Time-delayed auto-synchronous swarm control

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Abstract

In this paper a general Morse potential model of self-propelling particles is considered in the presence of a time-delayed term and a spring potential. It is shown that the emergent swarm behavior is dependent on the delay term and weights of the time-delayed function which can be set to induce a stationary swarm, a rotating swarm with uniform translation and a rotating swarm with a stationary center-of-mass. An analysis of the mean field equations shows that without a spring potential the motion of the center-of-mass is determined explicitly by a multi-valued function. For a non-zero spring potential the swarm converges to a vortex formation about a stationary center-of-mass, except at discrete bifurcation points where the center-of-mass will periodically trace an ellipse. The analytical results defining the behavior of the center-of-mass are shown to correspond with the numerical swarm simulations.

I. INTRODUCTION

In nature swarms of social entities such as insects, birds and fish, self-organize through local communications as opposed to centralized behavioral control. Mathematical investigations into the emergent spatio-temporal patterns of such swarms have been used to gain an understanding of the mechanism that drives this natural phenomena[1–7]. In turn this research has led to a number of efficient algorithms designed to control swarms of autonomous systems[8–13].

Many different mathematical approaches have been used to describe de-centralized swarm behavior. A common approach to modeling coherent swarms is in the use of Artificial Potential Functions (APFs)[12–19]. APFs have gained popularity in algorithms for de-centralized swarm control of autonomous systems as they are simple to implement, their emergent behavior is often verifiable analytically, see for example,[21] and they can be used for obstacle avoidance[22].

This paper focusses on approaches that have been used to model rotation in swarms of self-propelled particles which are either translating or with a stationary center-of-mass[23–25]. These models[23–25] all use APFs combined with additional terms to induce rotating swarms. In McInnes[23] a Morse APF was combined with a velocity alignment function requiring information on the relative velocity of each particle to induce vortex formations. In Ebeling. et al.[24] it was shown that a translating swarm induced by a harmonic attractive APF transitioned to a rotational motion in the presence of noise (with a large enough intensity) and in Schwartz and Forgotson[25] a purely attractive APF in the presence of noise and the addition of a communication time-delay was investigated. This showed that the delay induced transition from translational to rotational motion was associated with a super-critical hopf bifurcation as the value of a coupling parameter was increased. The models used by[24,25] have a computational advantage over the model in[23] as the swarm control algorithms do not require information on the relative velocity. However, the model in[23] is deterministic and the mean field equations can be investigated without imposing assumptions such as the equivalence of deterministic averaging and statistical averaging or simply ignoring the stochastic perturbations.

In this paper a method for inducing rotational motion of a swarm that interacts via APFs and time-delay auto synchronization (T-DAS)[26] is presented. Similarly to[25] a delay parameter is introduced into the equations, but in this case it is a delay in a velocity term rather than a delay in the relative position of each particle. The delay in[25] is introduced to account for communication time delays. However, the delay term here is considered purely as a feedback mechanism[26,27] requiring the ability to sense current state and store information on historical state. An investigation of the effect of a time-delay directly on an APF without the presence of noise is undertaken. It is shown that noise is not required to induce rotational motion with a stationary center-of-mass and can be a purely delay dependent phenomena. In comparison to previous deterministic algorithms to induce vortex formations in self-propelled particles this method does not require relative velocity information so is computationally more efficient. Furthermore, the completely deterministic mean field equations are shown to be linear delay differential equations that allow a complete stability analysis to be undertaken without the need for sophisticated numerical tools.

We consider a two-dimensional (2-D) model of a swarm that consists of homogeneous, self-propelled agents ($1 \leq i \leq N$) that are interacting through the following APF, $U(\mathbf{x}_i)$:

$$U(\mathbf{x}_i) = \sum_{j,j\neq i} \left[C_r \exp^{-\frac{|\mathbf{x}_{ij}|}{L_r}} - C_a \exp^{-\frac{|\mathbf{x}_{ij}|}{L_a}} \right] + \beta \frac{m_i}{2} |\mathbf{x}_i(t)|^2 \tag{1}$$

where, $\mathbf{x}_i$ is the position vector of agent i with corresponding mass m_i and $\mathbf{x}_{ij}$ is the relative position vector of agents i and j, C_a, C_r and L_a,L_r represent the amplitude and range of the attractive and repulsive potential respectively. Two cases of the APF are considered when $\beta = 0$ and $\beta = 1$. The Morse potential (Equ. (1) with $\beta = 0$) is used to provide long-range attraction and weak short-range repulsion (collision-free motion) for the swarm of agents[?] . The spring potential ($\frac{m_i}{2} |\mathbf{x}_i(t)|^2$) is used to bound the motion of the swarm about the origin. The swarm behavior is induced by the following equations of motion

$$\dot{\mathbf{x}}_i = \mathbf{v}_i \tag{2}$$

where $\mathbf{v}_i$ defines the mechanism of self-propulsion and

$$m_i \dot{\mathbf{v}}_i = -\nabla_i U(\mathbf{x}_i) + \mathbf{u}_i(t), \tag{3}$$

