Analysis of a stochastic predator-prey system with foraging arena scheme

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Abstract

This paper focuses on a predator-prey system with foraging arena scheme incorporating stochastic noises. This SDE model is generated from a deterministic framework by the stochastic parameter perturbation. We then study how the correlations of the environmental noises affect the long-time behaviours of the SDE model. Later on the existence of a stationary distribution is pointed out under certain parametric restrictions. Numerical simulations are carried out to substantiate the analytical results.

Keywords: stochastic predator-prey model, Brownian motion, asymptotic pathwise estimation, stationary distribution

1 Introduction

A population model clarifies the mathematical relationships among consumer strategies and ecological generalities [1]. An essential element of the population models is called the functional response, which describes the density-dependent uptake response of consumers [2, 3]. There are different types of functional responses. For example, Holling type II response $\lambda_2(x) = u_1x/(u_2 + x)$, Holling type III response $\lambda_2(x) = u_1x^2/(u_2^2 + x^2)$ [4], the ratio-dependent response $\lambda_2(x, y) = u_1x/(u_2y + x)$ [2, 5–7] and foraging arena response $\lambda_2(x, y) = u_1x/(w + u_3y) = sx/(\beta + y)$ [3, 8, 9], where u_1 is a maximum uptake rate by the predator and u_2 is a prey half-saturation coefficient, $\beta = w/u_3$ = consumer density at half maximum per capita uptake rate and $u_1/w = s/\beta$ = maximum per capita uptake rate by predator. The two-dimensional foraging arena predator-prey model is in a form

$$\begin{aligned} d\bar{x}_1(t) &= \bar{x}_1(t)\Big(a - b\bar{x}_1(t) - \frac{s\bar{x}_2(t)}{\beta + \bar{x}_2(t)}\Big)dt, \\ d\bar{x}_2(t) &= \bar{x}_2(t)\Big(\frac{h\bar{x}_1(t)}{\beta + \bar{x}_2(t)} - c - f\bar{x}_2(t)\Big)dt, \end{aligned} \tag{1.1}$$

where $\bar{x}_1(t)$ and $\bar{x}_2(t)$ represent the population densities of prey and predator in model (1.1) at time t and a, b, s, β, h, c and f are all positive constants. More precisely, a is the intrinsic growth rate of prey, c is the density-dependent mortality rate of consumer, $h = \phi s$, b and f are the quadratic mortality rates of prey and predator respectively. We set $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))^T$ as the solution of model (1.1) with the initial value $\bar{x}_0 = (\bar{x}_1(0), \bar{x}_2(0))^T$. In model (1.1), there are two non-negative trivial equilibrium points $\bar{E}_0 = (0, 0)$ and $\bar{E}_1 = (\frac{a}{b}, 0)$. Also an unique interior equilibrium point $\bar{E}^*(\bar{x}_1^*, \bar{x}_2^*)$ with the nullclines

$$\begin{aligned} (a - b\bar{x}_1^*)(\beta + \bar{x}_2^*) &= s\bar{x}_2^*, \\ (\beta + \bar{x}_2^*)(c + f\bar{x}_2^*) &= h\bar{x}_1^* \end{aligned}$$

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exists and is globally asymptotically stable provided that $a > \frac{b\beta c}{h}$ [10].

The deterministic models have been widely applied to explain and predict the population dynamics (e.g.[11–13]), as well as for population management and conservation (e.g.[14–20]). However, such models have their limitations in dealing with biological populations in the real world. As a result an increasing number of researchers have been studying the stochastic population systems. Mao [21] found a surprising fact that the presence of even a tiny amount of environmental noise can suppress a potential population explosion in a classical n-dimensional Lotka-Volterra model. Mao [22] compared three types of delay Lotka-Volterra models and revealed their unique properties individually. Moreover, in [23] the conditions for the SDE Lotka-Volterra model having a stationary distribution were explored and a useful method was introduced to compute the mean and variance of the stationary distribution. Further studies on the n-dimensional Lotka-Volterra models can be found in e.g.[24, 25]. Holling type II model was also well studied by many authors, e.g.[26–29]. Especially, Ji et al.[26] considered a stochastic predator-prey model with modified Leslie-Gower and Holling type II schemes and produced conditions for the system to be either extinctive or persistent. [30] studied a three-dimensional stochastic ratio-dependent model with Markov switching and Holling type III functional response. The stochastic permanence was proved mainly using the M-matrix analysis introduced by [31] and applied in [24, 25]. In [32], the asymptotic behaviours of the predator-prey system with Beddington-DeAngelis response were investigated and the conditions of having a stationary distribution were produced. However to the best of our knowledge, there has not been much work about the foraging arena model incorporating the environmental variabilities. Throughout this paper, unless otherwise specified, we let $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions (i.e.it is right continuous and increasing while $\{\mathcal{F}_0\}$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_6(t))^T$ and $\breve{B}(t) = (B_1(t), \cdots, B_4(t))^T$ be six-dimensional and four-dimensional Brownian motions defined on this probability space respectively. The SDE models are then formulated as follows: Due to the environmental changes such as temperature and rainfall, we may stochastically perturb the parameters a, c, s and h in model (1.1) with

$$a \to a + \sigma_1 \dot{B}_1(t), \quad c \to c + \sigma_2 \dot{B}_2(t),$$
$$s \to s + r_1 \dot{B}_3(t) \quad \text{and} \quad h \to h + r_2 \dot{B}_4(t),$$

where σ_1, σ_2, r_1 and r_2 denote the intensities of the corresponding white noise. As a result the perturbed system is given by

$$dx_1(t) = x_1(t)\Big(a - bx_1(t) - \frac{sx_2(t)}{\beta + x_2(t)}\Big)dt + \sigma_1 x_1(t)dB_1(t) - \frac{r_1 x_1(t)x_2(t)}{\beta + x_2(t)}dB_3(t) \tag{1.2a}$$

$$dx_2(t) = x_2(t)\Big(\frac{hx_1(t)}{\beta + x_2(t)} - c - fx_2(t)\Big)dt - \sigma_2 x_2(t)dB_2(t) + \frac{r_2 x_1(t)x_2(t)}{\beta + x_2(t)}dB_4(t). \tag{1.2b}$$

Furthermore, we would also like to incorporate perturbation into b and f:

$$b \to b + \delta_1 \dot{B}_5(t) \quad \text{and} \quad f \to f + \delta_2 \dot{B}_6(t),$$

where δ_1 and δ_2 represent the intensities of the corresponding white noise. We then obtain

$$dx_1(t) = x_1(t)\Big(a - bx_1(t) - \frac{sx_2(t)}{\beta + x_2(t)}\Big)dt + \sigma_1 x_1(t)dB_1(t) - \frac{r_1 x_1(t)x_2(t)}{\beta + x_2(t)}dB_3(t) - \delta_1 x_1^2(t)dB_5(t) \tag{1.3a}$$

$$dx_2(t) = x_2(t)\Big(\frac{hx_1(t)}{\beta + x_2(t)} - c - fx_2(t)\Big)dt - \sigma_2 x_2(t)dB_2(t) + \frac{r_2 x_1(t)x_2(t)}{\beta + x_2(t)}dB_4(t) - \delta_2 x_2^2(t)dB_6(t). \tag{1.3b}$$

In ecology, the spatial synchrony often occurs in the population dynamics, resulting from a synchronous random environmental factors including temperature, rainfall and sunlight etc.[33–35]. Moran effect, known as the synchronizing effect of environmental stochasticity, has been observed in multiple population models. By taking Moran effect into account, the prey and predator populations in our systems (1.2) and (1.3) can be influenced by the same external factors [35]. This phenomena can be characterised by the correlations between the Brownian motions affecting different species (see e.g. [35–38]). On the other hand, some environmental factors such as a disease, temperature and pollution might simultaneously affect several system parameters of a species. Therefore the correlations between the Brownian motions affecting a specific population group are also considered. As a result, we let $B(t) = \varpi Z(t)$, where $Z(t) = (Z_1(t), \cdots, Z_6(t))^T$ is a six-dimensional independent standard Brownian motion and $\varpi^T \varpi = \mathcal{R} = (\rho_{ij})_{6\times 6}$ is a constant correlation matrix with $\rho_{ij} \in [-1, 1]$ represents the correlation coefficient between $B_i(t)$ and $B_j(t)$ for $i, j = 1, 2, \cdots, 6$. And $\tilde{B}(t)$ can be defined in the same way. We also denote

$$\bar{\rho}_{ij} = \begin{cases} \rho_{ij}, & \text{if } \rho_{ij} > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\rho}_{ij} = \begin{cases} 0, & \text{if } \rho_{ij} > 0 \\ -\rho_{ij}, & \text{otherwise.} \end{cases}$$

We set $x(t) = (x_1(t), x_2(t))^T$ as the solution of model (1.2) or model (1.3) representing the population densities of prey and predator at time t with the initial value $x_0 = (x_1(0), x_2(0))^T$. Let $\mathbb{R}^2_+$ be the positive cone in $\mathbb{R}^2$, that is $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\}$. We also set $\inf \emptyset = \infty$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is $|A| = \sqrt{\text{trace}(A^T A)}$ whilst its operator norm is $\|A\| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, its smallest and largest eigenvalue are denoted by $\lambda_{min}(A)$ and $\lambda_{max}(A)$. Consider the n-dimensional stochastic differential equation

$$dz(t) = \bar{f}(t)dt + \bar{g}(t)dw(t) \tag{1.4}$$

for $t \geqslant 0$, where $z(t) = (z_1(t), \cdots, z_n(t))^T$ and $w(t) = (w_1(t), \cdots, w_n(t))^T$ be an n-dimensional Brownian motion defined on the complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ be the family of all real-valued functions $V(z, t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in z and once in t. Given $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, define an operator $LV : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$LV(z, t) = V_t(z, t) + V_z(z, t)\bar{f}(t) + \frac{1}{2}\text{trace}(\bar{g}^T(t)V_{zz}(z, t)\bar{g}(t)),$$

which is called the diffusion operator of the Itô process (1.4) associated with the $C^{2,1}$-function V (see e.g.[31, p. 41]). With the diffusion operator, the Itô formula (1.4) can be written as

$$dV(z(t), t) = LV(z(t), t)dt + V_z(z(t), t)\bar{g}(t)dw(t) \quad \text{a.s.}$$

This paper is divided into four main parts. In the first two parts, the unique properties of model (1.2) and (1.3) are discussed respectively including the existence and uniqueness of the positive global solution, asymptotic moment estimate and some long-time behaviours of the two species. In the third part, the parametric restrictions for either model (1.2) or (1.3) to have a stationary distribution are studied. In the final part, computer simulations based on the Euler-Maruyama scheme are performed to illustrate our theory

2 Model (1.2)

2.1 Global positive solution

Theorem 2.1. *For any given initial value $x_0 \in \mathbb{R}^2_+$, there is a unique solution $x(t)$ to equation (1.2) on $t \geqslant 0$ and the solution will remain in $\mathbb{R}^2_+$ with probability 1, namely $x(t) \in \mathbb{R}^2_+$ for all $t \geqslant 0$ almost surely.*

By defining $V(x) = x_1^2 - 2\log x_1 + x_2^2 - 2\log x_2$, this theorem is then proved in the same routine as in [21, 39].

2.2 Asymptotic moment estimate

Theorem 2.2. *For any $\theta > 0$, there exists a positive constant $K(\theta)$ such that for any initial value $x_0 \in \mathbb{R}^2_+$, the solution of model* (1.2) *has the property that*

$$\limsup_{t\to\infty} \mathbb{E}|x(t)|^\theta \leqslant K(\theta).$$

Proof. Applying the Itô formula to $e^t(x_1^\theta(t) + x_2^\theta(t))$ for $\theta > 0$,

$$\begin{aligned} & e^t(x_1^\theta(t) + x_2^\theta(t)) \\ &= x_1^\theta(0) + x_2^\theta(0) + \int_0^t e^s f(x(s))ds + \theta\sigma_1 \int_0^t e^s x_1^\theta(s) dB_1(s) - \theta\sigma_2 \int_0^t e^s x_2^\theta(s) dB_2(s) \\ &- \theta r_1 \int_0^t \frac{e^s x_1^\theta(s) x_2(s)}{\beta + x_2(s)} dB_3(s) + \theta r_2 \int_0^t \frac{e^s x_2^\theta(s) x_1(s)}{\beta + x_2(s)} dB_4(s), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} f(x) &= \theta x_1^\theta\Big(a - bx_1 - \frac{sx_2}{\beta + x_2}\Big) + \theta x_2^\theta\Big(\frac{hx_1}{\beta + x_2} - c - fx_2\Big) + \frac{1}{2}\theta(\theta-1)x_1^\theta\Big(\sigma_1^2 + \frac{r_1^2 x_2^2}{(\beta + x_2)^2} \\ &- \frac{2\sigma_1 r_1 \rho_{13} x_2}{\beta + x_2}\Big) + \frac{1}{2}\theta(\theta-1)x_2^\theta\Big(\sigma_2^2 + \frac{r_2^2 x_1^2}{(\beta + x_2)^2} - \frac{2r_2\sigma_2\rho_{24}x_1}{\beta + x_2}\Big) + x_1^\theta + x_2^\theta. \end{aligned}$$

