Online Supplement for
Valid Inequalities for Two-Period Relaxations of Big-Bucket Lot-Sizing Problems: Zero Setup Case[☆]

Mahdi Doostmohammadi[a], Kerem Akartunalı[b,*]

[a] *School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, UK*
[b] *Department of Management Science, University of Strathclyde, Glasgow G4 0GE, UK*

Abstract

This online supplement provides some key rigorous and lengthy proofs that were not included in the paper entitled "Valid Inequalities for Two-Period Relaxations of Big-Bucket Lot-Sizing Problems: Zero Setup Case".

Keywords: Lot-Sizing, Integer Programming, Polyhedral Analysis, Valid Inequalities
2000 MSC: 90B30, 90C11, 90C57

PROOF OF PROPOSITION 8. First, we prove the claim regarding the cover inequalities. Let j^1, j^2 be any two members of S^+, and $\epsilon > 0$ be a sufficiently small number. We present the following $3NI$ points that satisfy $\sum_{i\in S} x^i - \sum_{i\in S}(d^i-\lambda)^+ y^i = \sum_{i\in S} s^i + C - \sum_{i\in S}(d^i-\lambda)^+$:

1. For every $i' \in S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S\backslash\{i'\}$, and set all other variables to zero. **($|S^+|$ points)**
2. For every $i' \in S^+$, set $x^{i'} = d^{i'} - \lambda$ and $y^{i'} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S\backslash\{i'\}$, and set all other variables to zero. **($|S^+|$ points)**
3. For every $i' \in S\backslash S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^{j^1} = d^{j^1} - \lambda + d^{i'}$ and $y^{j^1} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S\backslash\{i', j^1\}$, and set all other variables to zero. **($|S\backslash S^+|$ points)**

[☆] The research presented in this paper is supported by the EPSRC grant EP/L000911/1, entitled "Multi-Item Production Planning: Theory, Computation and Practice".

[*] Corresponding author
Email addresses: mahdi.doostmohammadi@ed.ac.uk (Mahdi Doostmohammadi), kerem.akartunali@strath.ac.uk (Kerem Akartunalı)

4. For every $i' \in S \backslash S^+$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^2} = d^{j^2} - \lambda + d^{i'}$, and $y^{j^2} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i', j^2\}$, and set all other variables to zero. **(**$|S \backslash S^+|$ **points)**
5. For every $i' \in S \backslash \{j^1\}$, set $x^{i'} = d^{i'} + \epsilon$, $y^{i'} = 1$ and $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i', j^1\}$, and set all other variables to zero. **(**$|S| - 1$ **points)**
6. Set $x^{j^1} = d^{j^1} + \epsilon$, $y^{j^1} = 1$ and $s^{j^1} = \epsilon$, set $x^{j^2} = 0 = y^{j^2}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1, j^2\}$, and set all other variables to zero. **(1 point)**
7. For every $i' \in I \backslash S$, set $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **(**$NI - |S|$ **points)**
8. For every $i' \in I \backslash S$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **(**$NI - |S|$ **points)**
9. For every $i' \in I \backslash S$, set $x^{i'} = \epsilon$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **(**$NI - |S|$ **points)**

It is easy to observe that these $3NI$ points are affinely independent. This suffices to prove the claim about the facet-defining conditions for cover inequalities.

Next, we note that the condition for item-extended cover inequalities requires that $\bar{d}^i = \bar{d}, \forall i \in K$. Let $\bar{d} = d^{j^1}$, i.e., j^1 has the highest demand in set S. Next, we note that a majority of the affinely independent points provided in the above proof for cover inequalities can be used here: the first 7 sets of the points are valid, and the last two sets are valid for $i' \notin S \cup K$. Therefore we need $2|K|$ new points, which we present as follows, where $\epsilon > 0$ is a sufficiently small number:

1. For every $i' \in K$, set $x^{i'} = \bar{d} - \lambda$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, and set all other variables to zero. **(**$|K|$ **points)**
2. For every $i' \in K$, set $x^{i'} = \bar{d} - \lambda + \epsilon$ for a sufficiently small $\epsilon > 0$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^{j^2} = d^{j^2} - \epsilon$ and $y^{j^2} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1, j^2\}$, and set all other variables to zero. **(**$|K|$ **points)**

These $3NI$ affinely independent points suffice to prove the claim about the facet-defining conditions for item-extended cover inequalities. $\square$

PROOF OF PROPOSITION 10. Let j^1 be any member of T'^+, and $\epsilon > 0$ be a sufficiently small number. We present the following $3|I|$ points that satisfy the partition inequality as an equation:

