 1$, and γ is the unique root to the equation $\gamma c_2 + e^{\gamma\tau}\gamma_2 = \gamma_1$. Hence, the G-SDDE (4.1) is exponentially stable in mean square with growth constant $K = 1.1309$ and rate constant $\alpha = 2.6912$. On the other hand, based on the EM scheme (2.3), we use the algorithm for simulating G-expectation from reference [12] to estimate $\hat{\mathbb{E}}|y(t)|^2$. Let $B(t) \sim \mathcal{N}(0,[\underline{\sigma}^2,\overline{\sigma}^2]t)$, we first construct an equidistant partition $\underline{\sigma} = \sigma_1 < \cdots < \sigma_i < \cdots \sigma_I = \overline{\sigma}$. For the i-th $(1 \leq i \leq I)$ round random sampling, $\xi_j^i(k)$ $(j = 1,2,\cdots J; k = 1,2,\cdots)$ is from the classical normal distribution $\mathcal{N}(0,\sigma_i^2\Delta)$. From (2.3), we define $y_j^i(t_k)$ by

$$y_j^i(t_{k+1}) = y_j^i(t_k) + (-3y_j^i(t_k) + y_j^i(t_{k-m}))\Delta + \frac{\sqrt{2}}{2}y_j^i(t_k)\xi_j^i(k) + \sin(y_j^i(t_k))\sigma_k^2\Delta, \ k \geq 0,$$
$$y_j^i(t_k) = 1, \ -\tau/\Delta \leq k \leq 0,$$

for $1 \leq i \leq I$, $1 \leq j \leq J$. Inspired by the idea of φ-max-mean in [13], we use the estimator

$$\mathbf{\Theta}|y(t_k)|^2 := \max_{1\leq i\leq I}\frac{1}{J}\sum_{j=1}^{J}|y_j^i(t_k)|^2, \ for\ k = 0,1,2,\cdots,$$

to approximate $\hat{\mathbb{E}}|y(t_k)|^2$. The operator $\mathbf{\Theta}$ is known as the maximum sample second moment. Now, taking $\Delta = 0.005$, $J = 500$ and $I = 20$, we give a simulation result plotted in Fig.1 on the evolution of the maximum sample second moment concerning EM solution $y(t)$ with time t. It seems that $\hat{\mathbb{E}}|y(t)|^2$ is decayed exponentially with time. Therefore, we further assume that an exponent law relation $\hat{\mathbb{E}}|y(t)|^2 = He^{-\beta t}$ exists for some constants H and β. A nonlinear fitting for H and β in least-squares sense gives that $H = 1.0090$ and $\beta = 2.8381$. We see from Fig.1

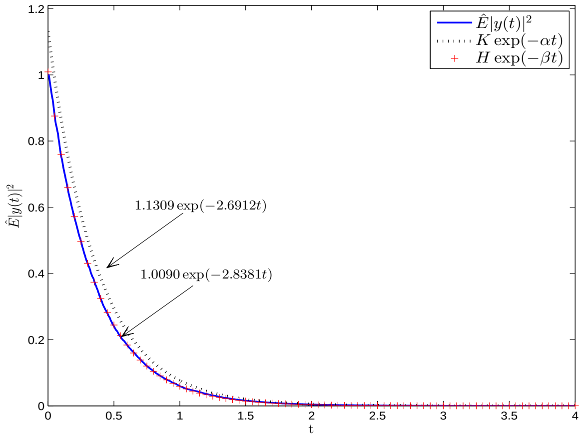

Fig. 1. Simulation results for G-SDDE (4.1)

that the two curves representing $\hat{\mathbb{E}}|y(t)|^2$ and $He^{-\beta t}$ respectively appear to fit well, suggesting that the equation $\hat{\mathbb{E}}|y(t)|^2 = He^{-\beta t}$ is valid and the EM method is exponentially stable in mean square. Hence, by Theorem 3.2, the G-SDDE (4.1) *is also exponentially stable in mean square with rate constant* 0.5β*. If we are interested only in the decay rate, then we can prove that for sufficiently small step size* Δ *the mean square exponential stability of EM method with rate constant* β *and growth constant H implies that the G-SDDE* (4.1) *has a rate constant bounded by* $\beta - \epsilon$*, which is close to the rate constant* α *obtained by the preceding method, where* ϵ *is an arbitrary constant in* $(0, \beta/2)$*. But the price we paid for this was an increase in the growth constant (see [7]). Fig.1 interprets the decay rate of G-SDDE* (4.1)*. It follows from this example that it is feasible to judge the mean square exponential stability of the G-SDDE by the careful numerical simulations for EM method under the given conditions.*

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