Using the elementary inequality

$$v_1^\kappa v_2^{1-\kappa} \leqslant \kappa v_1 + (1-\kappa)v_2 \quad \text{for } v_1, v_2 \geqslant 0 \text{ and } 0 \leqslant \kappa < 1,$$

for $\theta \geqslant 2$ we obtain

$$\frac{x_1 x_2^\theta}{\beta + x_2} \leqslant x_1 x_2^{\theta-1} \leqslant \frac{1}{\theta}x_1^\theta + \frac{\theta - 1}{\theta}x_2^\theta$$

and

$$\frac{x_1^2 x_2^\theta}{(\beta + x_2)^2} \leqslant x_1^2 x_2^{\theta-2} \leqslant \frac{2}{\theta}x_1^\theta + \frac{\theta-2}{\theta}x_2^\theta.$$

Hence

$$\begin{aligned} f(x) &\leqslant \Big(h + 1 + a\theta + (\theta-1)\Big(\frac{1}{2}\theta(\sigma_1^2 + r_1^2 + 2\sigma_1 r_1\tilde{\rho}_{13}) + r_2^2 + r_2\sigma_2\tilde{\rho}_{24}\Big)\Big)x_1^\theta + \Big(1 - c\theta \\ &+ (\theta - 1)\Big(h + \frac{1}{2}\theta\sigma_2^2 + \frac{1}{2}(\theta-2)r_2^2 + (\theta-1)r_2\sigma_2\tilde{\rho}_{24}\Big)\Big)x_2^\theta - b\theta x_1^{\theta+1} - f\theta x_2^{\theta+1}. \end{aligned}$$

is bounded, say by $K^*(\theta)$. Moreover, it follows from (2.1) that

$$\mathbb{E}\Big[e^{t\wedge\tau_k}\big(x_1^\theta(t\wedge\tau_k) + x_2^\theta(t\wedge\tau_k)\big)\Big] \leqslant x_1^\theta(0) + x_2^\theta(0) + K^*(\theta)\int_0^{t\wedge\tau_k} e^s ds.$$

Letting $k \to \infty$ and then $t \to \infty$ yields

$$\limsup_{t\to\infty} \mathbb{E}\big[x_1^\theta(t) + x_2^\theta(t)\big] \leqslant \lim_{t\to\infty}\frac{1}{e^t}\Big(x_1^\theta(0) + x_2^\theta(0) + K^*(\theta)(e^t - 1)\Big) = K^*(\theta).$$

On the other hand, we have

$$|x|^2 \leqslant 2(x_1^2 \vee x_2^2), \text{ so } |x|^\theta \leqslant 2^{\theta/2}(x_1^\theta \vee x_2^\theta) \leqslant 2^{\theta/2}(x_1^\theta + x_2^\theta).$$

As a result,

$$\limsup_{t\to\infty}\mathbb{E}|x(t)|^\theta \leqslant 2^{\theta/2}\limsup_{t\to\infty}\mathbb{E}[x_1^\theta(t) + x_2^\theta(t)] \leqslant 2^{\theta/2}K^*(\theta) = K(\theta). \tag{2.2}$$

For $0 < \theta < 2$, Hölder's inequality yields

$$\mathbb{E}|x(t)|^\theta \leqslant \big(\mathbb{E}|x(t)|^2\big)^{\frac{\theta}{2}}.$$

Hence from (2.2)

$$\limsup_{t\to\infty}\mathbb{E}|x(t)|^\theta \leqslant \limsup_{t\to\infty}(\mathbb{E}|x(t)|^2)^{\frac{\theta}{2}} \leqslant K(\theta).$$

□

2.3 Asymptotic pathwise estimation

In order to study the asymptotic properties of model (1.2), we first introduce a lemma.

Lemma 2.3. *For any initial value $x_0 \in \mathbb{R}^2_+$, the solution of model (1.2) has the property that*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_1^2(u)du \leqslant \frac{4a^2}{b^2} \quad a.s.$$

Proof. According to (1.2a),

$$x_1(t) = x_1(0) + a\int_0^t x_1(u)du - b\int_0^t x_1^2(u)du - s\int_0^t \frac{x_1(u)x_2(u)}{\beta + x_2(u)}du + m_1(t) + m_3(t) \tag{2.3}$$

where

$$m_1(t) = \sigma_1 \int_0^t x_1(u)dB_1(u) \quad \text{and} \quad m_3(t) = -r_1\int_0^t \frac{x_1(u)x_2(u)}{\beta + x_2(u)}dB_3(u)$$

are two continuous local martingales with the quadratic variations

$$\langle m_1(t)\rangle = \sigma_1^2\int_0^t x_1^2(u)du \quad \text{and} \quad \langle m_3(t)\rangle = r_1^2\int_0^t \frac{x_1^2(u)x_2^2(u)}{(\beta + x_2(u))^2}du \leqslant r_1^2\int_0^t x_1^2(u)du.$$

By the exponential martingle inequality, we have

$$\mathbb{P}\Big(\sup_{o\leqslant t\leqslant n}\big(m_i(t) - 0.5\alpha\langle m_i(t)\rangle\big) > \frac{2\log n}{\alpha}\Big) \leqslant \frac{1}{n^2} \quad \text{for } i = 1,3 \text{ and } n = 1,2,\cdots,$$

where

$$\alpha = \frac{b}{\sigma_1^2 + r_1^2}. \tag{2.4}$$

An application of the Borel-Cantelli lemma suggests that for almost all $\omega \in \Omega$ there is a random integer $n_0 = n_0(\omega) \geqslant 1$ such that

$$\sup_{o\leqslant t\leqslant n}\Big(m_i(t) - 0.5\alpha\langle m_i(t)\rangle\Big) \leqslant \frac{2\log n}{\alpha} \text{ whenever } n \geqslant n_0 \text{ for } i = 1,3.$$

Hence for $t \in [0,n]$ and $n \geqslant n_0$,

$$m_i(t) \leqslant \frac{2\log n}{\alpha} + 0.5\alpha\langle m_i(t)\rangle \quad \text{a.s.}$$

And then (2.3) and (2.4) imply that for $t \in [0,n]$ and $n \geqslant n_0$,

$$\begin{aligned}
x_1(t) &\leqslant x_1(0) + a\int_0^t x_1(u)du - \Big(b - 0.5\alpha(\sigma_1^2 + r_1^2)\Big)\int_0^t x_1^2(u)du + \frac{4\log n}{\alpha}\\
&= x_1(0) + a\int_0^t x_1(u)du - \frac{b}{2}\int_0^t x_1^2(u)du + \frac{4\log n}{\alpha} \quad \text{a.s.}
\end{aligned}$$

Therefore it follows that for $t \in [0,n]$ and $n \geqslant n_0$,

$$\begin{aligned}
\frac{b}{4}\int_0^t x_1^2(u)du &\leqslant x_1(0) + a\int_0^t x_1(u)du - \frac{b}{4}\int_0^t x_1^2(u)du + \frac{4\log n}{\alpha}\\
&\leqslant x_1(0) + \frac{a^2t}{b} + \frac{4\log n}{\alpha} \quad \text{a.s.}
\end{aligned}$$

Consequently, for almost all $\omega \in \Omega$, if $n \geqslant n_0$ and $n-1 \leqslant t \leqslant n$,

$$\frac{1}{t}\int_0^t x_1^2(u)du \leqslant \frac{4}{(n-1)b}\Big(x_1(0) + \frac{a^2n}{b} + \frac{4\log n}{\alpha}\Big).$$

Letting $t \to \infty$ and hence $n \to \infty$ we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_1^2(u)du \leqslant \frac{4a^2}{b^2} \quad \text{a.s.}$$

□

Theorem 2.4. *For any initial value $x_0 \in \mathbb{R}^2_+$,*
(a) if

$$2a < \sigma_1^2 - 2r_1\sigma_1\bar{\rho}_{13} \tag{2.5}$$

then both $x_1(t)$ and $x_2(t)$ of model (1.2) tend to zero exponentially as $t \to \infty$ with probability 1;

(b) if

$$\sigma_1^2 + 2r_1\sigma_1\tilde{\rho}_{13} < 2a < \sigma_1^2 - 2r_1\sigma_1\bar{\rho}_{13} + \frac{2b\beta c}{h + r_2\sigma_2\rho_{24}} + \frac{b\beta\sigma_2^2}{h + r_2\sigma_2\rho_{24}} \quad \textit{for } \rho_{24} > -\frac{h}{r_2\sigma_2} \tag{2.6}$$

or

$$2a > \sigma_1^2 + 2r_1\sigma_1\tilde{\rho}_{13} \quad \textit{for } -1 \leqslant \rho_{24} \leqslant -\frac{h}{r_2\sigma_2}, \tag{2.7}$$

then $x_1(t)$ of model (1.2) obeys

$$\frac{2a - \sigma_1^2}{2b} \leqslant \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du \leqslant \frac{2a - \sigma_1^2 + 2r_1\sigma_1\bar{\rho}_{13}}{2b} \quad \textit{a.s.}$$

and $x_2(t)$ tends to zero exponentially as $t \to \infty$ with probability 1.

Proof. (a) Applying Itô's formula on $\log x_1$ yields

$$\begin{aligned} d\log x_1(t) &= \Big(a - bx_1(t) - \frac{1}{2}\sigma_1^2 - \frac{sx_2(t)}{\beta + x_2(t)} - \frac{r_1^2 x_2^2(t)}{2(\beta + x_2(t))^2} + \frac{r_1\sigma_1\rho_{13}x_2(t)}{\beta + x_2(t)}\Big)dt + \sigma_1 dB_1(t) \\ &\quad - \frac{r_1 x_2(t)}{\beta + x_2(t)}dB_3(t) \\ &\leqslant \big(a - \frac{1}{2}\sigma_1^2 + r_1\sigma_1\bar{\rho}_{13}\big)dt + \sigma_1 dB_1(t) - \frac{r_1 x_2(t)}{\beta + x_2(t)}dB_3(t). \end{aligned} \tag{2.8}$$

Integrating from 0 to t and dividing by t infers

$$\frac{1}{t}\log x_1(t) \leqslant \frac{1}{t}\log x_1(0) + a - \frac{1}{2}\sigma_1^2 + r_1\sigma_1\bar{\rho}_{13} + \frac{M_1(t)}{t} + \frac{M_3(t)}{t},$$

where

$$M_1(t) = \sigma_1 B_1(t) \quad \text{and} \quad M_3(t) = -r_1\int_0^t \frac{x_2(u)}{\beta + x_2(u)}dB_3(u)$$

are two continuous martingales with the quadratic variations

$$\langle M_1(t)\rangle = \sigma_1^2 t \quad \text{and} \quad \langle M_3(t)\rangle = r_1^2\int_0^t \frac{x_2^2(u)}{(\beta + x_2(u))^2}dt \leqslant r_1^2 t.$$

By the strong law of large numbers for martingales [31, 40],

$$\lim_{t\to\infty}\frac{M_1(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{M_3(t)}{t} = 0 \quad \text{a.s.}$$

and thus from condition (2.5)

$$\limsup_{t\to\infty}\frac{1}{t}\log x_1(t) \leqslant a - \frac{1}{2}\sigma_1^2 + r_1\sigma_1\bar{\rho}_{13} < 0 \quad \text{a.s.}$$

as required. Therefore we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_1(u)du = 0 \quad \text{a.s.} \tag{2.9}$$

Meanwhile

$$d\log x_2(t) = \Big(\frac{h + r_2\sigma_2\rho_{24}}{\beta + x_2(t)}x_1(t) - c - \frac{\sigma_2^2}{2} - fx_2(t) - \frac{r_2^2x_1^2(t)}{2(\beta + x_2(t))^2}\Big)dt - \sigma_2 dB_2(t)$$
$$+ \frac{r_2x_1(t)}{\beta + x_2(t)}dB_4(t). \tag{2.10}$$

It follows that

$$\frac{\log x_2(t)}{t} \leqslant \frac{1}{t}\Big(\log x_2(0) + \frac{h + r_2\sigma_2\bar{\rho}_{24}}{\beta}\int_0^t x_1(u)du\Big) - \big(c + \frac{\sigma_2^2}{2}\big) + \frac{M_2(t)}{t} + \frac{M_4(t)}{t},$$

where

$$M_2(t) = -\sigma_2 B_2(t) \quad \text{and} \quad M_4(t) = r_2\int_0^t \frac{x_1(u)}{\beta + x_2(u)}dB_4(u)$$

are two martingales with the quadratic variations

$$\langle M_2(t)\rangle = \sigma_2^2 t \quad \text{and} \quad \langle M_4(t)\rangle = r_2^2\int_0^t \frac{x_1^2(u)}{(\beta + x_2(u))^2}du.$$

Hence from Lemma 2.3,

$$\limsup_{t\to\infty}\frac{\langle M_4(t)\rangle}{t} \leqslant \limsup_{t\to\infty}\frac{r_2^2}{\beta^2 t}\int_0^t x_1^2(u)du \leqslant \frac{4r_2^2a^2}{\beta^2b^2} \quad \text{a.s.}$$

By the strong law of large numbers for martingales,

$$\lim_{t\to\infty}\frac{M_2(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{M_4(t)}{t} = 0 \quad \text{a.s.}$$

Letting $t \to \infty$ and recalling equation (2.9) indicates

$$\limsup_{t\to\infty}\frac{\log x_2(t)}{t} \leqslant -\Big(c + \frac{\sigma_2^2}{2}\Big) < 0 \qquad \text{a.s.}$$