1. For every $i' \in S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T'$, and set all other variables to zero. **($|S^+|$ points)**
2. For every $i' \in S^+$, set $x^{i'} = d^{i'} - \xi$ and $y^{i'} = 1$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T'$, and set all other variables to zero. **($|S^+|$ points)**
3. For every $i' \in T'^+$, set $x^{i'} = 0 = y^{i'}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i'\}$, and set all other variables to zero. **($|T'^+|$ points)**
4. For every $i' \in T'^+$, set $x^{i'} = M^{i'} - \xi$, $y^{i'} = 1$ and $s^{i'} = (M^{i'} - \xi - d^{i'})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i'\}$, and set all other variables to zero. **($|T'^+|$ points)**
5. For every $i' \in S \backslash S^+$, set $x^{i'} = 0 = y^{i'}$, set $x^{j^1} = M^{j^1} - \xi + d^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + d^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|S \backslash S^+|$ points)**
6. For every $i' \in S \backslash S^+$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = M^{j^1} - \xi + d^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + d^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|S \backslash S^+|$ points)**
7. For every $i' \in T' \backslash T'^+$, set $x^{i'} = 0 = y^{i'}$, set $x^{j^1} = M^{j^1} - \xi + M^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + M^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i', j^1\}$, and set all other variables to zero. **($|T' \backslash T'^+|$ points)**
8. For every $i' \in T' \backslash T'^+$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = M^{j^1} - \xi + M^{i'}$, $y^{j^1} = 1$ and $s^{j^1} = (M^{j^1} - \xi + M^{i'} - d^{j^1})^+$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i', j^1\}$, and set all other variables to zero. **($|T' \backslash T'^+|$ points)**
9. For every $i' \in S$, set $x^{i'} = d^{i'} + \epsilon$, $y^{i'} = 1$ and $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{i'\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|S|$ points)**
10. Set $x^{j^1} = 0 = y^{j^1}$ and $s^{j^1} = \epsilon$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **(1 point)**

11. For every $i' \in T''$, set $s^{i'} = \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|T''|$ points)**
12. For every $i' \in T' \backslash \{j^1\}$, set $x^{i'} = M^{i'}$, $y^{i'} = 1$ and $s^{i'} = (M^{i'} - d^{i'})^+ + \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{i', j^1\}$, and set all other variables to zero. **($|T'| - 1$ points)**
13. For every $i' \in T''$, set $x^{i'} = 0$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|T''|$ points)**
14. For every $i' \in T''$, set $x^{i'} = \epsilon$ and $y^{i'} = 1$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|T''|$ points)**

We note that the affine independence of these $3|I|$ points is straightforward and therefore omitted here for the sake of brevity. □

PROOF OF PROPOSITION 12. First, note that the condition requires that $\bar{p}^i = \bar{p}, \forall i \in K$. Next, we note that a majority of the affinely independent points provided in the proof of Proposition 10 are also valid for this proof: we can use the first 12 sets of the points without any change, and the last two sets are valid for $i' \in T'' \backslash K$. Therefore we need to present $2|K|$ new points, which we list as follows. For these points, we let $\bar{p} = p^{j^1}$, i.e., j^1 has the highest p^i value in set $S \cup T'$, and also define $\epsilon > 0$, which is a sufficiently small number. We also identify another element $j^2 \in S \cup T'$, where $j^2 \in S$ if $j^1 \in T'^+$ or $j^2 \in T'$ if $j^1 \in S^+$.

1. For every $i' \in K$, set $x^{i'} = \bar{p} - \xi$, $y^{i'} = 1$ and $s^{i'} = (\bar{p} - \xi - d^{i'})^+$, set $x^{j^1} = 0 = y^{j^1}$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1\}$, and set all other variables to zero. **($|K|$ points)**
2. For every $i' \in K$, set $x^{i'} = \bar{p} - \xi + \epsilon$, $y^{i'} = 1$ and $s^{i'} = (\bar{p} - \xi - d^{i'})^+ + \epsilon$, set $x^{j^1} = 0 = y^{j^1}$, set $x^{j^2} = M^{j^2} - \epsilon$, $y^{j^2} = 1$ and $s^{j^2} = (M^{j^2} - \epsilon - d^{j^2})^+$ if $j^2 \in T'$ or $x^{j^2} = d^{j^2} - \epsilon$, $y^{j^2} = 1$ and $s^{j^2} = (d^{j^2} - \epsilon - d^{j^2})^+$ if $j^2 \in S$, set $x^i = d^i$ and $y^i = 1$ for all $i \in S \backslash \{j^1, j^2\}$, set $x^i = M^i$, $y^i = 1$ and $s^i = (M^i - d^i)^+$ for all $i \in T' \backslash \{j^1, j^2\}$, and set all other variables to zero. **($|K|$ points)**