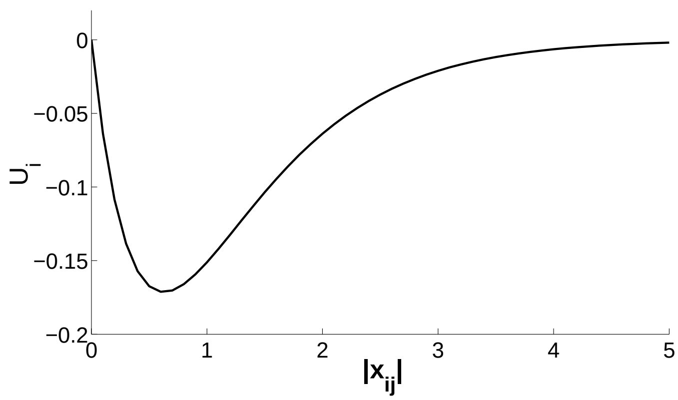
where,

$$\mathbf{u}_i(t) = a m_i \ \mathbf{v}_i(t-\tau) - b m_i \ \mathbf{v}_i(t), \tag{4}$$

with a and b are arbitrary constants and τ a delay term. The dissipation term (4) is of the form of a time-delayed feedback control or time-delayed auto synchronization (T-DAS) a method originally posed by Pyragas[26]. The following section considers the case when $\beta = 0$ (no spring potential) and investigates the interaction between T-DAS and the Morse potential function.

II. SIMULATION AND ANALYSIS FOR $\beta = 0$

For this study the parameters of the potential function are taken to be $\beta = 0, C_a = 1, L_a = 0.8, C_r = 1, L_r = 0.5$ which yields the potential function illustrated in Figure 1 Numerical



(i)

FIG. 1: The Morse potential as a function of agent separation

simulations were undertaken for agents in the x-y plane. An example is given in Fig. 2 where the velocity of each agent is illustrated. In Figure 2 (i) $b > a$ whereby the feedback control magnitude and direction is dominated by its current velocity. As the feedback control acts in the opposite direction to the current velocity it will act as a dissipative force and the speed of each agent will converges to zero i.e. the center-of-mass stops. In Figure 2 (ii) $a > b$ the feedback control mechanism is dominated by the delayed velocity. As this component of the feedback acts in the same direction as the agent's motion the magnitude of velocity will continuously increase. In this case the center-of-mass diverges exponentially. At the bifurcation point $a = b$ the velocity of each agent is non-zero yet bounded, as illustrated in Figure 2 (iii), with the swarm converging to a uniform rotating and translating motion. This qualitative behavior can be characterized by the stability of the center-of-mass. Furthermore, the behavior of the center-of-mass can be verified analytically by analyzing the swarms mean