(b) Applying Itô's formula on $\frac{1}{x_1(t)}$ gives

$$d\Big(\frac{1}{x_1(t)}\Big) = \Big(\frac{1}{x_1}\Big(\frac{sx_2}{\beta + x_2} - a + \sigma_1^2 + \frac{r_1^2x_2^2}{(\beta + x_2)^2} - \frac{2\sigma_1 r_1\rho_{13}x_2}{\beta + x_2}\Big) + b\Big)dt - \frac{\sigma_1}{x_1}dB_1(t)$$
$$+ \frac{r_1x_2}{x_1(\beta + x_2)}dB_3(t),$$

where we write $x(t) = x$. Hence by the variation-of-constants formula (see e.g. [40, pp. 98-99]),

$$\begin{aligned}
\frac{1}{x_1(t)} &= \exp\Big(\int_0^t\Big(\frac{1}{2}\sigma_1^2 - a + \frac{sx_2(u)}{\beta + x_2(u)} + \frac{r_1^2x_2^2(u)}{2(\beta + x_2(u))^2} - \frac{2r_1\sigma_1\rho_{13}x_2(u)}{\beta + x_2(u)}\Big)du - M_1(t) \\
&\quad - M_3(t)\Big)\Big(\frac{1}{x_1(0)} + b\int_0^t \exp\Big(\int_0^u\Big(a - \frac{sx_2(v)}{\beta + x_2(v)} - \frac{1}{2}\sigma_1^2 - \frac{r_1^2x_2^2(v)}{2(\beta + x_2(v))^2} \\
&\quad + \frac{2r_1\sigma_1\rho_{13}x_2(v)}{\beta + x_2(v)}\Big)dv + M_1(u) + M_3(u)\Big)du\Big) \\
&= \exp\big(-M_1(t) - M_3(t)\big)\Big(\frac{1}{x_1(0)}\exp\Big(-\big(a - \frac{1}{2}\sigma_1^2\big)t + s\int_0^t \frac{x_2(u)}{\beta + x_2(u)}du \\
&\quad + \frac{r_1^2}{2}\int_0^t \frac{x_2^2(u)}{(\beta + x_2(u))^2}du - 2r_1\sigma_1\rho_{13}\int_0^t \frac{x_2(u)}{\beta + x_2(u)}du\Big) + b\int_0^t \exp\Big(-\big(a - \frac{1}{2}\sigma_1^2\big)(t-u) \\
&\quad + s\int_u^t \frac{x_2(v)}{\beta + x_2(v)}dv + \frac{r_1^2}{2}\int_u^t \frac{x_2^2(v)}{(\beta + x_2(v))^2}dv - 2r_1\sigma_1\rho_{13}\int_u^t \frac{x_2(v)}{\beta + x_2(v)}dv + M_1(u) \\
&\quad + M_3(u)\Big)du\Big).
\end{aligned} \tag{2.11}$$

On the one hand, (2.11) leads to

$$
\begin{aligned}
\frac{1}{x_1(t)} &\leqslant \exp\big(-M_1(t)-M_3(t)\big)\Big(\frac{1}{x_1(0)}\exp\Big(-(a-\frac{1}{2}\sigma_1^2)t+s\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\\
&+\frac{r_1^2}{2}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du+2r_1\sigma_1\tilde{\rho}_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)+b\exp\Big(\max_{0\leqslant u\leqslant t}M_1(u)+\max_{0\leqslant u\leqslant t}M_3(u)\\
&+s\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du+\frac{r_1^2}{2}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du+r_1\sigma_1\tilde{\rho}_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)\cdot\\
&\int_0^t\exp(-q(t-u))du\Big)\\
&\leqslant \exp\Big(\max_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\max_{0\leqslant u\leqslant t}M_3(u)-M_3(t)+s\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\\
&+\frac{r_1^2}{2}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du+r_1\sigma_1\tilde{\rho}_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)\Big(\frac{1}{x_1(0)}\exp(-qt)\\
&+b\int_0^t\exp(-q(t-u))du\Big)\\
&=\exp\Big(\max_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\max_{0\leqslant u\leqslant t}M_3(u)-M_3(t)+s\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\\
&+\frac{r_1^2}{2}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du+r_1\sigma_1\tilde{\rho}_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)\cdot K_1(t),
\end{aligned}
$$

where

$$q:=a-\frac{1}{2}\sigma_1^2-r_1\sigma_1\tilde{\rho}_{13}\quad\text{and}\quad K_1(t)=\frac{1}{x_1(0)}\exp(-qt)+\frac{2b\big(1-\exp(-qt)\big)}{2a-\sigma_1^2}.$$

Under condition (2.6) or (2.7), we obtain that $q>0$ and therefore $\sup_{0\leqslant t<\infty}K_1(t)<\infty$. It then follows that

$$
\begin{aligned}
\frac{\log x_1(t)}{t} &\geqslant -\frac{\log K_1(t)}{t}-\frac{\max_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\max_{0\leqslant u\leqslant t}M_3(u)-M_3(t)}{t}\\
&-\frac{s}{t}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du-\frac{r_1^2}{2t}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du-\frac{r_1\sigma_1\tilde{\rho}_{13}}{t}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du.
\end{aligned}
\tag{2.12}
$$

By (2.8) and (2.12),

$$
\begin{aligned}
\frac{1}{t}\int_0^t x_1(u)du &= \frac{2a-\sigma_1^2}{2b}-\frac{\log x_1(t)}{bt}+\frac{\log x_1(0)}{bt}-\frac{s}{bt}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\\
&-\frac{r_1^2}{2bt}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du+\frac{r_1\sigma_1\rho_{13}}{bt}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du+\frac{M_1(t)}{bt}+\frac{M_3(t)}{bt}\\
&\leqslant \frac{2a-\sigma_1^2}{2b}+\frac{\log K_1(t)}{bt}+\frac{\max_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\max_{0\leqslant u\leqslant t}M_3(u)-M_3(t)}{bt}\\
&+\frac{r_1\sigma_1\bar{\rho}_{13}}{b}+\frac{\log x_1(0)}{bt}+\frac{M_1(t)}{bt}+\frac{M_3(t)}{bt}.
\end{aligned}
\tag{2.13}
$$

As $t\to\infty$ and from the strong law of large numbers for martingales,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_1(u)du\leqslant\frac{2a-\sigma_1^2+2r_1\sigma_1\bar{\rho}_{13}}{2b}\quad\text{a.s.}\tag{2.14}$$

Assume that $\rho_{24}>-\frac{h}{r_2\sigma_2}$. From equation (2.10),

$$d\log x_2(t)\leqslant\Big(\frac{h+r_2\sigma_2\rho_{24}}{\beta}x_1(t)-c-\frac{\sigma_2^2}{2}\Big)dt-\sigma_2dB_2(t)+\frac{r_2x_1(t)}{\beta+x_2(t)}dB_4(t).$$

It is then followed from (2.14) and the strong law of large numbers for martingales that

$$\limsup_{t\to\infty}\frac{1}{t}\log x_2(t)\leqslant\frac{h+r_2\sigma_2\rho_{24}}{\beta}\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_1(u)du-\left(c+\frac{\sigma_2^2}{2}\right)$$
$$\leqslant\frac{(h+r_2\sigma_2\rho_{24})(2a-\sigma_1^2+2r_1\sigma_1\bar\rho_{13})}{2\beta b}-\left(c+\frac{\sigma_2^2}{2}\right)<0$$

in view of (2.6). If $\rho_{24}\leqslant-\frac{h}{r_2\sigma_2}$, it immediately indicates that

$$\limsup_{t\to\infty}\frac{1}{t}\log x_2(t)\leqslant-\left(c+\frac{\sigma_2^2}{2}\right)<0\quad\text{a.s.}$$

Hence for arbitrary small $\zeta>0$, there exists $t_\zeta(\omega)$ such that

$$\mathbb{P}(\Omega_\zeta)\geqslant1-\zeta\text{ where }\Omega_\zeta=\left\{\omega:\frac{(s+r_1\sigma_1|\rho_{13}|)x_2(t,\omega)}{b(\beta+x_2(t,\omega))}+\frac{r_1^2x_2^2(t,\omega)}{2b(\beta+x_2(t,\omega))^2}\leqslant\zeta\text{ for all }t\geqslant t_\zeta\right\}.$$

On the other hand, (2.11) yields

$$\begin{aligned}\frac{1}{x_1(t)}&\geqslant\exp\big(-M_1(t)-M_3(t)\big)\Big(\frac{1}{x_1(0)}\exp\Big(-\big(a-\frac12\sigma_1^2\big)t-2r_1\sigma_1\bar\rho_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)\\&+b\exp\Big(\min_{0\leqslant u\leqslant t}M_1(u)+\min_{0\leqslant u\leqslant t}M_3(u)-2r_1\sigma_1\bar\rho_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)\cdot\\&\int_0^t\exp\big(-\big(a-\frac12\sigma_1^2\big)(t-u)\big)du\Big)\\&\geqslant\exp\Big(\min_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\min_{0\leqslant u\leqslant t}M_3(u)-M_3(t)-2r_1\sigma_1\bar\rho_{13}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du\Big)\cdot K_2(t),\end{aligned}$$

where

$$K_2(t)=\frac{1}{x_1(0)}\exp\Big(-\big(a-\frac12\sigma_1^2\big)t\Big)+\frac{2b\Big(1-\exp\big(-(a-\frac12\sigma_1^2)t\big)\Big)}{2a-\sigma_1^2}$$

and $\sup_{0\leqslant t<\infty}K_2(t)<\infty$ if either condition (2.6) or (2.7) holds. Then

$$\begin{aligned}\frac{\log x_1(t)}{t}\leqslant&-\frac{\log K_2(t)}{t}-\frac{\min_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\min_{0\leqslant u\leqslant t}M_3(u)-M_3(t)}{t}\\&+\frac{2r_1\sigma_1\bar\rho_{13}}{t}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du.\end{aligned}$$

Hence we obtain from (2.13) that

$$\begin{aligned}\frac1t\int_0^t x_1(u)du\geqslant&\frac{2a-\sigma_1^2}{2b}+\frac{\log K_2(t)}{bt}+\frac{\min_{0\leqslant u\leqslant t}M_1(u)-M_1(t)+\min_{0\leqslant u\leqslant t}M_3(u)-M_3(t)}{bt}\\&+\frac{\log x_1(0)}{bt}-\frac{s}{bt}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du-\frac{r_1^2}{2bt}\int_0^t\frac{x_2^2(u)}{(\beta+x_2(u))^2}du\\&-\frac{r_1\sigma_1|\rho_{13}|}{bt}\int_0^t\frac{x_2(u)}{\beta+x_2(u)}du+\frac{M_1(t)}{bt}+\frac{M_3(t)}{bt}.\end{aligned}\tag{2.15}$$

For any $\omega\in\Omega_\zeta$, (2.15) indicates

$$\liminf_{t\to\infty}\frac1t\int_0^t x_1(u)du\geqslant\frac{2a-\sigma_1^2}{2b}-\zeta\quad\text{a.s.}$$

Letting $\zeta\to0$ and together with (2.14) implies the required assertion. □

Remark 2.5. *Let all the Brownian motions in model (1.2) be uncorrelated. Then Theorem 2.2 is still obtained. Moreover, Theorem 2.4(a) or (b) holds if assertion (2.5) or (2.6) is satisfied with $\rho_{ij}=0$ for all $i,j=1,\cdots,4$ and $i\neq j$.*

Remark 2.6. *Assume that $\rho_{13} \leqslant 0$. Then under condition* (2.6) *or* (2.7)*, $x_1(t)$ of model* (1.2) *obeys*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du = \frac{2a-\sigma_1^2}{2b} \quad a.s.$$

and $x_2(t)$ tends to zero exponentially as $t \to \infty$ with probability 1.

Theorem 2.4(a) shows that large white noise intensity σ_1^2 may let the populations die out. In Theorem 2.4(b), the situation where a becomes larger is discussed. There are generally two cases, depending on the value of ρ_{24}. In the first case, $B_2(t)$ and $B_4(t)$ are strongly negatively correlated $(-1 \leqslant \rho_{24} \leqslant -\frac{h}{r_2\sigma_2})$. Then under condition (2.7), the prey species keep persistent while the consumers become extinct ultimately. On the other hand, we let $\rho_{24} > -\frac{h}{r_2\sigma_2}$. Then system (1.2) has the same behaviours as in the first case provided that (2.6) is fulfilled. It is then interesting to examine how the population system behaves when a gets larger in the case $\rho_{24} > -\frac{h}{r_2\sigma_2}$. This is further developed in section 4.

3 Model (1.3)

In this section, we investigate the long-time behaviours of model (1.3). Notice that if δ_1 and δ_2 are zero, model (1.3) is then degenerated to model (1.2) which has been analysed as above. Hence this section only focuses on the unique properties of model (1.3) with two positive constants δ_1 and δ_2.

3.1 Global positive solution

Theorem 3.1. *For any given initial value $x_0 \in \mathbb{R}_+^2$, there is a unique solution $x(t)$ to equation (1.3) on $t \geqslant 0$ and the solution will remain in $\mathbb{R}_+^2$ with probability 1, namely $x(t) \in \mathbb{R}_+^2$ for all $t \geqslant 0$ a.s.*

By defining $V(x) = x_1^{0.5} - 0.5\log x_1 + x_2^{0.5} - 0.5\log x_2$, this theorem is then proved in the same routine as in [21, 39].