These $3|I|$ affinely independent points suffice to prove the claim. □

PROOF OF PROPOSITION 14. Let $(x,y,s)\in X^{2PL}$, and $T_k=\{i\in I|y_k^i=1\}$, for $k\in\{1,2\}$. Then we consider two cases as follows:
Case I: $|S_t^+\backslash T_t|\le|K_t\cap T_t|$. Then, we have

$$\sum_{i\in S_t\cup K_t}x_t^i+\sum_{i\in L_{t'}}x_{t'}^i+\sum_{i\in S_t}(\widetilde{d}_t^i-\lambda_t)^+(1-y_t^i)-\sum_{i\in K_t}(\bar{d}_t^i-\lambda_t)y_t^i-\sum_{i\in L_{t'}}\widetilde{d}_{t'}^i y_{t'}^i$$

$$=\sum_{i\in S_t\cup K_t}x_t^i+\sum_{i\in L_{t'}}x_{t'}^i+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\lambda_t)-\sum_{i\in K_t}(\bar{d}_t^i-\lambda_t)y_t^i-\sum_{i\in L_{t'}}\widetilde{d}_{t'}^i y_{t'}^i$$

$$=\sum_{i\in S_t\cup K_t}x_t^i+\sum_{i\in L_{t'}\cap T_{t'}}x_{t'}^i+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\lambda_t)-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i-\lambda_t)-\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i$$

$$\le\widetilde{C}_t+\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i+\sum_{i\in L_{t'}\cap T_{t'}}s^i+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\lambda_t)-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i-\lambda_t)-\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i$$

$$\le\widetilde{C}_t+\sum_{i\in S_t}s^i+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\lambda_t)-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i-\lambda_t)\le\widetilde{C}_t+\sum_{i\in S_t}s^i+\sum_{i\in S_t^+\backslash T_t}(\bar{d}_t-\lambda_t)$$

$$-\sum_{i\in K_t\cap T_t}(\bar{d}_t-\lambda_t)=\widetilde{C}_t+\sum_{i\in S_t}s^i+\Big(|S_t^+\backslash T_t|-|K_t\cap T_t|\Big)(\bar{d}_t-\lambda_t)\le\widetilde{C}_t+\sum_{i\in S_t}s^i$$

where the first inequality follows the capacity constraint in period t, $x_{t'}^i\le\widetilde{d}_{t'}^i y_{t'}^i+s^i$ and the property $y_{t'}^i=1, i\in T_{t'}$, the second inequality follows the fact that $L_{t'}\subseteq S_t$, the third inequality follows $\widetilde{d}_t^i\le\bar{d}_t\le\bar{d}_t^i$, and the last inequality uses $|S_t^+\backslash T_t|\le|K_t\cap T_t|$ and $\bar{d}_t\ge\lambda_t$.
Case II: $|S_t^+\backslash T_t|\ge|K_t\cap T_t|+1$. Then, we have

$$\sum_{i\in S_t\cup K_t}x_t^i+\sum_{i\in L_{t'}}x_{t'}^i+\sum_{i\in S_t}(\widetilde{d}_t^i-\lambda_t)^+(1-y_t^i)-\sum_{i\in K_t}(\bar{d}_t^i-\lambda_t)y_t^i-\sum_{i\in L_{t'}}\widetilde{d}_{t'}^i y_{t'}^i$$

$$=\sum_{i\in S_t}x_t^i+\sum_{i\in L_{t'}}x_{t'}^i+\sum_{i\in K_t\cap T_t}x_t^i+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\lambda_t)-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i-\lambda_t)-\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i$$

$$=\sum_{i\in S_t\backslash L_{t'}}x_t^i+\sum_{i\in L_{t'}}(x_t^i+x_{t'}^i)+\sum_{i\in K_t\cap T_t}x_t^i+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\lambda_t)-\sum_{i\in K_t\cap T_t}(\bar{d}_t^i-\lambda_t)$$

$$-\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i\le\sum_{i\in(S_t\backslash L_{t'})\cap T_t}\widetilde{d}_t^i+\sum_{i\in S_t\backslash L_{t'}}s^i+\sum_{i\in L_{t'}\cap T_t}\widetilde{d}_t^i+\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i+\sum_{i\in L_{t'}}s^i$$

