Stabilization of Highly Nonlinear Continuous-time Hybrid Stochastic Differential Delay Equations by Discrete-time Feedback Control

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Abstract

In this paper, we consider how to use discrete-time state feedback to stabilize hybrid stochastic differential delay equations. The coefficients of these stochastic differential delay equations do not satisfy the conventional linear growth conditions, but are highly nonlinear. Using the Lyapunov functional method, we show that a discrete feedback controller $u(x([t/\tau]\tau), r(t), t)$ can be designed to make the solutions of such controlled hybrid stochastic differential delay equations asymptotically stable and exponentially stable. The upper bound of the discrete observation interval τ is also given in the article. Finally, a numerical examples are given to illustrate our theory.

Keywords: Delay system, Highly nonlinear, Markov chain, Discrete-time feedback control, Lypunov functional

1. Introduction

In power system, economic system and ecosystem, because of its structure and parameters are prone to change suddenly, people always use discrete-time Markov chain-driven stochastic differential equation (also known as hybrid SDEs) to model it. The hybrid SDEs can be described by

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t). \tag{1.1}$$

Here the state $x(t)$ takes values in R^n and the mode $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, \cdots, N\}$, $w(t)$ is a Brownian motion and f and g are referred to as the drift and diffusion coefficients, respectively. The important issues in the study of hybrid SDEs are the analysis of stability and stabilization (see, e.g., Ji, and Chizeck (1990); Mao (1999, 2002); Mariton (1990); Shaikhet (1996); Shi, Mahmoud, Yi, and Ismail (2006); Sun, Lam, Xu, and Zou (2007)).

If the given hybrid SDE (1.1) is unstable, then Mao (2013) first proposed that we can design a feedback control $u(x([t/\tau]\tau), r(t), t)$, based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$, so that the controlled system

$$\begin{aligned} dx(t) =& (f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t))dt \\ &+ g(x(t), r(t), t)dw(t) \end{aligned} \tag{1.2}$$

becomes stable. Here $\tau > 0$ is a constant which stands for the duration between two consecutive state observations, and $[t/\tau]$ is the integer part of t/τ. Compared with continuous-time feedback control, discrete-time observation feedback control has great advantages (lower cost and more practical etc.). Subsequently, many scholars began to study the feedback control of discrete-time observations (see, e.g., You, Liu, Lu, Mao, and Qiu (2015); Mao (2016); Shao (2017); Fei, Fei, Mao, Xia, and Yan (2019)).

On the other hand, the evolution process of a stochastic system is not only related to the current state, but also to a certain period of history before the system. Therefore, hybrid stochastic delay systems have received considerable attention (see, e.g., Mao, Matasov, and Piunovskiy (2000); Mao, and Yuan (2006); Wei, Wang, Shu, and Fang (2006); Yue, and Han (2005)). Recently, many papers have taken into account the stability of hybrid stochastic delay systems with highly nonlinear (see, e.g., Fei, Shen, Fei, Mao, and Yan (2019); Fei, Hu, Mao, and Shen (2017, 2018); Hu, Mao, and Shen (2013); Shen, Fei, Mao, and Liang (2018)). In the real world, there are many hybrid stochastic systems with high nonlinearity and delay (see, e.g., Lewis (2000); Yuan, Mao, and Lygeros (2009)). For example, the following scalar hybrid SDDE

$$\begin{aligned} dx(t) =& f(x(t), x(t-\tau_0), r(t), t)dt \\ &+ g(x(t), x(t-\tau_0), r(t), t)dw(t), \end{aligned} \tag{1.3}$$

where the coefficients f and g are defined by

$$\begin{aligned} &f(x, y, 1, t) = -2x^5 + y^3,\ g(x, y, 1, t) = 0.5y^2, \\ &f(x, y, 2, t) = -3x^5 + 2y^3,\ g(x, y, 2, t) = y^2, \end{aligned} \tag{1.4}$$

$w(t)$ is a scalar Brownian motion, τ_0 is time lag of the system, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its