3.2 Asymptotic moment estimate

Theorem 3.2. *Let η_1 and η_2 be positive numbers satisfying*

$$\begin{aligned} \eta_1, \eta_2 &< \frac{1}{2} \quad \textit{for } \rho_{45} \geqslant 0 \textit{ and } \rho_{56} \leqslant 0; \\ \eta_1 + 2\eta_2 &< \frac{1}{2} \quad \textit{for } \rho_{45} < 0 \textit{ and } \rho_{56} > 0; \\ \eta_1 + \eta_2 &< \frac{1}{2} \quad \textit{otherwise.} \end{aligned}$$

Then for any initial value $x_0 \in \mathbb{R}_+^2$, the solution of model (1.3) *has the property that*

$$\limsup_{t\to\infty} \mathbb{E}[x_1^{\eta_1}(t)\, x_2^{\eta_2}(t)] \leqslant e^{c_1/c_2},$$

where c_1 and c_2 are two constants determined in (3.5) *and* (3.6) *below in the case $\rho_{45} < 0$ and $\rho_{56} > 0$.*

In order to prove this theorem, let us first consider the following lemma.

Lemma 3.3. *Let η_1 and η_2 be positive numbers satisfying*

$$\eta_1, \eta_2 < 1 \quad \textit{for } \rho_{45} \geqslant 0 \textit{ and } \rho_{56} \leqslant 0; \tag{3.1a}$$

$$\eta_1 + 2\eta_2 < 1 \quad \textit{for } \rho_{45} < 0 \textit{ and } \rho_{56} > 0; \tag{3.1b}$$

$$\eta_1 + \eta_2 < 1 \quad \textit{otherwise,} \tag{3.1c}$$

Then for any initial value $x_0 \in \mathbb{R}_+^2$, the solution of model (1.3) *has the property that*

$$\mathbb{E}[x_1^{\eta_1}(t)\, x_2^{\eta_2}(t)] < \infty \quad \textit{for all } t \geqslant 0.$$

Proof. Define a C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$ by $V(x) = x_1^{\eta_1} x_2^{\eta_2}$. And we obtain

$$\begin{aligned}LV(x) \leqslant V(x)\Big[A_1 + A_2x_1 + A_3x_2 + \frac{1}{2}\delta_1^2\eta_1(\eta_1-1)x_1^2 + \frac{1}{2}\delta_2^2\eta_2(\eta_2-1)x_2^2 + \frac{\eta_2(\eta_2-1)r_2^2}{2(\beta+x_2)^2}x_1^2 \\ - \frac{\delta_1 r_2\eta_1\eta_2\rho_{45}}{\beta+x_2}x_1^2 + \delta_1\delta_2\eta_1\eta_2\rho_{56}x_1x_2\Big],\end{aligned}$$

where

$$A_1 = a\eta_1 - c\eta_2 - \sigma_1\sigma_2\eta_1\eta_2\rho_{12} + \sigma_1 r_1\eta_1(1-\eta_1)\bar{\rho}_{13} + r_1\sigma_2\eta_1\eta_2\bar{\rho}_{23}, \tag{3.2}$$

$$\begin{aligned}A_2 = -b\eta_1 + \frac{h\eta_2}{\beta} + \frac{\sigma_1 r_2\eta_1\eta_2\bar{\rho}_{14}}{\beta} + \delta_1\sigma_1\eta_1(1-\eta_1)\rho_{15} + \frac{\sigma_2 r_2\eta_2(1-\eta_2)\bar{\rho}_{24}}{\beta} + \sigma_2\delta_1\eta_1\eta_2\rho_{25} \\ + \frac{r_1 r_2\eta_1\eta_2\tilde{\rho}_{34}}{\beta} + \delta_1 r_1\eta_1(1-\eta_1)\tilde{\rho}_{35} + \delta_2 r_2\eta_2(1-\eta_2)\bar{\rho}_{46}\end{aligned} \tag{3.3}$$

and

$$A_3 = -f\eta_2 - \sigma_1\delta_2\eta_2\eta_1\rho_{16} - \delta_2\sigma_2\eta_2(1-\eta_2)\rho_{26} + r_1\delta_2\eta_1\eta_2\bar{\rho}_{36}. \tag{3.4}$$

Assuming that $\rho_{45} < 0$ and $\rho_{56} > 0$, we obtain

$$\begin{aligned}&\frac{1}{4}\delta_1^2\eta_1(\eta_1-1)x_1^2 + \frac{\eta_2(\eta_2-1)r_2^2}{2(\beta+x_2)^2}x_1^2 - \frac{\delta_1 r_2\eta_1\eta_2\rho_{45}}{\beta+x_2}x_1^2 \\ \leqslant\ & \frac{1}{4}\delta_1^2\eta_1^2x_1^2 - \frac{1}{4}\delta_1^2\eta_1x_1^2 + \frac{\eta_2^2r_2^2x_1^2}{2(\beta+x_2)^2} - \frac{\eta_2r_2^2x_1^2}{2(\beta+x_2)^2} + \frac{1}{2}\eta_1\eta_2\Big(\frac{r_2^2x_1^2}{(\beta+x_2)^2} + \delta_1^2x_1^2\Big) \\ =\ & \frac{1}{4}\eta_1\big(\eta_1+2\eta_2-1\big)\delta_1^2x_1^2 + \frac{1}{2}\eta_2(\eta_1+\eta_2-1)\frac{r_2^2x_1^2}{(\beta+x_2)^2}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{4}\delta_1^2\eta_1(\eta_1-1)x_1^2 + \frac{1}{2}\delta_2^2\eta_2(\eta_2-1)x_2^2 + \delta_1\delta_2\eta_1\eta_2\rho_{56}x_1x_2 \\ \leqslant\ & \frac{1}{4}\delta_1^2\eta_1^2x_1^2 - \frac{1}{4}\delta_1^2\eta_1x_1^2 + \frac{1}{2}\delta_2^2\eta_2^2x_2^2 - \frac{1}{2}\delta_2^2\eta_2x_2^2 + \frac{1}{2}\eta_1\eta_2(\delta_1^2x_1^2 + \delta_2^2x_2^2) \\ =\ & \frac{1}{4}\eta_1(\eta_1+2\eta_2-1)\delta_1^2x_1^2 + \frac{1}{2}\eta_2(\eta_1+\eta_2-1)\delta_2^2x_2^2.\end{aligned}$$

Hence

$$LV(x) \leqslant V(x)\Big(A_1 + A_2x_1 + A_3x_2 - \frac{1}{2}\eta_1\big(1-(\eta_1+2\eta_2)\big)\delta_1^2x_1^2 - \frac{1}{2}\eta_2\big(1-(\eta_1+\eta_2)\big)\delta_2^2x_2^2\Big).$$

As the polynomial

$$A_1 + A_2x_1 + A_3x_2 - \frac{1}{4}\eta_1\big(1-(\eta_1+2\eta_2)\big)\delta_1^2x_1^2 - \frac{1}{4}\eta_2\big(1-(\eta_1+\eta_2)\big)\delta_2^2x_2^2$$

is bounded by

$$c_1 = \frac{\eta_1\big(1-(\eta_1+2\eta_2)\big)\delta_1^2A_1 + A_2^2}{\eta_1(1-(\eta_1+2\eta_2))\delta_1^2} + \frac{A_3^2}{\eta_2\big(1-(\eta_1+\eta_2)\big)\delta_2^2}, \tag{3.5}$$

we obtain

$$LV(x) \leqslant V(x)\big(c_1 - c_2|x|^2\big),$$

where

$$c_2 = \frac{1}{4}\big(\eta_1(1-(\eta_1+2\eta_2))\delta_1^2 \wedge \eta_2\big(1-(\eta_1+\eta_2)\big)\delta_2^2\big). \tag{3.6}$$

This leads to

$$\mathbb{E}V(x(t\wedge\tau_k)) = V(x_0) + \mathbb{E}\int_0^{t\wedge\tau_k} LV(x(s))ds \leqslant V(x_0) + c_1\int_0^t \mathbb{E}V(x(s\wedge\tau_k))ds.$$

It then follows from the Gronwall inequality that

$$\mathbb{E}V(x(t\wedge\tau_k)) \leqslant V(x_0)e^{c_1 t}.$$

Letting $k\to\infty$ implies

$$\mathbb{E}V(x(t)) \leqslant V(x_0)e^{c_1 t} < \infty \quad \text{for all } t\geqslant 0.$$

Similarly, one can deduce the same results under condition (3.1a) or (3.1c) with the corresponding values of c_1 and c_2. Here it is omitted. □

Proof of Theorem 3.2. This proof is standard by using the results of Lemma 3.3. One can refer to [21, pp. 104-105] for details. □

We can obtain from the Chebyshev's inequality and Theorem 3.2 that

$$\begin{aligned}&\mathbb{P}(x_1\geqslant D_1 \text{ and } x_2\geqslant D_2)\\ =&\mathbb{E}\left[\mathbf{I}_{\{x_1\geqslant D_1\}}\mathbf{I}_{\{x_2\geqslant D_2\}}\right] \leqslant \mathbb{E}\left[\frac{x_1^{\eta_1}}{D_1^{\eta_1}}\mathbf{I}_{\{x_1\geqslant D_1\}}\frac{x_2^{\eta_2}}{D_2^{\eta_2}}\mathbf{I}_{\{x_2\geqslant D_2\}}\right] \leqslant \frac{\mathbb{E}\left[x_1^{\eta_1}x_2^{\eta_2}\right]}{D_1^{\eta_1}D_2^{\eta_2}} \leqslant \frac{e^{c_1/c_2}}{D_1^{\eta_1}D_2^{\eta_2}},\end{aligned}$$

where $\mathbf{I}$ is the indicator function. From the biological point of view, this implies that it is unlikely that the amount of two populations will become very large simultaneously.

3.3 Asymptotic pathwise estimation

Theorem 3.4. *For any initial value $x_0\in\mathbb{R}_+^2$, the solution of model* (1.3) *has the property that*

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log t} \leqslant 6 \quad a.s. \tag{3.7}$$

Proof. Defining V: $\mathbb{R}_+^2\to\mathbb{R}_+$ by $V(x)=x_1+x_2$, for any constant $\gamma>0$ we obtain

$$e^{\gamma t}\log V(x(t)) = \log V(x(0)) + \int_0^t e^{\gamma u}g(x(u))du + \sum_{i=1}^{6}\breve{M}_i(t), \tag{3.8}$$

where

$$\begin{aligned}g(x) =& \gamma\log V(x) + \frac{1}{V(x)}\Big(ax_1 - cx_2 - bx_1^2 - fx_2^2 - \frac{sx_1x_2}{\beta+x_2} + \frac{hx_1x_2}{\beta+x_2}\Big) - \frac{1}{2V^2(x)}\Big(\sigma_1^2x_1^2+\sigma_2^2x_2^2\\ &+\frac{r_1^2x_1^2x_2^2}{(\beta+x_2)^2} + \frac{r_2^2x_1^2x_2^2}{(\beta+x_2)^2} + \delta_1^2x_1^4 + \delta_2^2x_2^4 - 2\sigma_1\sigma_2\rho_{12}x_1x_2 + \frac{2\sigma_1r_2\rho_{14}x_1^2x_2}{\beta+x_2} - 2\sigma_1\delta_2\rho_{16}x_1x_2^2\\ &+\frac{2r_1\sigma_2\rho_{23}x_1x_2^2}{\beta+x_2} - \frac{2r_1r_2\rho_{34}x_1^2x_2^2}{(\beta+x_2)^2} + \frac{2r_1\delta_2\rho_{36}x_1x_2^3}{\beta+x_2} + 2\delta_1\sigma_2\rho_{25}x_1^2x_2 - \frac{2\delta_1r_2\rho_{45}x_1^3x_2}{\beta+x_2}\\ &+2\delta_1\delta_2\rho_{56}x_1^2x_2^2\Big)\end{aligned}$$

and

$$\begin{aligned}&\breve{M}_1(t) = \sigma_1\int_0^t \frac{e^{\gamma u}x_1(u)}{V(x(u))}dB_1(u), \qquad \breve{M}_2(t) = -\sigma_2\int_0^t \frac{e^{\gamma u}x_2(u)}{V(x(u))}dB_2(u),\\ &\breve{M}_3(t) = -r_1\int_0^t \frac{e^{\gamma u}x_1(u)x_2(u)}{(\beta+x_2(u))V(x(u))}dB_3(u),\\ &\breve{M}_4(t) = r_2\int_0^t \frac{e^{\gamma u}x_1(u)x_2(u)}{(\beta+x_2(u))V(x(u))}dB_4(u),\\ &\breve{M}_5(t) = -\delta_1\int_0^t \frac{e^{\gamma u}x_1^2(u)}{V(x(u))}dB_5(u), \qquad \breve{M}_6(t) = -\delta_2\int_0^t \frac{e^{\gamma u}x_2^2(u)}{V(x(u))}dB_6(u)\end{aligned}$$

are local martingales with quadratic variations

$$\langle \breve{M}_1(t)\rangle = \sigma_1^2 \int_0^t \frac{e^{2\gamma u}x_1^2(u)}{V^2(x(u))}du, \qquad \langle \breve{M}_2(t)\rangle = \sigma_2^2 \int_0^t \frac{e^{2\gamma u}x_1^2(u)}{V^2(x(u))}du,$$
$$\langle \breve{M}_3(t)\rangle = r_1^2 \int_0^t \frac{e^{2\gamma u}x_1^2(u)x_2^2(u)}{(\beta+x_2(u))^2V^2(x(u))}du, \qquad \langle \breve{M}_4(t)\rangle = r_2^2 \int_0^t \frac{e^{2\gamma u}x_1^2(u)x_2^2(u)}{(\beta+x_2(u))^2V^2(x(u))}du,$$
$$\langle \breve{M}_5(t)\rangle = \delta_1^2 \int_0^t \frac{e^{2\gamma u}x_1^4(u)}{V^2(x(u))}du, \qquad \langle \breve{M}_6(t)\rangle = \delta_2^2 \int_0^t \frac{e^{2\gamma u}x_2^4(u)}{V^2(x(u))}du.$$