$$+\sum_{i\in K_t\cap T_t}\widetilde{M}_t^i+\sum_{i\in S_t^+\backslash T_t}\widetilde{d}_t^i-|S_t^+\backslash T_t|\lambda_t-\sum_{i\in K_t\cap T_t}\bar{d}_t^i+|K_t\cap T_t|\lambda_t-\sum_{i\in L_{t'}\cap T_{t'}}\widetilde{d}_{t'}^i$$

$$\leq\sum_{i\in S_t}\widetilde{d}_t^i+\sum_{i\in S_t}s^i+\Big(|K_t\cap T_t|-|S_t^+\backslash T_t|\Big)\lambda_t+\lambda_t-\lambda_t+\sum_{i\in K_t\cap T_t}(\widetilde{M}_t^i-\bar{d}_t^i)$$

$$\leq\widetilde{C}_t+\sum_{i\in S_t}s^i+\Big(|K_t\cap T_t|-|S_t^+\backslash T_t|+1\Big)\lambda_t\leq\widetilde{C}_t+\sum_{i\in S_t}s^i$$

where the first inequality uses $x_t^i\leq\widetilde{d}_t^iy_t^i+s^i$, $x_t^i+x_{t'}^i\leq\widetilde{d}_t^iy_t^i+\widetilde{d}_{t'}^iy_{t'}^i+s^i$, $x_t^i\leq\widetilde{M}_t^iy_t^i$, and the property $y_k^i=1, i\in T_k, k\in\{1,2\}$, the second inequality exploits the disjoint sets in the previous expression, the third inequality uses the definition of λ_t and $\widetilde{M}_t^i\leq\bar{d}_t^i$, and finally the last inequality uses $\lambda_t>0$ and $|S_t^+\backslash T_t|\geq|K_t\cap T_t|+1$. □

PROOF OF PROPOSITION 17. Let $(x,y,s)\in X^{2PL}$. For $k\in\{t,t'\}$, we define $T_k=\{i\in I|y_k^i=1\}$, $S_k^+=\{i\in S_k|\widetilde{d}_k^i>\theta_k\}$ and $T_k'^+=\{i\in T_k'|\widetilde{M}_k^i>\theta_k\}$. By using these new definitions, first we rewrite the left-hand side of the inequality as follows:

$$\sum_{i\in S_t\cup T_t'}x_t^i+\sum_{i\in S_{t'}\cup T_{t'}'}x_{t'}^i+\sum_{i\in S_t}(\widetilde{d}_t^i-\theta_t)^+(1-y_t^i)+\sum_{i\in T_t'}(\widetilde{M}_t^i-\theta_t)^+(1-y_t^i)$$

$$+\sum_{i\in S_{t'}}(\widetilde{d}_{t'}^i-\theta_{t'})^+(1-y_{t'}^i)+\sum_{i\in T_{t'}'}(\widetilde{M}_{t'}^i-\theta_{t'})^+(1-y_{t'}^i)=\sum_{i\in S_t\cup T_t'}x_t^i+\sum_{i\in S_{t'}\cup T_{t'}'}x_{t'}^i$$

$$+\sum_{i\in S_t^+\backslash T_t}(\widetilde{d}_t^i-\theta_t)+\sum_{i\in T_t'^+\backslash T_t}(\widetilde{M}_t^i-\theta_t)+\sum_{i\in S_{t'}^+\backslash T_{t'}}(\widetilde{d}_{t'}^i-\theta_{t'})+\sum_{i\in T_{t'}'^+\backslash T_{t'}}(\widetilde{M}_{t'}^i-\theta_{t'})$$

This equation is valid since all other terms will be zero. We evaluate four cases next, where we first rewrite this expression we derived above:

Case I: $|S_t^+\backslash T_t|+|T_t'^+\backslash T_t|+|S_{t'}^+\backslash T_{t'}|+|T_{t'}'^+\backslash T_{t'}|=0$.

$$=\sum_{i\in S_t\cup T_t'}x_t^i+\sum_{i\in S_{t'}\cup T_{t'}'}x_{t'}^i\leq\widetilde{C}_t+\widetilde{C}_{t'}+\sum_{i\in S_t\cup S_{t'}}s^i$$

where the first equality holds as all these subsets are empty, and the first inequality uses the capacity limitations for both periods and $s^i\geq 0$.