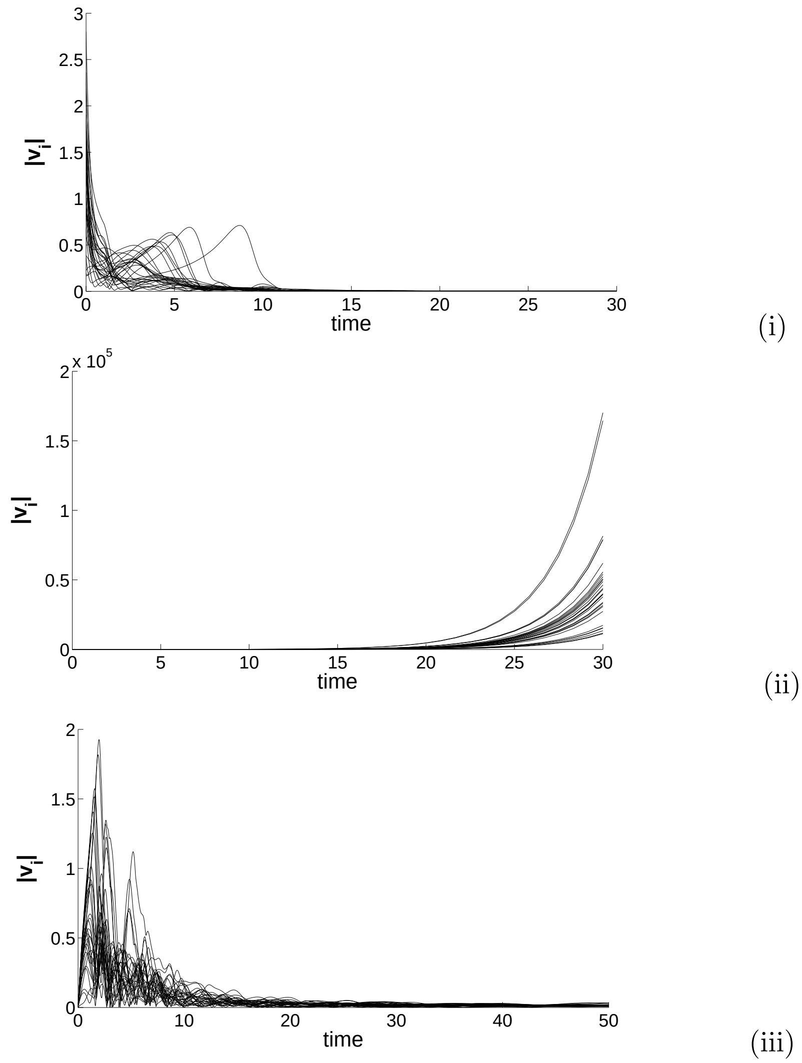


FIG. 2: The magnitude of the velocity history for 30 agents in the swarm given by numerical simulation with random initial conditions (i) $b > a$ all velocities converge to zero (asymptotically stable) (ii) $a > b$ the velocities diverge rapidly (unstable) (iii) $a = b$ the velocities are non-zero but bounded (marginally-stable)

field equations. The mean field equations are derived by defining the position $\mathbf{R}_c$, velocity $\dot{\mathbf{R}}_c$ and acceleration $\ddot{\mathbf{R}}_c$ of the center-of-mass of the swarm by Eq. (5)

$$\mathbf{R}_c = \frac{\sum_i m_i \mathbf{x}_i}{\sum_i m_i}, \quad \dot{\mathbf{R}}_c = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i}, \quad \ddot{\mathbf{R}}_c = \frac{\sum_i m_i \dot{\mathbf{v}}_i}{\sum_i m_i}. \tag{5}$$

then summing over all agents in Eq. (3), with delay term (4) included, yields

$$\sum_i m_i \dot{\mathbf{v}}_i(t) = -b \sum_i m_i \mathbf{v}_i(t) + a \sum_i m_i \mathbf{v}_i(t-\tau), \tag{6}$$

where, $\sum_i \nabla_i U(\mathbf{x}_{ij}) = 0$ due to internal symmetry in the swarm. The center-of-mass of the swarm can, thus, be expressed combining equations (5) and (6) to yield the mean field equations:

$$\ddot{\mathbf{R}}_c(t) = a\dot{\mathbf{R}}_c(t-\tau) - b\dot{\mathbf{R}}_c(t), \tag{7}$$

which after using the change of variable,

$$x(t) = \dot{\mathbf{R}}_c(t) \text{ and } \dot{x}(t) = \ddot{\mathbf{R}}_c(t), \tag{8}$$

is rewritten as

$$\dot{x}(t) = -bx(t) + ax\left(t-\tau\right). \tag{9}$$

The stability analysis of this equation will then determine the behavior of the center-of-mass of the swarm. Assuming equation (9) to have a wave function as a solution of the form, $x(t) = e^{\lambda_k t}$ with λ_k a complex number, then the characteristic equation associated with equation (9) is:

$$\lambda_k = -b + ae^{-\lambda_k \tau} \tag{10}$$

The solution to the transcendental equation (10) can be given analytically in terms of a Lambert function, as is well known for a one-dimensional linear time-delay differential equation[27]. By definition, the Lambert function $W(z)$, is a multi-valued function given implicitly by equation

$$z = W(z)e^{W(z)}, \tag{11}$$

with z any complex number.
So, equation (10) is first rewritten as

$$\tau\lambda_k e^{\lambda_k \tau} = \tau\left(-be^{\lambda_k \tau} + a\right), \tag{12}$$

then into

$$\left(b\tau + \lambda_k \tau\right) e^{\lambda_k \tau} e^{b\tau} = a\tau e^{b\tau}, \tag{13}$$

or

$$\left(b\tau + \lambda_k \tau\right) e^{\lambda_k \tau + b\tau} = a\tau e^{b\tau}. \tag{14}$$

From the definition of the Lambert function in equation (11), the solution to equation (14), is