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generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1.5}$$

This article attempts to design the feedback controls based on the discrete-time state observations in order to stabilize highly nonlinear hybrid SDDEs. In comparison with Fei, Fei, Mao, Xia, and Yan (2019), the key contributions in this paper are highlighted below:

- We can observe that the controlled system (2.8) itself has a delay of τ_0, and the time interval of discrete observation τ is also the upper bound of variable delay $\varphi(t)$. Unlike Fei, Fei, Mao, Xia, and Yan (2019), the controlled system (2.8) is actually a hybrid stochastic differential multiple-delay equation (2.13). In general these two delays should not be equal. Different from the general stochastic differential delay equation, the variable delay in our equation (2.13) is piecewise differentiable and the derivative $\dot{\varphi}(t)$ is equal to 1 in $t \in (k\tau, (k+1)\tau)$, thus the previous stability results will no longer apply here.
- Moreover, the system (2.8) itself has time lag, and many methods in Fei, Fei, Mao, Xia, and Yan (2019) will no longer be applicable. In fact, we need not only new theories to illustrate various kinds of stability, but also some new techniques to solve the existence, uniqueness and asymptotic boundedness of solutions .
- Compared with Fei, Fei, Mao, Xia, and Yan (2019), besides the results of moment stability and exponential stability of controlled system (2.8), the almost sure stability of the system is also investigated in this paper.

2. Standing Hypotheses and Boundedness

Throughout this article, unless otherwise specified, we use the following notation. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $R_+ = [0, \infty)$. For $x \in R^n$, $|x|$ denotes its Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T. For $A \in R^{n\times m}$, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. If A is a symmetric real-valued matrix ($A = A^T$), denote by $\lambda_{\max}(A)$and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. By $A \le 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). If A is a subset of Ω, denote by I_A its indicator function; that is, $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while

$$\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}.$$

We always assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$.

For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions ϕ from $[-h, 0] \to R^n$ with the norm $\|\phi\| = \sup_{-h\le u\le 0} |\phi(u)|$. Let $C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-h, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) \in -h \le \theta \le 0\}$.

Consider a nonlinear hybrid SDDE

$$\begin{aligned} dx(t) =& f(x(t), x(t-\tau_0), r(t), t)dt \\ &+ g(x(t), x(t-\tau_0), r(t), t)dw(t) \end{aligned} \tag{2.1}$$

on $t \ge 0$ with the initial value $x(\theta) = \{x : -\tau_0 \le \theta \le 0\} = \xi \in C([-\tau_0, 0]; R^n)$, where $x(t) \in R^n$ is the state vector; positive scalar constant τ_0 is time lag of the system; and

$$f : R^n \times R^n \times S \times R_+ \to R^n \text{ and } g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$$

are Borel measurable functions. In this paper, we maintain the local Lipschitz condition. But as mentioned in the preceding section, we will not confine the coefficient f or g to the linear growth condition, but to a condition similar to the polynomial growth condition. For this reason, we give the following hypothesis.

Assumption 2.1. *For each integer $k > 0$, there is a constant $L_k > 0$ such that*

$$\begin{aligned} &|f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)| \vee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)| \\ \le& L_k(|x - \bar{x}| + (y - \bar{y})) \end{aligned} \tag{2.2}$$

for all $x, \bar{x}, y, \bar{y} \in R^n$ with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \le k$ and all $(i, t) \in S \times R_+$.

Assumption 2.2. *Assume that there are three constants $L > 0$, $p_1 > 1$ and $p_2 \ge 1$ such that*

$$\begin{aligned} &|f(x, y, i, t)| \le L(|x|^{p_1} + |x| + |y|^{p_1} + |y|) \\ \text{and} \quad &|g(x, y, i, t)| \le L(|x|^{p_2} + |x| + |y|^{p_2} + |y|) \end{aligned} \tag{2.3}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Considering highly nonlinear hybrid SDDEs, we always assume $p_1 > 1$ in condition (2.3). For the hybrid SDDE (1.4), it is easy to see that $p_1 = 5$ and $p_2 = 3$. We also observe condition (2.3) implies that $f(0, 0, i, t) \equiv 0$ and $g(0, 0, i, t) \equiv 0$, which are required for the stability purpose of this paper. Obviously, the Assumption 2.2 may cause the hybrid SDDE (2.1) to explode in finite random time. In order to guarantee on the existence of the global unique solution of the SDDE (2.1), we need to impose another Khasminskii-type condition.