Given any $\alpha_1 \in (0,1)$ and $p>1$. By the exponential martingale inequality, we have

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant \psi}\Big(\breve{M}_i(t) - \frac{\alpha_1}{2}e^{-\gamma\psi}\langle \breve{M}_i(t)\rangle\Big) > \frac{pe^{\gamma\psi}}{\alpha_1}\log\psi\Big) \leqslant \frac{1}{\psi^p}, \quad \psi = 1,2,\cdots.$$

Then by the Borel-Cantelli lemma, for almost all $\omega\in\Omega$, there exists $\psi_i=\psi_i(\omega)\geqslant 1$ such that

$$\breve{M}_i(t) \leqslant \frac{\alpha_1}{2}e^{-\gamma\psi}\langle \breve{M}_i(t)\rangle + \frac{pe^{\gamma\psi}}{\alpha_1}\log\psi \quad \text{for all } 0\leqslant t\leqslant\psi \text{ and } \psi\geqslant\psi_i.$$

Thus substituting this into (3.8) indicates that for almost every $\omega\in\Omega$,

$$\begin{aligned}&e^{\gamma t}\log V(x(t))\\&\leqslant \log V(x(0)) + \int_0^t e^{\gamma u}\Big(\gamma\log V(x(u)) + a + h + g_1(x(u)) - \frac{1-\alpha_1 e^{\gamma(u-\psi)}}{2V^2(x(u))}\big(\delta_1^2x_1^4(u)+\delta_2^2x_2^4(u)\big)\Big)du\\&\quad + \frac{6pe^{\gamma\psi}}{\alpha_1}\log\psi \end{aligned} \tag{3.9}$$

for all $0\leqslant t\leqslant\psi$ and $\psi\geqslant\psi_0 := \max(\psi_1,\psi_2,\cdots,\psi_6)$, where $g_1(x)$ is a first-order polynomial about x. By the elementary inequality

$$\frac{V^2(x)}{2} \leqslant |x|^2 \leqslant 2V^2(x), \tag{3.10}$$

we obtain

$$\frac{1}{V^2(x)}(\delta_1^2x_1^4+\delta_2^2x_2^4) \geqslant \frac{1}{4}(\delta_1^2\wedge\delta_2^2)|x|^2.$$

Therefore (3.9) is rewritten as

$$\begin{aligned}&e^{\gamma t}\log V(x(t))\\&\leqslant \log V(x(0)) + \int_0^t e^{\gamma u}\Big(\gamma\log V(x(u)) + a + h + g_1(x(u)) - \frac{1}{8}(1-\alpha_1)(\delta_1^2\wedge\delta_2^2)|x(u)|^2\Big)du\\&\quad + \frac{6pe^{\gamma\psi}}{\alpha_1}\log\psi.\end{aligned}$$

Obviously, there exists a positive constant K_3 such that for almost every $\omega\in\Omega$,

$$\begin{aligned}&e^{\gamma t}\log V(x(t))\\&\leqslant \log V(x(0)) + K_3\int_0^t e^{\gamma u}du + \frac{6pe^{\gamma\psi}}{\alpha_1}\log\psi \leqslant \log V(x(0)) + \frac{K_3}{\gamma}e^{\gamma t} - \frac{K_3}{\gamma} + \frac{6pe^{\gamma\psi}}{\alpha_1}\log\psi\end{aligned}$$

for all $0\leqslant t\leqslant\psi$ and $\psi\geqslant\psi_0 := \max(\psi_1,\psi_2,\cdots,\psi_6)$. Consequently, for $\psi-1\leqslant t\leqslant\psi$ and $\psi\geqslant\psi_0$, it follows that

$$\frac{\log V(x(t))}{\log t} \leqslant \frac{1}{\log(\psi-1)}\Big(e^{-\gamma(\psi-1)}\log(x_1(0)x_2(0)) + \frac{K_3}{\gamma} + \frac{6pe^{\gamma}}{\alpha_1}\log\psi\Big).$$

This implies

$$\limsup_{t\to\infty}\frac{\log V(x(t))}{\log t}\leqslant\frac{6pe^{\gamma}}{\alpha_1}\quad\text{a.s.}$$

Letting $\alpha_1\to 1, p\to 1$ and $\gamma\to 0$ implies

$$\limsup_{t\to\infty}\frac{\log V(x(t))}{\log t}\leqslant 6\quad\text{a.s.}$$

Recalling inequality (3.10) gives the required assertion (3.7). □

Remark 3.5. *Let all the Brownian motions in model (1.3) be uncorrelated. Then Theorem 3.4 still holds. Besides, Theorem 3.2 is fulfilled provided that η_1 and η_2 satisfy*

$$\eta_1,\eta_2<\frac{1}{2},$$

with c_1 and c_2 defined by

$$c_1=\frac{(h\eta_2/\beta-b\eta_1)^2+\eta_1\delta_1^2(1-\eta_1)(a\eta_1-c\eta_2)}{\eta_1(1-\eta_1)\delta_1^2}$$

and

$$c_2=\frac{1}{4}\big(\eta_1(1-\eta_1)\delta_1^2\wedge\eta_2(1-\eta_2)\delta_2^2\big).$$

4 Stationary distribution

In this section, the stationary distributions of the solutions of model (1.2) and (1.3) are established. Let $P_{x_0,t}$ denote the probability measure induced by $x(t)$ with initial value $x(0)=x_0$, that is

$$P_{x_0,t}(D)=\mathbb{P}(x(t)\in D),\quad D\in\mathcal{B}(\mathbb{R}_+^2),$$

where $\mathcal{B}(\mathbb{R}_+^2)$ is the σ-algebra of all the Borel sets $D\subseteq\mathbb{R}_+^2$. If there is a probability measure $\mu(\cdot)$ on the measurable space $(\mathbb{R}_+^2,\mathcal{B}(\mathbb{R}_+^2))$ such that

$$P_{x_0,t}(\cdot)\to\mu(\cdot)\text{ in distribution for any }x_0\in\mathbb{R}_+^2,$$

we then say that the SDE model (1.2) (or (1.3)) has a stationary distribution $\mu(\cdot)$ [23, 31, 41]. To show the existence of a stationary distribution, let us first cite a known result from Khasminskii [41, pp. 107-109, Theorem 4.1] as a lemma.

Lemma 4.1. *The SDE model (1.3) has a unique stationary distribution if*
(i) the matrix

$$U(x)=A(x)\mathcal{R}A(x)^T$$

is positive definite for $x\in\mathbb{R}_+^2$, where

$$A(x)=\begin{bmatrix}\sigma_1x_1 & 0 & \frac{-r_1x_1x_2}{\beta+x_2} & 0 & -\delta_1x_1^2 & 0\\ 0 & -\sigma_2x_2 & 0 & \frac{r_2x_1x_2}{\beta+x_2} & 0 & -\delta_2x_2^2\end{bmatrix};$$

(ii) there is a bounded openset G of $\mathbb{R}_+^2$ and

$$\sup_{x_0\in Q-G}\mathbb{E}(\tau_G)<\infty$$

for every compact subset Q of $\mathbb{R}_+^2$ such that $G\subset Q$ where $\tau_G=\inf\{t\geqslant 0:x(t)\in G\}$.

Theorem 4.2. *If*

$$\begin{aligned}&\rho_{i_1 i_2} \neq \pm 1; \rho_{1i_3}, \rho_{4i_3} < 1/2; \rho_{26}, \rho_{35} > -1/2;\\ &\rho_{1i_3} \leqslant \rho_{1i_2}\rho_{i_2 i_3}; \rho_{35} \geqslant \rho_{3i_2}\rho_{5i_2}; \rho_{4i_4} \leqslant \rho_{i_1 4}\rho_{i_1 i_4}; \rho_{26} \geqslant \rho_{2i_1}\rho_{6i_1};\\ &2\rho_{1i_3}\rho_{26} \leqslant \rho_{12}\rho_{6i_3} + \rho_{16}\rho_{2i_3}; 2\rho_{35}\rho_{4i_4} \leqslant \rho_{3i_4}\rho_{45} + \rho_{34}\rho_{5i_4};\\ &2\rho_{1i_3}\rho_{4i_4} \geqslant \rho_{14}\rho_{i_3 i_4} + \rho_{1i_4}\rho_{4i_3}; 2\rho_{35}\rho_{26} \geqslant \rho_{23}\rho_{56} + \rho_{36}\rho_{25}\\ &\text{for } i_1 = 1, 3 \text{ or } 5, i_2 = 2, 4 \text{ or } 6, i_3 = 3 \text{ or } 5 \text{ and } i_4 = 2 \text{ or } 6,\end{aligned} \tag{4.1}$$

$$h_0 := \sigma_2 r_2 \tilde{\rho}_{24} + \delta_2 r_2 \beta \tilde{\rho}_{46} < h, \tag{4.2}$$

$$2a\Big(1 - \frac{1+\beta}{2(h-h_0)}\sigma_2^2 - \frac{(1+\beta)c}{h-h_0}\Big) > \sigma_1^2 + 2r_1\sigma_1\tilde{\rho}_{13} + \frac{2(b+b_0)\beta c}{h-h_0} + \frac{(b+b_0)\beta}{h-h_0}\sigma_2^2 \tag{4.3}$$

and

$$b \geqslant \frac{1}{2}\delta_1^2 + \frac{E_1}{2h\beta^2}r_2^2 + \frac{a+E_2}{\beta}, \tag{4.4}$$

where

$$b_0 = \sigma_1\delta_1\tilde{\rho}_{15} + r_1\delta_1\bar{\rho}_{35}, \quad E_1 = \frac{ah + \big(a+b+b_0\big)\beta h}{h-h_0} \quad \text{and} \quad E_2 = b_0 + \frac{h_0 E_1}{\beta h},$$

then for any initial value $x_0 \in \mathbb{R}_+^2$, model (1.3) has a unique stationary distribution.

Proof. (i) We compute

$$U(x) = (U_{ij}(x))_{2\times 2},$$

where

$$U_{11}(x) = \sigma_1^2 x_1^2 - \frac{2\rho_{13}\sigma_1 r_1 x_1^2 x_2}{\beta + x_2} - 2\sigma_1\delta_1\rho_{15}x_1^3 + \frac{2r_1\delta_1\rho_{35}x_1^3 x_2}{\beta + x_2} + \frac{r_1^2 x_1^2 x_2^2}{(\beta+x_2)^2} + \delta_1^2 x_1^4,$$

$$U_{22}(x) = \sigma_2^2 x_2^2 - \frac{2r_2\sigma_2\rho_{24}x_1x_2^2}{\beta + x_2} + 2\rho_{26}\sigma_2\delta_2 x_2^3 - \frac{2\rho_{46}\delta_2 r_2 x_1 x_2^3}{\beta + x_2} + \frac{r_2^2 x_1^2 x_2^2}{(\beta+x_2)^2} + \delta_2^2 x_2^4$$

and

$$\begin{aligned}U_{12}(x) = U_{21}(x) = &-\sigma_1\sigma_2\rho_{21}x_1x_2 + \frac{\rho_{14}r_2\sigma_1 x_1^2 x_2}{\beta+x_2} - \delta_2\sigma_1\rho_{16}x_1x_2^2 + \frac{r_1\sigma_2\rho_{23}x_1x_2^2}{\beta+x_2}\\ &- \frac{\rho_{34}r_1r_2x_1^2x_2^2}{(\beta+x_2)^2} + \frac{r_2\delta_2\rho_{36}x_1x_2^3}{\beta+x_2} + \delta_1\sigma_2\rho_{25}x_1^2x_2 - \frac{\rho_{45}\delta_1 r_2 x_1^3 x_2}{\beta+x_2} + \rho_{56}\delta_1\delta_2 x_1^2 x_2^2.\end{aligned}$$

The sufficient conditions (4.1) guarantee that $U_{11}(x) > 0$, $U_{22}(x) > 0$ and $U_{11}(x)U_{22}(x) - U_{12}^2(x) > 0$. Hence $U(x)$ is a positive-definite matrix.