$$b\tau + \lambda_k \tau = W(a\tau e^{b\tau}). \tag{15}$$

or

$$\lambda_k = \frac{-b\tau + W(a\tau e^{b\tau})}{\tau}. \tag{16}$$

Therefore, knowing properties of the Lambert function one can analyze the solution of equation (16) of the characteristic equation (10) and extract stability criteria which is primarily defined as $Re[\lambda_k] < 0$ for all λ_k. As a multi-valued function, the branches or the set of Lambert functions are denoted $W_k(z)$ with $k \in \mathbb{Z}$. For a given triplet, (b, a, τ), the set of solutions in equation (16) admits a clear leading eigenvalue, the rightmost eigenvalue. The value of this rightmost eigenvalue, that is given by λ_0, by conjecture determines the stability i.e. $Re[\lambda_0] < 0$ implies the center-of mass will converge. Fig. 3 illustrates a surface ($a = 1$) with the vertical axes corresponding to the real part of the right-most eigenvalue of the system, and the horizontal axis the parameter b and the delay τ. This illustrates the stable and unstable regions of the swarm, that is, when the center-of-mass stops and when it diverges rapidly. The eigen-modes for a subset of these values are also illustrated in Figure 4. This indicates that, in all cases, all of the eigen-modes converge to zero except in the case of the right-most eigen-mode which is the controlling mode. However, the right-most eigen-mode is dependent on the values of the parameters a and b as is illustrated. The equation of the velocity of the centre-of-mass (recall $x(t) = \dot{\mathbf{R}}_c(t)$) can also be explicitly defined as a solution of the Delay Differential Equation (DDE) equation (9) by,

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{\lambda_k t}. \tag{17}$$

where λ_k is defined by Equ. (16) and the coefficients C_k are dependent on the initial conditions.

III. SIMULATION AND ANALYSIS FOR $\beta = 1$

It has been shown in the previous section that the T-DAS term can be augmented to induce either stable (stationary), marginally stable (uniformly rotating and translating -

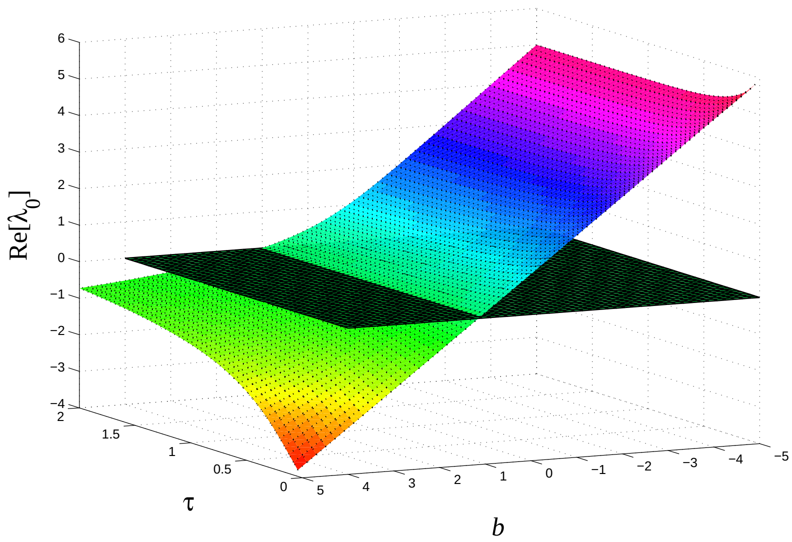


FIG. 3: (Color online) Real part of the right-most eigenvalue is represented by the multi-colored surface (grayscale surface in print) with varying b and τ intersecting the plane defined by $Re[\lambda_0] = 0$ represented by the single colored surface (black surface in print)

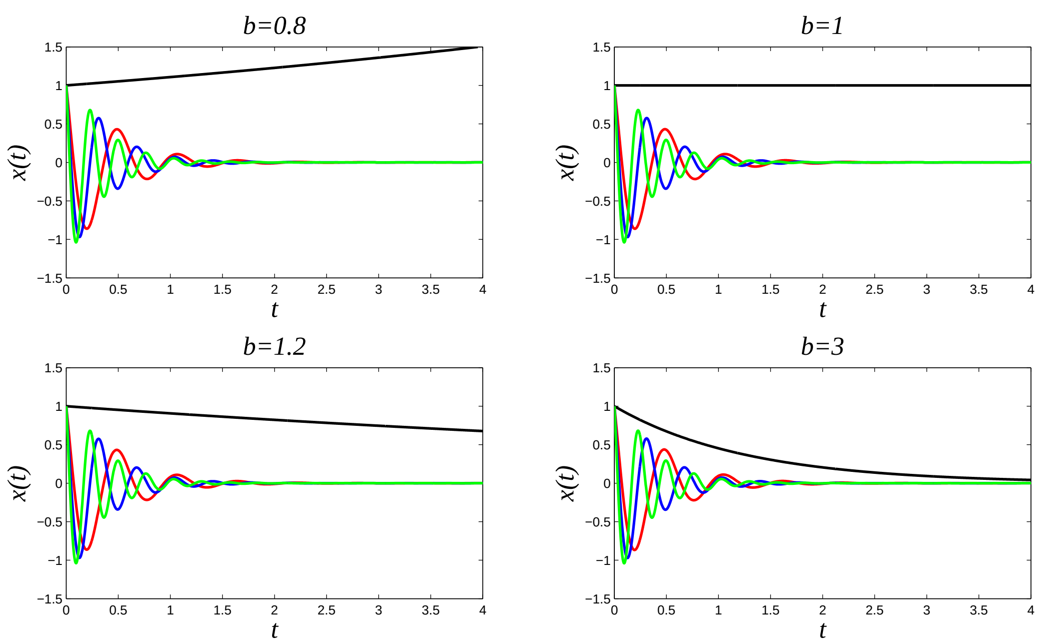


FIG. 4: (Color online) Evolution of the first 4 modes for different values of b and $a = \tau = 1$