Assumption 2.3. *Assume that there are some nonnegative constants $p, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, q_1, q_2 such that*

$$p > (p_1 + 1) \vee (2p_2), \quad q_1 \ge (p_1 + 1) \vee (2p_2 - p_1 + 1)$$

$$\text{and} \quad q_1 > q_2 \geq 2 \tag{2.4}$$

(where p_1 and p_2 are the same as in Assumption 2.2) while

$$x^T f(x,y,i,t) + \frac{p-1}{2}|g(x,y,i,t)|^2$$
$$\leq -\alpha_1|x|^{q_1} + \alpha_2|x|^2 + \alpha_3|y|^{q_2} + \alpha_4|y|^2 \tag{2.5}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$.

It is useful to point out that the range of q and p in Assumption 2.3 is very wide in many hybrid SDDEs, and sometimes p can even be arbitrarily large. For example, consider the hybrid SDDE (1.4) and let p be arbitrarily large. Then

$$x^T f(x,y,i,t) + \frac{p-1}{2}|g(x,y,i,t)|^2$$
$$= \begin{cases} -2x^6 + xy^3 + 0.125(p-1)y^4 & \text{if } i=1, \\ -3x^6 + 2xy^3 + 0.5(p-1)y^4 & \text{if } i=2. \end{cases} \tag{2.6}$$

By inequalities

$$xy^3 \leq 0.25x^4 + 0.75y^4 \text{ and } x^4 \leq 0.5x^2 + 0.5x^6.$$

Hence

$$x^T f(x,y,i,t) + \frac{p-1}{2}|g(x,y,i,t)|^2$$
$$\leq -1.875x^6 + 0.25x^2 + \big(1.5 + 0.5(p-1)\big)y^4 + y^2. \tag{2.7}$$

That is, the hybrid SDDE (1.4) satisfies Assumption 2.3 with any large p and $q_1 = 6$, $q_2 = 4$, $\alpha_1 = 1.875$, $\alpha_2 = 0.25$, $\alpha_3 = 1.5 + 0.5(p-1)$, $\alpha_4 = 1$.

Under Assumptions 2.1, 2.2 and 2.3, the hybrid SDDE (2.1) has a unique global solution such that $\sup_{-\tau_0 \leq t < \infty} \mathbb{E}|x(t,\xi)|^p < \infty$ with any initial value ξ (see, e.g., Hu, Mao, and Shen (2013)). But boundedness does not mean stability. When the given SDDE (2.1) is unstable, we are required to design a feedback control $u(x([t/\tau]\tau), r(t), t)$, based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$, in the drift part so that the controlled system

$$dx(t) = [f(x(t), x(t-\tau_0), r(t), t) + u(x(\eta_t), r(t), t)]dt$$
$$+ g(x(t), x(t-\tau_0), r(t), t)dw(t), \quad t \geq 0, \tag{2.8}$$

becomes stable, where $\eta_t = [t/\tau]\tau$ and the control function $u : R^n \times S \times R_+ \to R^n$ is a Borel measurable. In this paper, we will design the control functions to satisfy the following assumption.

Assumption 2.4. *Assume that there is a nonnegative number ϖ such that*

$$|u(x,i,t) - u(y,i,t)| \leq \varpi|x-y| \tag{2.9}$$

for all $x, y \in R^n$, $i \in S$ and $t \geq 0$. Moreover, assume that $u(0,i,t) \equiv 0$ for all $(i,t) \in S \times R_+$.