Case II: $|S_t^+ \backslash T_t| + |T_t'^+ \backslash T_t| \geq 1$ *and* $|S_{t'}^+ \backslash T_{t'}| + |T_{t'}'^+ \backslash T_{t'}| = 0$.

$$= \sum_{i \in S_t \cup T_t'} x_t^i + \sum_{i \in S_{t'} \cup T_{t'}'} x_{t'}^i + \sum_{i \in S_t^+ \backslash T_t} (\widetilde{d}_t^i - \theta_t) + \sum_{i \in T_t'^+ \backslash T_t} (\widetilde{M}_t^i - \theta_t) \leq \sum_{i \in S_t} \widetilde{d}_t^i y_t^i$$

$$+ \sum_{i \in S_t} s^i + \sum_{i \in T_t'} \widetilde{M}_t^i y_t^i + \sum_{i \in S_{t'} \cup T_{t'}'} x_{t'}^i + \sum_{i \in S_t^+ \backslash T_t} \widetilde{d}_t^i - \sum_{i \in S_t^+ \backslash T_t} \theta_t + \sum_{i \in T_t'^+ \backslash T_t} \widetilde{M}_t^i - \sum_{i \in T_t'^+ \backslash T_t} \theta_t$$

$$= \sum_{i \in S_t \cap T_t} \widetilde{d}_t^i + \sum_{i \in S_t} s^i + \sum_{i \in T_t' \cap T_t} \widetilde{M}_t^i + \sum_{i \in S_{t'} \cup T_{t'}'} x_{t'}^i + \sum_{i \in S_t^+ \backslash T_t} \widetilde{d}_t^i - |S_t^+ \backslash T_t| \theta_t + \sum_{i \in T_t'^+ \backslash T_t} \widetilde{M}_t^i$$

$$- |T_t'^+ \backslash T_t| \theta_t \leq \sum_{i \in S_t} \widetilde{d}_t^i + \sum_{i \in S_t \cup S_{t'}} s^i + \sum_{i \in T_t'} \widetilde{M}_t^i + \widetilde{C}_{t'} - \Big(|S_t^+ \backslash T_t| + |T_t'^+ \backslash T_t| - 1 \Big) \theta_t - \theta_t$$

$$= \widetilde{C}_t + \widetilde{C}_{t'} + \sum_{i \in S_t \cup S_{t'}} s^i - \Big(|S_t^+ \backslash T_t| + |T_t'^+ \backslash T_t| - 1 \Big) \theta_t \leq \widetilde{C}_t + \widetilde{C}_{t'} + \sum_{i \in S_t \cup S_{t'}} s^i$$

where the first equality holds due to $|S_{t'}^+ \backslash T_{t'}| = 0 = |T_{t'}'^+ \backslash T_{t'}|$, the first inequality exploits the basic relations $x_t^i \leq \widetilde{M}_t^i y_t^i$ and $x_t^i \leq \widetilde{d}_t^i y_t^i + s^i$, the second equality uses the definition of T_t, the second inequality uses basic set operations, capacity constraint in period t' and the fact that $s^i \geq 0$, the third equality uses the definition of θ_t, and finally the last inequality exploits $|S_t^+ \backslash T_t| + |T_t'^+ \backslash T_t| \geq 1$ and $\theta_t \geq 0$.

Case III: $|S_t^+ \backslash T_t| + |T_t'^+ \backslash T_t| = 0$ *and* $|S_{t'}^+ \backslash T_{t'}| + |T_{t'}'^+ \backslash T_{t'}| \geq 1$. We note that this case can be proven in an identical fashion to Case II.

Case IV: $|S_t^+ \backslash T_t| + |T_t'^+ \backslash T_t| \geq 1$ *and* $|S_{t'}^+ \backslash T_{t'}| + |T_{t'}'^+ \backslash T_{t'}| \geq 1$.