bounded velocity) or unstable (exponentially diverging velocity). In this section we investigate the transition of these swarm topologies to rotating swarms with a stationary center-of-mass due to the addition of a spring potential. It is shown that introducing a spring potential alongside the Morse potential, and used in combination with T-DAS, induces dynamic vortex formations about the origin. The spring potential function is purely attractive and grows linearly with the separation between each particle and the origin. Explicitly the APF (1) is used with $\beta = 1, C_a = 1, L_a = 0.8, C_r = 1, L_r = 0.5$ which yields the potential function surface in Figure 5. Note that for $b > a$ the velocities will always converge to zero

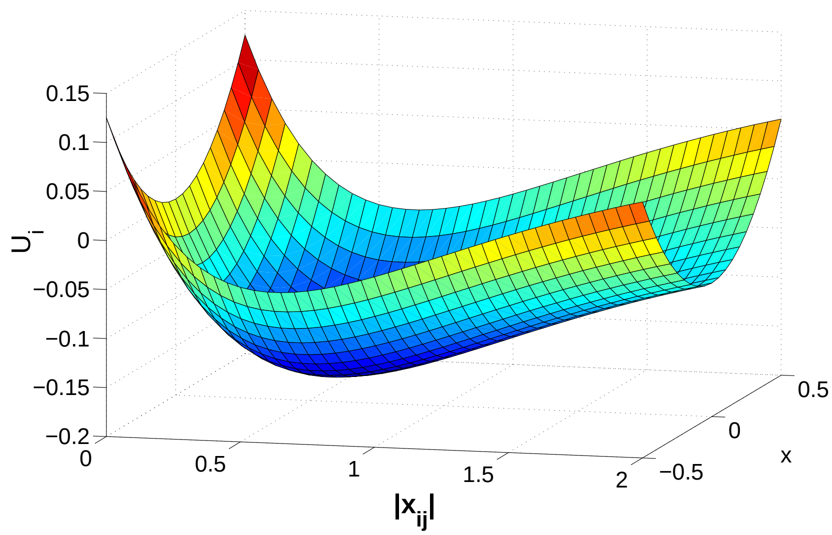
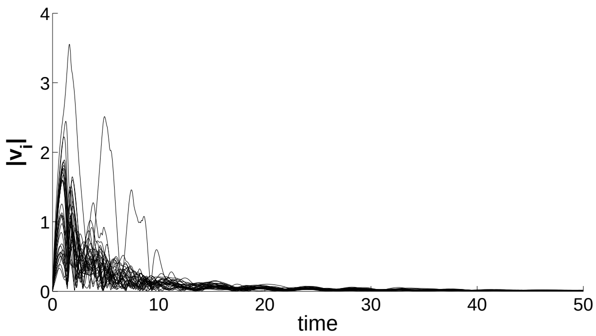
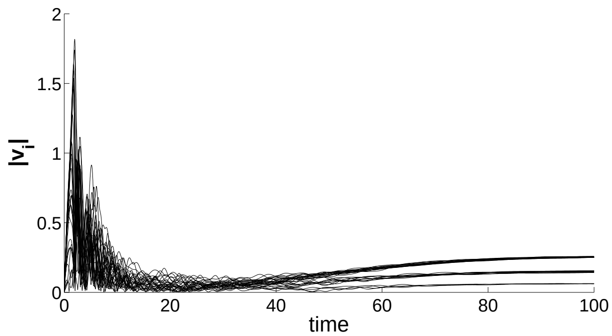


FIG. 5: (Color online) The potential surface as a function of agent separation and the distance x from the origin

as in the case when the spring potential is not included. Furthermore, when $b << a$ the velocity will diverge and for a slightly larger than b the velocity will be bounded but at a larger velocity than for $a = b$. In other words as b increases (above a) the final bounded velocity will increase until it reaches a critical value where it will diverge (escapes from the potential well). Examples of bounded velocities are illustrated in the Figures 6 (i) and (ii). The two behaviors are qualitatively unchanged with each agent converging to one of three constant velocity magnitudes (this is most clearly observed in Fig. 6 (ii)). From here on we assume $a = b = 1$ which corresponds to the marginally stable case for $\beta = 0$. Summing over



(i)



(ii)