(ii) We define a C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+$:

$$V(x) = MV_1(x) + V_2(x) + e,$$

where

$$V_1(x) = x_1 + \ln(\beta + x_1) - \ln(x_1) + \frac{l}{h}x_2 - \frac{E_1}{h}\ln x_2, \quad V_2(x) = x_1 + \frac{s}{h}x_2,$$

and e, l and M are three constants. $e = -\min(MV_1(x) + V_2(x))$ to keep the non-negativity of $V(x)$,

$$l = \Big(\frac{hs}{c\beta} + \frac{E_1 f}{c} + \frac{E_1\sigma_2\delta_2\bar{\rho}_{26}}{c}\Big) \bigvee \Big(\frac{E_1\delta_2^2}{2f} + \frac{hr_1^2}{2f\beta^2} + \frac{E_1 h}{4f\beta^2}\Big) \tag{4.5}$$

and M is to be defined later. First compute

$$\begin{aligned}
LV_1 =& \Big(x_1 + \frac{x_1}{\beta + x_1} - 1\Big)\Big(a - bx_1 - \frac{sx_2}{\beta + x_2}\Big) + \frac{1}{2}\Big(1 - \frac{x_1^2}{(\beta + x_1)^2}\Big)\Big(\sigma_1^2 + \frac{r_1^2 x_2^2}{(\beta + x_2)^2} + \delta_1^2 x_1^2 \\
&- \frac{2\sigma_1 r_1 \rho_{13} x_2}{\beta + x_2} - 2\sigma_1\delta_1\rho_{15}x_1 + \frac{2r_1\delta_1\rho_{35}x_1x_2}{\beta + x_2}\Big) + \Big(\frac{lx_2}{h} - \frac{E_1}{h}\Big)\Big(\frac{hx_1}{\beta + x_2} - c - fx_2\Big) \\
&+ \frac{E_1}{2h}\Big(\sigma_2^2 + \frac{r_2^2 x_1^2}{(\beta + x_2)^2} + \delta_2^2 x_2^2 - \frac{2r_2\sigma_2\rho_{24}x_1}{\beta + x_2} + 2\delta_2\sigma_2\rho_{26}x_2 - \frac{2\delta_2 r_2 \rho_{46}x_1x_2}{\beta + x_2}\Big) \\
\leqslant& \ ax_1 - bx_1^2 + \frac{ax_1}{\beta + x_1} - \frac{bx_1^2}{\beta + x_1} + bx_1 - \frac{E_1x_1}{\beta + x_2} - a + \frac{sx_2}{\beta + x_2} + \frac{1}{2}\sigma_1^2 + \frac{r_1^2x_2^2}{2\beta^2} + \frac{\delta_1^2 x_1^2}{2} \\
&+ r_1\sigma_1\tilde{\rho}_{13} + \big(\sigma_1\delta_1\tilde{\rho}_{15} + r_1\delta_1\bar{\rho}_{35}\big)x_1 + \frac{lx_1x_2}{\beta + x_2} - \frac{clx_2}{h} - \frac{flx_2^2}{h} + \frac{E_1c}{h} + \frac{E_1fx_2}{h} + \frac{E_1\sigma_2^2}{2h} \\
&+ \frac{E_1r_2^2x_1^2}{2h\beta^2} + \frac{E_1\delta_2^2x_2^2}{2h} + \frac{E_1\sigma_2r_2\tilde{\rho}_{24}x_1}{h\beta} + \frac{E_1\sigma_2\delta_2\rho_{26}x_2}{h} + \frac{E_1\delta_2r_2\tilde{\rho}_{46}x_1}{h} \\
\leqslant& \frac{(a + E_2)\beta x_1}{\beta + x_1} + \frac{ax_1}{\beta + x_1} + \frac{b\beta x_1}{\beta + x_1} - \frac{E_1x_1}{\beta + x_2} + \Big(- b + \frac{\delta_1^2}{2} + \frac{a + E_2}{\beta + x_1} + \frac{E_1r_2^2}{2h\beta^2}\Big)x_1^2 \\
&- a + \frac{1}{2}\sigma_1^2 + r_1\sigma_1\tilde{\rho}_{13} + \frac{E_1c}{h} + \frac{E_1\sigma_2^2}{2h} + \Big(\frac{s}{\beta} + \frac{E_1f}{h} - \frac{cl}{h} + \frac{E_1\sigma_2\delta_2\rho_{26}}{h}\Big)x_2 + \Big(\frac{E_1\delta_2^2}{2h} + \frac{r_1^2}{2\beta^2} \\
&- \frac{fl}{h}\Big)x_2^2 + \frac{lx_1x_2}{\beta + x_2} \\
\leqslant& \frac{E_1(x_1x_2 - x_1^2)}{(\beta + x_1)(\beta + x_2)} + \Big(- b + \frac{\delta_1^2}{2} + \frac{a + E_2}{\beta} + \frac{E_1r_2^2}{2h\beta^2}\Big)x_1^2 - \lambda + \Big(\frac{s}{\beta} + \frac{E_1f}{h} - \frac{cl}{h} \\
&+ \frac{E_1\sigma_2\delta_2\rho_{26}}{h}\Big)x_2 + \Big(\frac{E_1\delta_2^2}{2h} + \frac{r_1^2}{2\beta^2} - \frac{fl}{h}\Big)x_2^2 + \frac{lx_1x_2}{\beta + x_2},
\end{aligned}$$

where $\lambda = a - \frac{1}{2}\sigma_1^2 - r_1\sigma_1\tilde{\rho}_{13} - \frac{E_1c}{h} - \frac{E_1\sigma_2^2}{2h} > 0$ from condition (4.3). Under (4.4), (4.5) and the Young inequality,

$$\begin{aligned}
LV_1 \leqslant& -\lambda + \Big(- b + \frac{\delta_1^2}{2} + \frac{a + E_2}{\beta} + \frac{E_1r_2^2}{2h\beta^2}\Big)x_1^2 + \Big(\frac{s}{\beta} + \frac{E_1f}{h} - \frac{cl}{h} + \frac{E_1\sigma_2\delta_2\rho_{26}}{h}\Big)x_2 \\
&+ \Big(\frac{E_1\delta_2^2}{2h} + \frac{r_1^2}{2\beta^2} - \frac{fl}{h} + \frac{E_1}{4\beta^2}\Big)x_2^2 + \frac{lx_1x_2}{\beta + x_2} \\
\leqslant& -\lambda + \frac{lx_1x_2}{\beta + x_2}.
\end{aligned}$$

Then compute

$$LV_2 = ax_1 - bx_1^2 - \frac{sc}{h}x_2 - \frac{sf}{h}x_2^2 \leqslant ax_1 - bx_1^2 - \frac{sf}{h}x_2^2.$$

Hence

$$LV(x) \leqslant M\Big(- \lambda + \frac{lx_1x_2}{\beta + x_2}\Big) + ax_1 - bx_1^2 - \frac{sf}{h}x_2^2,$$

where M satisfies $M\lambda \geqslant a^2/(4b) + 2$. Now we aim to show

$$LV(x) \leqslant -1 \text{ for all } x \in \mathbb{R}_+^2 - G := G^c. \tag{4.6}$$

As if this holds, let $x \in G^c$ be arbitrary and τ_G be the stopping time as defined in Lemma 4.1. From (4.6), we have

$$0 \leqslant V(x_0) - \mathbb{E}(t \wedge \tau_G \wedge \tau_k), \quad \forall t > 0.$$

Letting $k \to \infty$ and then $t \to \infty$, we have

$$\mathbb{E}(\tau_G) \leqslant V(x_0), \quad \forall x_0 \in G^c$$

as required. To show that (4.6) actually holds, we define

$$G^c = G_1^c \cup G_2^c \cup G_3^c \cup G_4^c,$$

where

$$G_1^c = \{x|x_1 \in (0, \epsilon_1]\}; \qquad G_2^c = \left\{x\middle|x_1 \in \left(0, \frac{1}{\epsilon_1}\right], x_2 \in (0, \epsilon_2]\right\};$$
$$G_3^c = \left\{x\middle|x_1 \in \left[\frac{1}{\epsilon_1}, +\infty\right)\right\}; \qquad G_4^c = \left\{x\middle|x_2 \in \left[\frac{1}{\epsilon_2}, +\infty\right)\right\}$$

with two constants $\epsilon_1, \epsilon_2 \in (0, 1)$ satisfying

$$\epsilon_1^2 \leqslant \frac{1}{M^2 l^2} \bigwedge \frac{b}{2(N_1+1)}, \quad \epsilon_2^2 \leqslant \frac{sf}{2h(N_2+1)} \quad \text{and} \quad \epsilon_2 \leqslant \frac{\beta\epsilon_1}{Ml}, \tag{4.7}$$

where the constants N_1 and N_2 will be determined later. We then show that in any subset of G^c, (4.6) holds. From (4.7),
(a) if $x \in G_1^c$,

$$LV(x) \leqslant -M\lambda + Mlx_1 + ax_1 - bx_1^2 - \frac{sf}{h}x_2^2 \leqslant Ml\epsilon_1 - 2 \leqslant -1;$$

(b) if $x \in G_2^c$,

$$LV(x) \leqslant -M\lambda + \frac{Mlx_1x_2}{\beta} + ax_1 - bx_1^2 - \frac{sf}{h}x_2^2 \leqslant \frac{Ml\epsilon_2}{\beta\epsilon_1} - 2 \leqslant -1;$$

(c) if $x \in G_3^c$,

$$LV(x) \leqslant -M\lambda + (Ml+a)x_1 - \frac{bx_1^2}{2} - \frac{bx_1^2}{2} - \frac{sfx_2^2}{h}.$$

Note that the polynomials $-M\lambda + (Ml+a)x_1 - \frac{bx_1^2}{2} - \frac{sfx_2^2}{h}$ has an upper bound, say N_1, hence

$$LV(x) \leqslant N_1 - \frac{b}{2\epsilon_1^2} \leqslant -1;$$

(d) if $x \in G_4^c$,

$$LV(x) \leqslant -M\lambda + (Ml+a)x_1 - bx_1^2 - \frac{sfx_2^2}{2h} - \frac{sfx_2^2}{2h}.$$

Note that the polynomial $-M\lambda + (Ml+a)x_1 - bx_1^2 - \frac{sfx_2^2}{2h}$ is again bounded, say by N_2, we have

$$LV(x) \leqslant N_2 - \frac{sf}{2h\epsilon_2^2} \leqslant -1.$$

In all,

$$LV(x) \leqslant -1 \text{ for all } x \in G^c.$$

$\square$

Remark 4.3. *Assume that all the Brownian motions in model (1.3) are uncorrelated. Then letting $\rho_{ij} = 0$ for $i, j = 1, 2, \cdots, 6$ and $i \neq j$ in condition (4.2)-(4.4) gives the parametric conditions of model (1.3) to have a unique stationary distribution.*

Remark 4.4. *Given that condition (4.2)-(4.4) are satisfied with $\delta_1 = \delta_2 = 0$ and $\rho_{i5} = \rho_{i6} = 0$ for $i = 1, 2, \cdots, 4$, model (1.2) then has a unique stationary distribution.*

5 Simulations

The following examples are developed to illustrate our results. The system parameters are given in appropriate units. The Euler-Maruyama (EM) scheme is used for the computer simulations [42]. From the theory introduced in [43], the EM approximate solutions are convergent to the true solutions of model (1.2) and (1.3) in probability.

Example 5.1. We perform a computer simulation of 10000 iterations of model (1.2) with initial value $x(0) = (0.7, 0.15)^T$ using the Euler-Maruyama (EM) method [40, 42] with stepsize $\Delta = 0.01$ and the system parameters given by

$$\begin{aligned}&a = 1, b = 0.5, \beta = 5, s = 16, h = 0.9, c = 2, f = 3, \sigma_1 = 1.5, \sigma_2 = 1.0, r_1 = 0.5\\&r_2 = 0.95 \text{ and } \rho_{13} = 0.15.\end{aligned} \tag{5.1}$$

This group of parameters satisfies condition (2.5) clearly. Theorem 2.4 then indicates that both species die out ultimately with probability 1. This is illustrated in Figure 1.

Example 5.2. We keep the system parameters of model (1.2) the same as Example 5.1 but let $\sigma_1 = 0.5$. Moreover, we let $\rho_{13} = 0.15$ and $\rho_{24} = 0.9$. As a result, condition (2.6) is fulfilled. From Theorem 2.4(b), the prey abundance has the property that

$$1.75 \leqslant \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du \leqslant 1.825 \quad \text{a.s.}$$

and the consumers will tend to zero exponentially with probability 1. Figure 2 supports these results clearly.

Example 5.3. In this example, we remain the system parameters of model (1.2) the same as Example 5.2 except that we let $\rho_{24} = -0.95$. This group of parameters does not obey condition (2.6) but satisfy (2.7). Hence Theorem 2.4(b) suggests that

$$1.75 \leqslant \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t x_1(u)du \leqslant 1.825 \quad \text{a.s.}$$

and the consumers will tend to zero exponentially with probability 1. Figure 3 supports these results clearly.

Then we study the case when the SDE system (1.2) and (1.3) have a stationary distribution.

Example 5.4. We assume that the parameters of system (1.2) are the same as in Example 5.3 but let $\beta = 2.5, h = 10, \sigma_1 = 0.01$, and $\sigma_2 = 0.02$. Also the correlation matrix is given by

$$\mathcal{R} = (\rho_{ij})_{4\times 4} = \begin{bmatrix} 1 & 0 & -0.8 & 0 \\ 0 & 1 & 0 & -0.95 \\ -0.8 & 0 & 1 & 0 \\ 0 & -0.95 & 0 & 1 \end{bmatrix}.$$

The time series of the correlated Brownian motions is shown in Figure 4. It is found that these parameters obey conditions (4.1)-(4.4) with $\delta_1 = \delta_2 = 0$ and $\rho_{i5} = \rho_{i6} = 0 = \rho_{56}$ for $i = 1, 2, \cdots, 4$. From Theorem 2.4 and Remark 4.4, system (1.2) has a stationary distribution. The ergodic property enables us to obtain the approximate probability distribution for the stationary distribution by computer simulation of a single sample path of a solution to model (1.2). Therefore the histogram of the 10000 iterations shown in Figure 5(b)(d) can be regarded as approximate p.d.f.s of the stationary distribution.