$$= \sum_{i \in S_t \backslash S_{t'}} x_t^i + \sum_{i \in S_{t'} \backslash S_t} x_{t'}^i + \sum_{i \in S_{t'} \cap S_t} (x_t^i + x_{t'}^i) + \sum_{i \in T_t'} x_t^i + \sum_{i \in T_{t'}'} x_{t'}^i + \sum_{i \in S_t^+ \backslash T_t} (\widetilde{d}_t^i - \theta_t)$$

$$+ \sum_{i \in T_t'^+ \backslash T_t} (\widetilde{M}_t^i - \theta_t) + \sum_{i \in S_{t'}^+ \backslash T_{t'}} (\widetilde{d}_{t'}^i - \theta_{t'}) + \sum_{i \in T_{t'}'^+ \backslash T_{t'}} (\widetilde{M}_{t'}^i - \theta_{t'}) \leq \sum_{i \in S_t \backslash S_{t'}} \widetilde{d}_t^i y_t^i + \sum_{i \in S_t \backslash S_{t'}} s^i$$

$$+ \sum_{i \in S_{t'} \backslash S_t} \widetilde{d}_{t'}^i y_{t'}^i + \sum_{i \in S_{t'} \backslash S_t} s^i + \sum_{i \in S_{t'} \cap S_t} \widetilde{d}_t^i y_t^i + \sum_{i \in S_{t'} \cap S_t} \widetilde{d}_{t'}^i y_{t'}^i + \sum_{i \in S_{t'} \cap S_t} s^i + \sum_{i \in T_t'} \widetilde{M}_t^i y_t^i$$

$$
\begin{aligned}
&+\sum_{i\in T'_{t'}}\widetilde{M}^i_{t'}y^i_{t'}+\sum_{i\in S^+_t\setminus T_t}(\widetilde{d}^i_t-\theta_t)+\sum_{i\in T'^+_t\setminus T_t}(\widetilde{M}^i_t-\theta_t)+\sum_{i\in S^+_{t'}\setminus T_{t'}}(\widetilde{d}^i_{t'}-\theta_{t'})\\
&+\sum_{i\in T'^+_{t'}\setminus T_{t'}}(\widetilde{M}^i_{t'}-\theta_{t'})=\sum_{i\in(S_t\setminus S_{t'})\cap T_t}\widetilde{d}^i_t+\sum_{i\in(S_{t'}\setminus S_t)\cap T_{t'}}\widetilde{d}^i_{t'}+\sum_{i\in(S_t\cap S_{t'})\cap T_t}\widetilde{d}^i_t\\
&+\sum_{i\in(S_{t'}\cap S_t)\cap T_{t'}}\widetilde{d}^i_{t'}+\sum_{i\in S_{t'}\cup S_t}s^i+\sum_{i\in T'_t\cap T_t}\widetilde{M}^i_t+\sum_{i\in T'_{t'}\cap T_{t'}}\widetilde{M}^i_{t'}+\sum_{i\in S^+_t\setminus T_t}\widetilde{d}^i_t-|S^+_t\setminus T_t|\theta_t\\
&+\sum_{i\in T'^+_t\setminus T_t}\widetilde{M}^i_t-|T'^+_t\setminus T_t|\theta_t+\sum_{i\in S^+_{t'}\setminus T_{t'}}\widetilde{d}^i_{t'}-|S^+_{t'}\setminus T_{t'}|\theta_{t'}+\sum_{i\in T'^+_{t'}\setminus T_{t'}}\widetilde{M}^i_{t'}-|T'^+_{t'}\setminus T_{t'}|\theta_{t'}\\
&\leq\sum_{i\in S_t}\widetilde{d}^i_t+\sum_{i\in S_{t'}}\widetilde{d}^i_{t'}+\sum_{i\in S_{t'}\cup S_t}s^i+\sum_{i\in T'_t}\widetilde{M}^i_t+\sum_{i\in T'_{t'}}\widetilde{M}^i_{t'}-\Big(|S^+_t\setminus T_t|+|T'^+_t\setminus T_t|-1\Big)\theta_t\\
&-\Big(|S^+_{t'}\setminus T_{t'}|+|T'^+_{t'}\setminus T_{t'}|-1\Big)\theta_{t'}-\theta_t-\theta_{t'}\leq\widetilde{C}_t+\widetilde{C}_{t'}+\sum_{i\in S_t\cup S_{t'}}s^i
\end{aligned}
$$

where the first equality uses basic set operations, the first inequality uses the basic relations $x^i_k\leq\widetilde{M}^i_k y^i_k$ and $x^i_k\leq\widetilde{d}^i_k y^i_k+s^i$ for $k=\{t,t'\}$ as well as $x^i_t+x^i_{t'}\leq\widetilde{d}^i_t y^i_t+\widetilde{d}^i_{t'}y^i_{t'}+s^i$, the second equality uses the definitions of T_t and $T_{t'}$ as well as basic set operations, the second inequality exploits unions of subsets, and finally the last inequality uses the definitions of ξ_t and $\xi_{t'}$ as well as the conditions used to define this case as well as $\xi_t\geq 0$ and $\xi_{t'}\geq 0$. □