FIG. 6: The velocity magnitude history for 30 agents in the swarm (i) $a = b = 1$ all velocities are small and bounded (ii) $a = 1.1, b = 1$ velocities are bounded but their magnitudes become larger

all agents in Equ. (3) with $\beta = 1$ yields:

$$\sum_i m_i \dot{\mathbf{v}}_i(t) = -\sum_i m_i \mathbf{v}_i(t) + \sum_i m_i \mathbf{v}_i(t-\tau) - \sum_i m_i \mathbf{x}_i \tag{18}$$

where $\sum_i \mathbf{x}_i$ is the additional component to the previous case (6) corresponding to the addition of the spring potential and $\sum_i \nabla_i U(\mathbf{x}_{ij}) = 0$ due to internal symmetry in the swarm. The center-of-mass of the swarm can thus be expressed as:

$$\ddot{\mathbf{R}}_c(t) = -\dot{\mathbf{R}}_c(t) + \dot{\mathbf{R}}_c(t-\tau) - \mathbf{R}_c(t) \tag{19}$$

defining $\mathbf{X} = [\mathbf{R}_c(t), \dot{\mathbf{R}}_c(t)]^T$, this can be expressed as a linear time delay system of the form:

$$\dot{\mathbf{X}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{X}(t) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{X}(t-\tau) \tag{20}$$

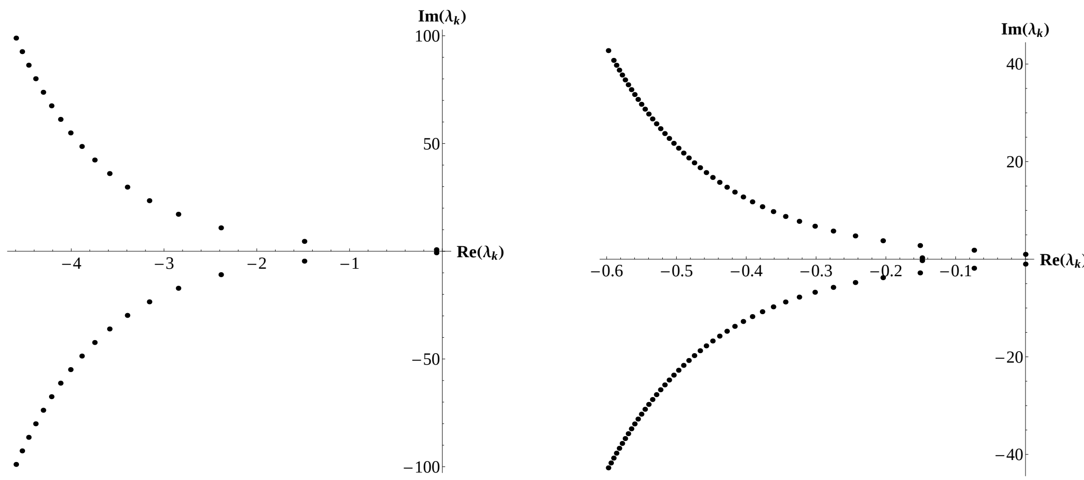
This system cannot be solved using the matrix generalization of the Lambert function as the two matrices A and B in $\dot{\mathbf{X}}(t) = A\mathbf{X}(t) + B\mathbf{X}(t-\tau)$ corresponding to (20) do not commute, see[28]. Therefore, the stability of the center-of-mass of the swarm is determined using a numerical eigenvalue based approach for time-delay systems[29]. It is well known (see[29]) that as the system, shown in equation (20), is of the form $\dot{\mathbf{X}}(t) = A_0\mathbf{X}(t) + A_1\mathbf{X}(t-\tau)$ where $\mathbf{X}(t) \in \mathbb{R}^2$ can be expressed as $\mathbf{X}(t) = \sum_{-\infty}^{\infty} C_k e^{\lambda_k t}$ and $A_0, A_1 \in \mathbb{R}^{2\times 2}$ are real matrices and $0 < \tau$ that the substitution of a sample solution of the form $e^{\lambda_k t}v$ where $v \in \mathbb{C}^{2\times 1}\backslash\{0\}$ leads to the characteristic equation:

$$\det \Delta(\lambda_k) = 0 \tag{21}$$

where,

$$\Delta(\lambda_k) = \lambda I - A_0 - A_1 e^{-\lambda_k \tau}. \tag{22}$$

The particular case when $\tau = 1$ and $\tau = 2\pi$ is illustrated in Figure 7 where the maximum real part of all the eigenvalues is $Re(\lambda_0) = -0.0638512$ and $Re(\lambda_0) = 0$ respectively. Figure



(i) (ii)

FIG. 7: Characteristic roots of equation (21) for (i) $\tau = 1$ the right most eigenvalue has negative real part (ii) $\tau = 2\pi$ the two right most eigenvalues lie on the imaginary axis