Example 5.5. In this example, the stationary distribution of model (1.3) is examined. We keep the system parameters the same as in Example 5.4 and let $\delta_1 = 0.01$ and $\delta_2 = 0.02$. Moreover the correlation matrix is given by

$$\mathcal{R} = (\rho_{ij})_{6\times 6} = \begin{bmatrix} 1 & 0 & -0.8 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & -0.95 & 0 & 0.7 \\ -0.8 & 0 & 1 & 0 & 0.9 & 0 \\ 0 & -0.95 & 0 & 1 & 0 & -0.8 \\ -0.5 & 0 & 0.9 & 0 & 1 & 0 \\ 0 & 0.7 & 0 & -0.8 & 0 & 1 \end{bmatrix}.$$

Obviously, these parameters obey conditions (4.1)-(4.4). From Theorem 4.2, model (1.3) has a stationary distribution. The approximate p.d.f.s of the stationary distribution could be identified from Figure 6(b)(d).

6 Summary

In this chapter, the different properties of the SDE population models (1.2) and (1.3) incorporating white noise were studied. The correlations between the Brownian motions do make an effect on the long-time behaviours of the systems. Especially, in model (1.2), a positive correlation between $B_1(t)$ and $B_3(t)$ leads to a slightly different condition for both populations to be extinct. Moreover, if $B_2(t)$ is strongly negatively correlated to $B_4(t)$, the population system always remains extinct (the prey populations become persistent while the consumers die out) and has no chance to have a multiple coexisting stationary status. In the contrast, provided that the correlation coefficient between $B_2(t)$ and $B_4(t)$ is bigger than $-\frac{h}{r_2\sigma_2}$, the system is possible to have a stationary distribution for both species with a larger value of a. In model (1.3), the correlations between $B_4(t)$ and $B_5(t)$ and between $B_5(t)$ and $B_6(t)$ also make a difference on the parametric conditions of the bounded properties. These imply how the correlations between the Brownian motions affect the dynamical behaviours of the populations. Theorem 4.2 reflects that a smaller amplitude environmental noise leads to a permanent population system. The ergodic property of the stationary distribution makes it possible to generate the approximate probability distribution using a single sample path of the solution to the SDE model by computer simulations.

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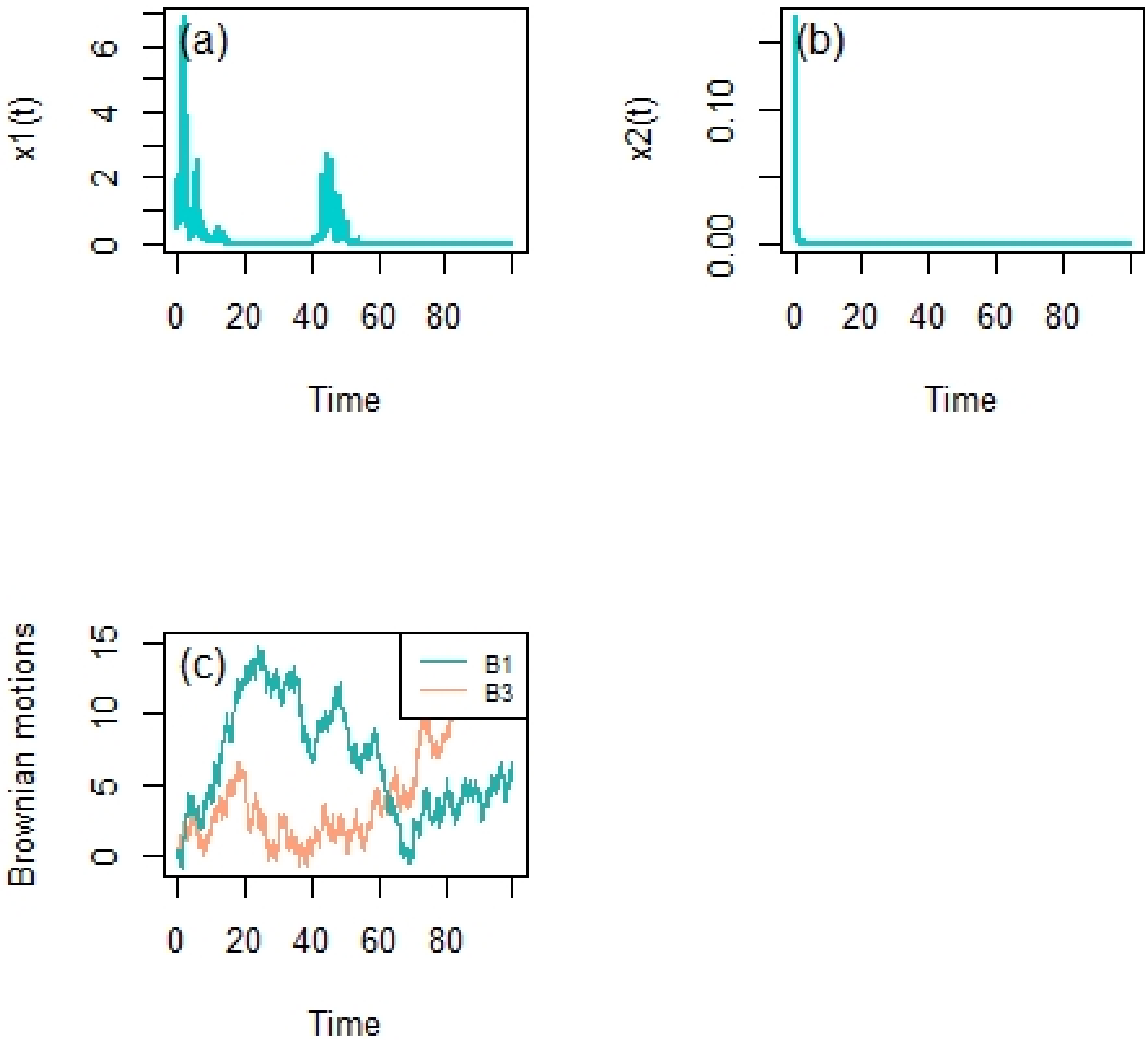

Figure 1: Numerical simulations of the paths (a) $x_1(t)$ and (b) $x_2(t)$ of SDE model (1.2) using the EM scheme with stepsize $\Delta = 0.01$ and initial value $x_0 = (0.7, 0.15)^T$ with the system parameters provided by (5.1). Times series of the correlated Brownian motions $B_1(t)$ and $B_3(t)$ is shown in (c).

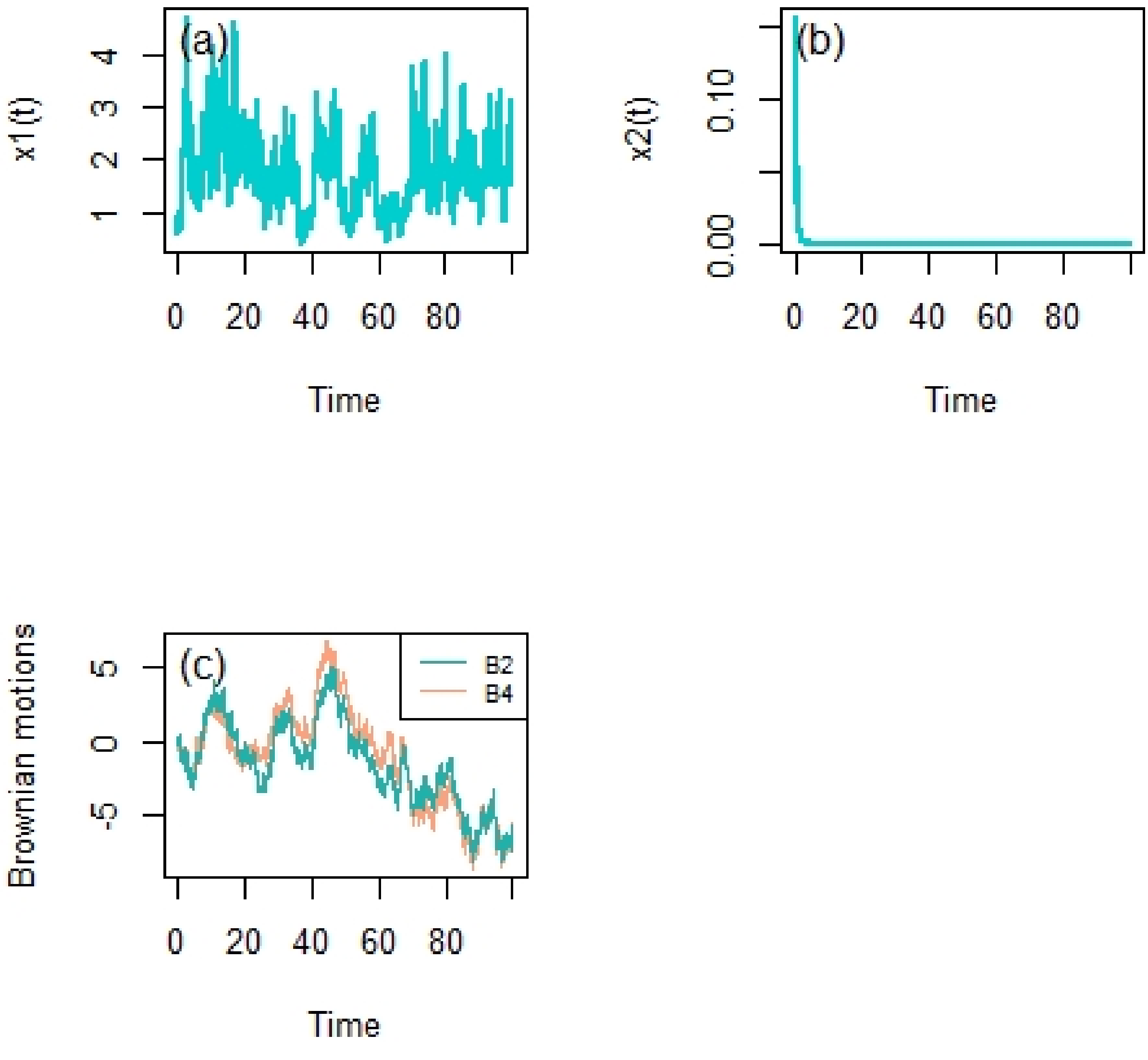

Figure 2: Under the system parameters described in Example 5.2, we obtain the numerical simulations of the paths (a) $x_1(t)$ and (b) $x_2(t)$ of SDE model (1.2) using the EM method with stepsize $\Delta = 0.01$ and initial value $x_0 = (0.7, 0.15)^T$. Times series of the correlated Brownian motions $B_2(t)$ and $B_4(t)$ is shown in (c).

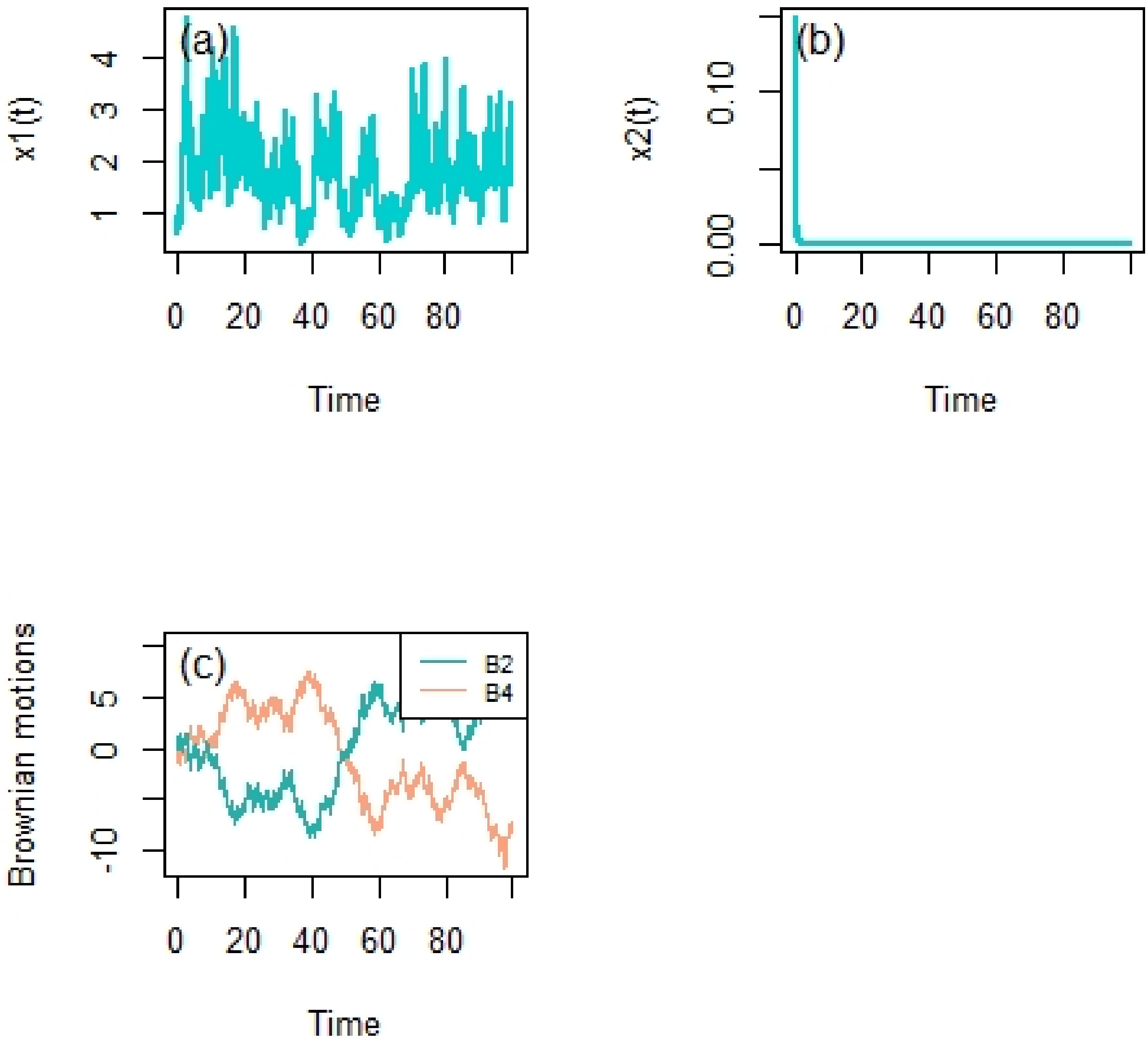

Figure 3: With the system parameters given in Example 5.3, we obtain the computer simulations of the paths (a) $x_1(t)$ and (b) $x_2(t)$ of SDE model (1.2) using the EM method with stepsize $\Delta = 0.01$ and initial value $x_0 = (0.7, 0.15)^T$. Times series of the correlated Brownian motions $B_2(t)$ and $B_4(t)$ is shown in (c).