Obviously this assumption implies

$$|u(x,i,t)| \leq \varpi|x|, \quad \forall (x,i,t) \in R^n \times S \times R_+. \tag{2.10}$$

As pointed out above, the pth moment of the solution of the given SDDE (2.1) is bounded. The following theorem shows that the controlled SDDE (2.8) preserves this good property.

Theorem 2.5. *Let Assumptions 2.1, 2.2, 2.3 and 2.4 hold. The controlled system (2.8) with any initial value $\xi \in C([-\tau_0, 0]; R^n)$ has a unique global solution $x(t)$ on $t \geq -\tau_0$. Moreover, the solution $x(t)$ obeys*

$$\sup_{-\tau_0 \leq t < \infty} \mathbb{E}|x(t,\xi)|^p < \infty. \tag{2.11}$$

Proof. To make the proof more understandable, we divide it into three steps.

Step 1. Let's define a bounded function $\varphi : R_+ \to [0, \tau]$ by

$$\varphi(t) = t - k\tau \quad \text{for } k\tau \leq t < (k+1)\tau, \ k = 0, 1, 2, \cdots. \tag{2.12}$$

Thus the controlled system (2.8) can be rewritten as

$$dx(t) = (f(x(t), x(t-\tau_0), r(t), t) + u(x(t-\varphi(t)), r(t), t))dt$$
$$+ g(x(t), x(t-\tau_0), r(t), t)dw(t) \tag{2.13}$$

on $t \geq 0$ with the initial value ξ. We observe that $\varphi(t)$ is a bounded variable delay, and the controlled system (2.8) is actually a hybrid stochastic differential multiple-delay equation.

Let $\bar{U}(x) = |x|^p$. Using the Itô formula,

$$d\bar{U}(x(t)) = \bar{L}\bar{U}(x(t), x(t-\tau_0), x(t-\varphi(t)), r(t), t)dt$$
$$+ p|x|^{p-2}x^T(t)g(x(t), x(t-\tau_0), r(t), t)dw(t), \tag{2.14}$$

where the operator $\bar{L}\bar{U} : R^n \times R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\bar{L}\bar{U}(x,y,z,i,t)$$
$$= p|x|^{p-2}x^T[f(x,y,i,t) + u(z,i,t)]$$
$$+ \frac{p}{2}|x|^{p-2}|g(x,y,i,t)|^2 + \frac{p(p-2)}{2}|x|^{p-4}|x^T g(x,y,i,t)|^2$$
$$\leq p|x|^{p-2}\Big[x^T[f(x,y,i,t) + u(z,i,t)] + \frac{p-1}{2}|g(x,y,i,t)|^2\Big].$$

By Assumptions 2.3 and 2.4, we derive that

$$\bar{L}\bar{U}(x,y,z,i,t) \leq -p\alpha_1|x|^{q_1+p-2} + p\alpha_2|x|^p + p\alpha_3|x|^{p-2}|y|^{q_2}$$
$$+ p\alpha_4|x|^{p-2}|y|^2 + p\varpi|x|^{p-1}|z|.$$

Choosing a constant $\varepsilon \in (0, 1/2)$ sufficiently small for

$$\varepsilon < (0.5 + 0.5e^{-\tau})^{\bar{k}} e^{-\tau}, \tag{2.15}$$

where

$$\bar{k} := \Big[\frac{\tau_0}{\tau}\Big] + 1.$$

From the Young inequality, we get

$$p\alpha_3|x|^{p-2}|y|^{q_2} \leq C|x|^{q_2+p-2} + \varepsilon|y|^{q_2+p-2},$$
$$p\alpha_4|x|^{p-2}|y|^2 \leq C|x|^p + \varepsilon|y|^p,$$
$$p\varpi|x|^{p-1}|z| \leq C|x|^p + \varepsilon|z|^p$$

where, here and in the remaining part of this paper, C denotes a positive constant that may change from line to line but its special form is of no use. Hence

$$\bar{L}\bar{U}(x,y,z,i,t)$$

$$\leq C_1 - |x|^{q_2+p-2} + \varepsilon|y|^{q_2+p-2} - 2|x|^p + \varepsilon|y|^p + \varepsilon|z|^p, \quad (2.16)$$

where

$$C_1 := \sup_{x\geq 0}\Big[- p\alpha_1 x^{q_1+p-2} + (C-1)x^{q_2+p-2} + (C-2)x^p\Big].$$