7 illustrates that the center-of-mass will always stop, independently of the number of agents in the swarm, for $\tau = 1$. Figure 8 shows a plot of just the right-most eigenvalue against τ and illustrates that the centre-of-mass will always stop for $\tau \in (0, 2\pi)$. Moreover, each agent's velocity has been shown to converge to a constant bounded velocity (with $a = b$) and that the center-of-mass will stop independently of initial conditions. This implies that for random initial conditions the swarm must converge to a rotating motion. Figure 9 illustrates convergence to the rotating (vortex) motion for a swarm of 30 agents projected on the x y plane. However, from Figure 7 (ii) ($\tau = 2\pi$) it can be seen that the right-most eigenvalues

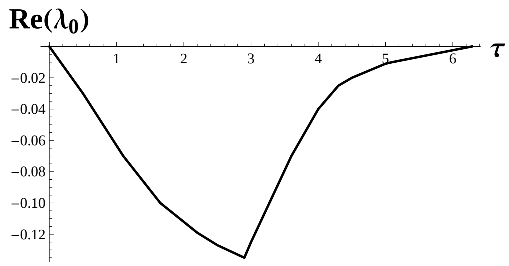


FIG. 8: The rightmost characteristic roots of the system (20) as a function of the delay term τ

lie on the imaginary axis. In this case as $t \to \infty$ all modes converge to zero except the right-most and therefore the solution in the limit is a periodic motion. This periodic solution exists for $\tau = 2n\pi$ where $n \in \mathbb{Z}$ and is easily shown to be:

$$\mathbf{R}_c(t) = \dot{\mathbf{R}}_c(0)\sin t + \mathbf{R}_c(0)\cos t. \tag{23}$$

This periodic motion can be considered stable in that all transient motion independently of initial conditions will converge to it (except for the trivial case $\dot{\mathbf{R}}_c(0) = \mathbf{R}_c = 0$). In this case each agent winds round the origin as illustrated in Figure 10 (ii) with the periodic motion of the center-of-mass tracing an ellipse. If a numerical continuation of the delay parameter is extended beyond $\tau = 2\pi$ it is seen that the real part of the right-most eigenvalue is always negative except at the discrete bifurcation points $\tau = 2n\pi$. Note that the bifurcations involve two stable delay-dependent steady states: an equilibrium point and a periodic orbit. However, the eigenvalues never cross the imaginary axis of the complex plane for any value of the delay parameter so it is different from the classical hopf bifurcation reported in Schwartz and Forgotson[25].

IV. CONCLUSION

This work has investigated the combined effect of an Artificial Potential Function (Morse potential and a spring potential) with a time-delayed auto-synchronous (T-DAS) term. The Morse potential is conventionally used to ensure collision avoidance and long-range attraction in swarms while it is shown that the T-DAS term can be used to induce stationary, uniformly rotating and translating swarms or swarms with exponentially increasing translational velocity. The corresponding center-of-mass motion of the swarm without a spring potential is shown to be explicitly defined by a multi-valued function. In the presence of