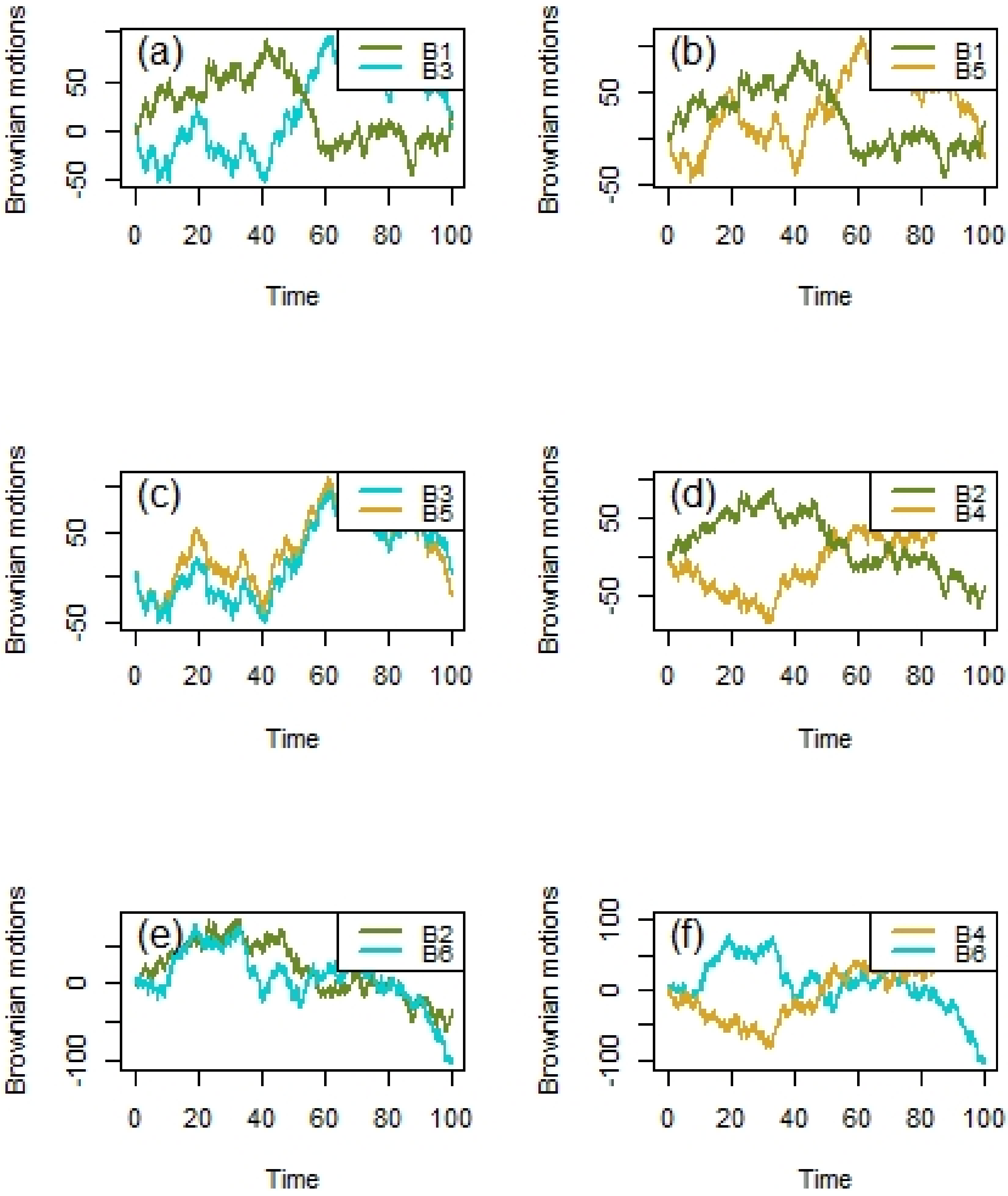

Figure 4: Time series of the correlated Brownian motions adopted in Example 5.4 and 5.5.

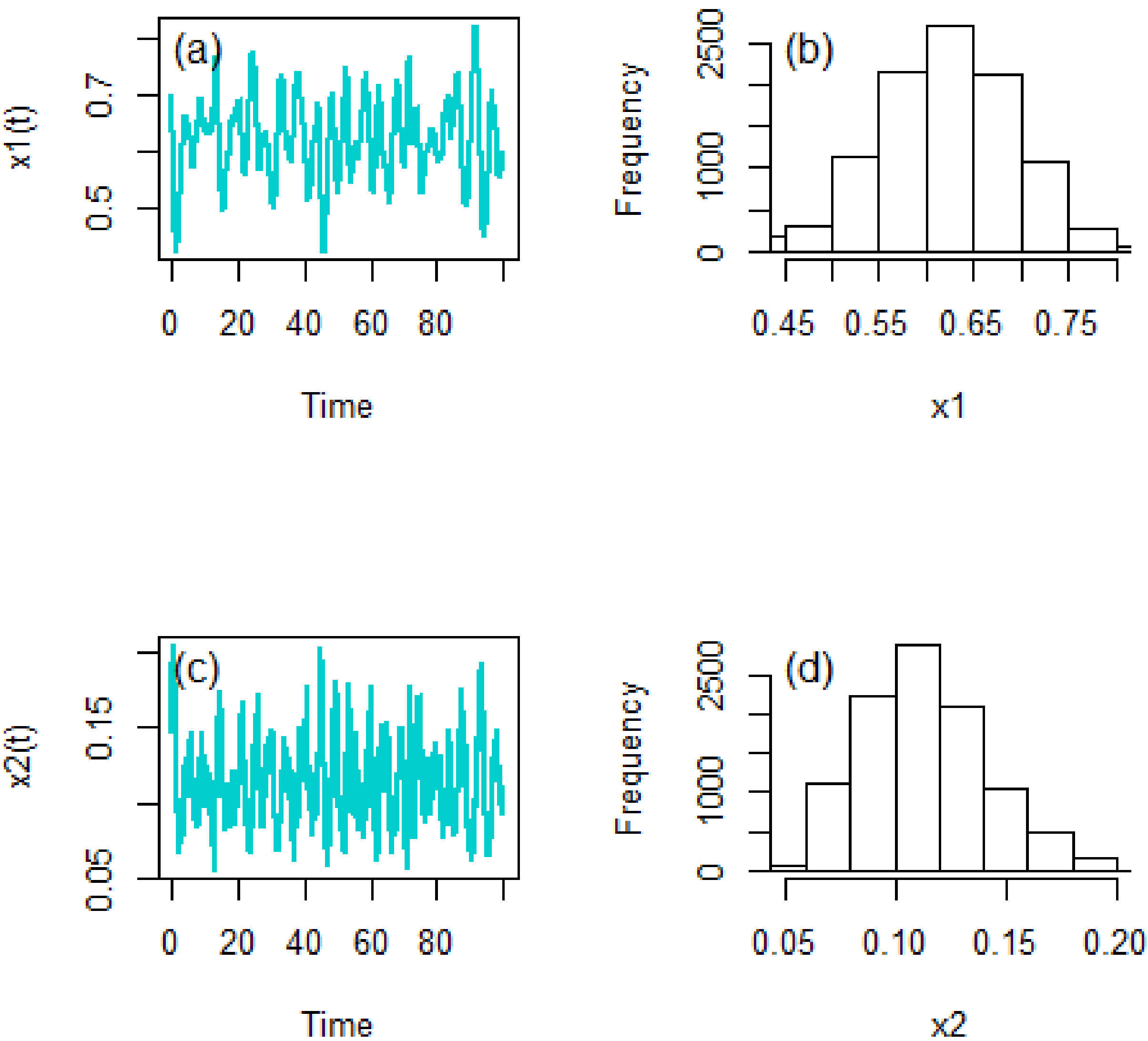

Figure 5: Numerical simulations of the paths (a) $x_1(t)$ and (c) $x_2(t)$ of SDE model (1.2) based on the model parameters described in Example 5.4 using the EM technique with stepsize $\Delta = 0.01$ and initial value $x_0 = (0.7, 0.15)^T$, followed by the histograms for the SDE paths (b) $x_1(t)$ and (d) $x_2(t)$.

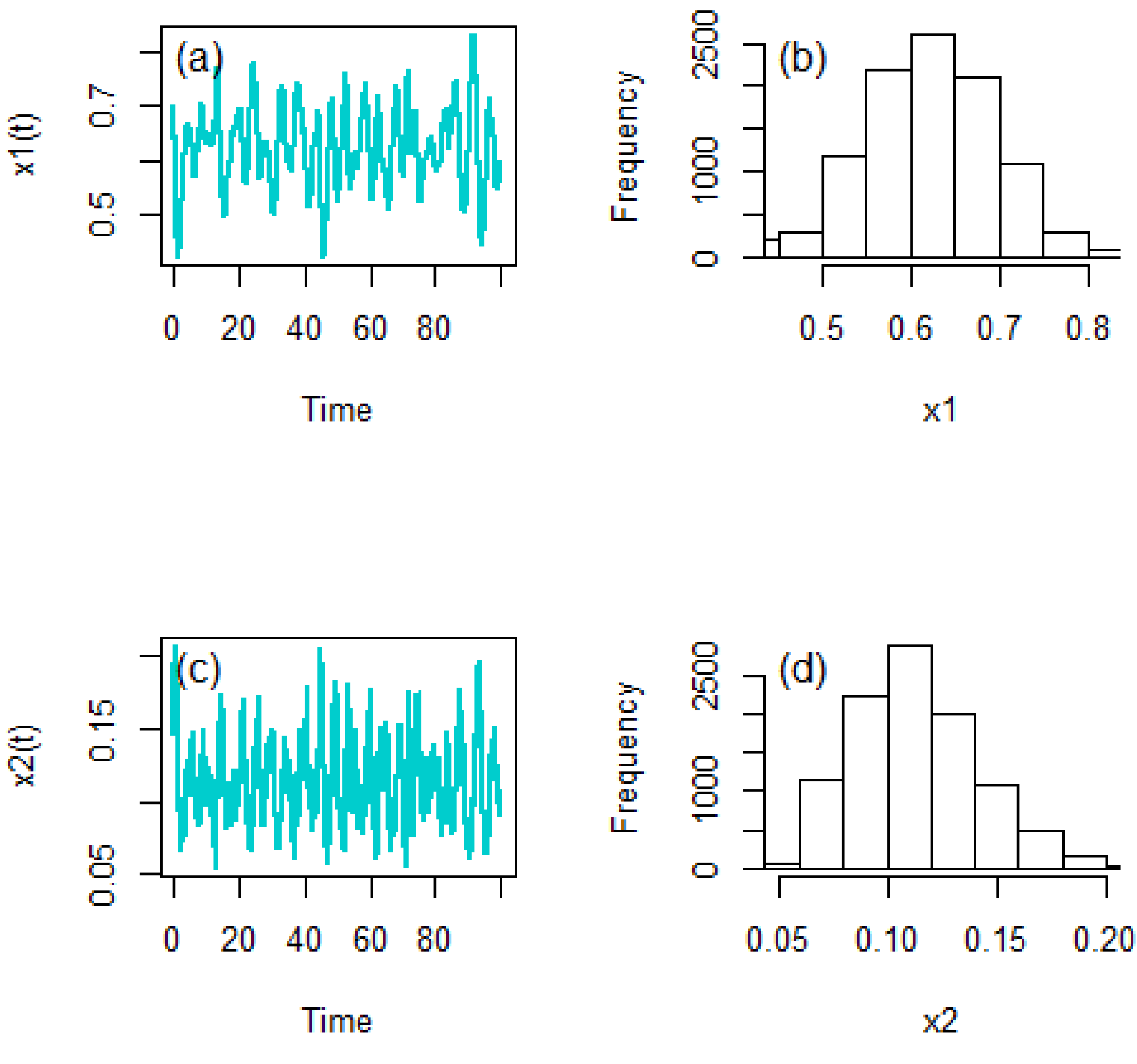

Figure 6: Computer simulations of the paths (a) $x_1(t)$ and (c) $x_2(t)$ of SDE model (1.3) based on the model parameters provided in Example 5.5 using the EM technique with stepsize $\Delta = 0.01$ and initial value $x_0 = (0.7, 0.15)^T$, followed by the histograms for the SDE paths (b) $x_1(t)$ and (d) $x_2(t)$.

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