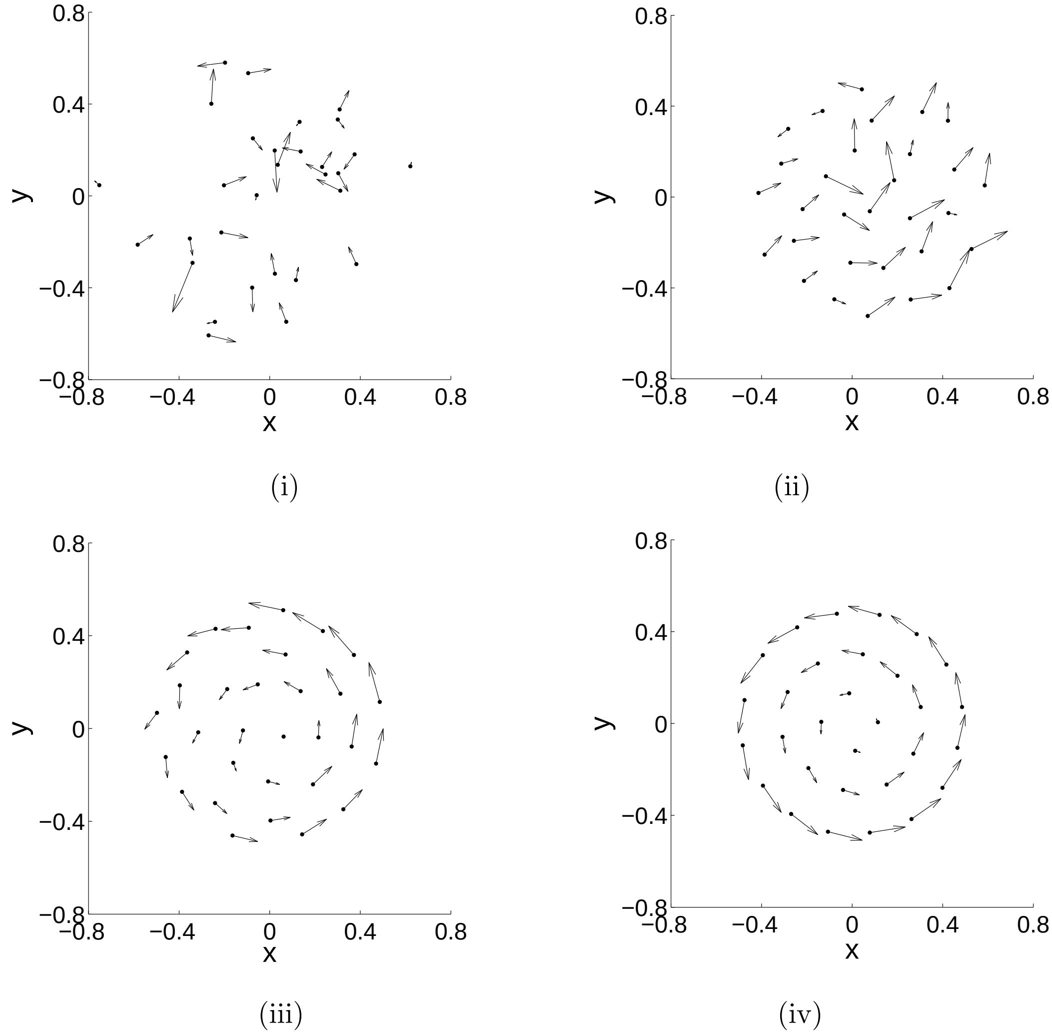
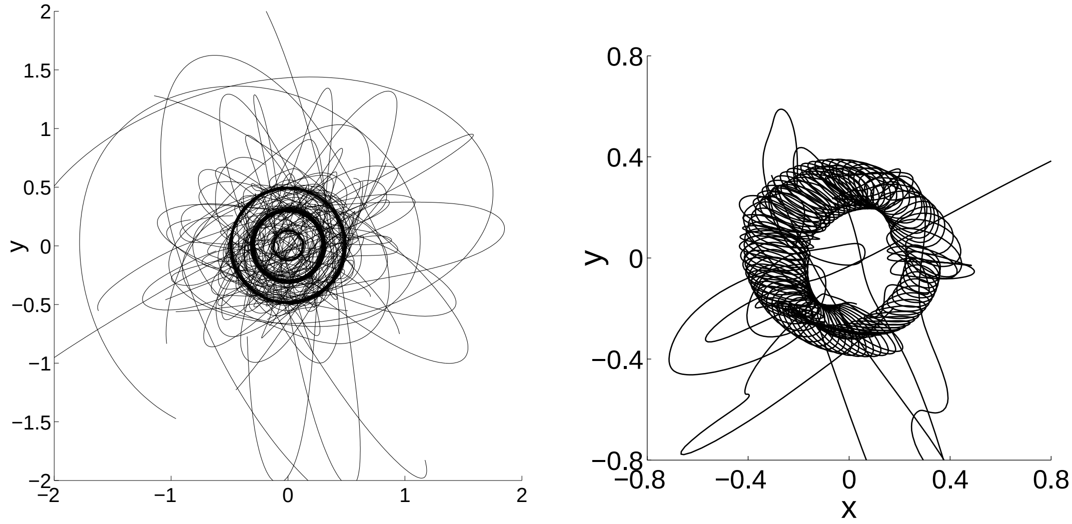


FIG. 9: Swarm of 30 agents forming a vortex independently of initial conditions (i) random initial conditions (ii) t=10 (iii) t=20 (iv) t=40

a spring potential the swarm converges to a vortex formation where the center-of-mass is guaranteed to stop, except at discrete bifurcation points where the delay term $\tau = 2n\pi$. At the discrete bifurcation points, after an initial transient, the center-of-mass will periodically trace an ellipse, whose semi-major and semi-minor axis are explicitly dependent on the initial position and velocity of the center-of-mass. For the purpose of engineering the presented model for vortex formation has advantages over noise induced rotations as it is completely deterministic. This implies that results can be repeated and the mean field equations can be analyzed without assumptions being placed on the stochastic perturbation. In contrast to previous deterministic models for vortex formations it has low-computational



(i) (ii)

FIG. 10: Trajectories of 30 agents with random initial conditions converging to a steady state (i) $\tau = 1$ the center-of-mass stops and the swarm forms a vortex formation (ii) $\tau = 2\pi$ the center-of-mass oscillates about the origin and each agent winds around the origin

requirement as the active interaction only requires that each agent is capable of sensing their relative position within their environment without the need for any relative velocity information. This shows that it is possible to induce rotational motion with a stationary center-of-mass without using noise or information on the relative velocity. Therefore, these results may prove useful in controlling swarms of autonomous vehicles which posses only low-computational power on-board. The model could also provide a deterministic insight into swarm alignment of biological systems such as vortex formation in schools of fish using a feedback mechanism that is a function of memory.

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