Step 2. Next we will show the existence and uniqueness of the solution of the hybrid SDDE (2.13) on $t \geq -\tau_0$. Under the local Lipschitz condition (2.2), there exists a unique maximal local solution $x(t)$ to Equation (2.13) on $t \in [0, \sigma_e)$ with any initial value ξ, where σ_e is the explosion time (see, e.g., Mao, and Yuan (2006), Theorem 7.12 on page 278). Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\sigma_k = \inf\{t \in [0, \sigma_e) : |x(t)| \geq k\},$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, σ_k increases as $k \to \infty$ and $\sigma_k \to \sigma_\infty \leq \sigma_e$ a.s. If we can get that $\sigma_\infty = \infty$ a.s., then $\sigma_e = \infty$ a.s. That is, the maximal local solution $x(t)$ is the unique global solution.

By the standard stopping time technique to (2.14), then using (2.16), we obtain

$$\begin{aligned}&\mathbb{E}\bar{U}(x(t\wedge\sigma_k))\\ &\leq |x_0|^p + \mathbb{E}\int_0^{t\wedge\sigma_k}(C_1 - |x(s)|^{q_2+p-2} + \varepsilon|x(s-\tau_0)|^{q_2+p-2}\\ &\quad - 2|x(s)|^p + \varepsilon|x(s-\tau_0)|^p + \varepsilon|x(s-\varphi(s))|^p)ds, \quad (2.17)\end{aligned}$$

which shows

$$\begin{aligned}&\mathbb{E}|x(t\wedge\sigma_k)|^p\\ \leq& K(t) - (1-\varepsilon)\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)|^{q_2+p-2}ds\\ &- (2-\varepsilon)\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)|^p ds + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s-\varphi(s))|^p ds\\ \leq& K(t) + \varepsilon\mathbb{E}\int_0^t |x(s-\varphi(s))|^p I_{[0,\sigma_k]}(s)ds\\ =& K(t) + \varepsilon\int_0^t \mathbb{E}\Big(|x(s-\varphi(s))|^p I_{[0,\sigma_k]}(s)\Big)ds,\end{aligned}$$

where

$$K(t) = |x_0|^p + C_1 t + \varepsilon\mathbb{E}\int_{-\tau_0}^0 |x(s)|^{q_2+p-2}ds + \varepsilon\mathbb{E}\int_{-\tau_0}^0 |x(s)|^p ds.$$

But, from (2.12), we observe that $0 \leq s - \varphi(s) \leq s$ for all $s \geq 0$. Then

$$\begin{aligned}\mathbb{E}\Big(|x(s-\varphi(s))|^p I_{[0,\sigma_k]}(s)\Big) &\leq \sup_{0\leq v\leq s}\mathbb{E}\Big(|x(v)|^p I_{[0,\sigma_k]}(s)\Big)\\ &\leq \sup_{0\leq v\leq s}\mathbb{E}|x(v\wedge\sigma_k)|^p.\end{aligned}$$

We therefore get

$$\mathbb{E}|x(t\wedge\sigma_k)|^p \leq K(t) + \varepsilon\int_0^t \sup_{0\leq v\leq s}\mathbb{E}|x(v\wedge\sigma_k)|^p ds.$$

Noting the sum of the right hand-side terms is increasing in t, we obtain

$$\sup_{0\leq v\leq t}\mathbb{E}|x(v\wedge\sigma_k)|^p \leq K(t) + \varepsilon\int_0^t \sup_{0\leq v\leq s}\mathbb{E}|x(v\wedge\sigma_k)|^p ds.$$

By the Gronwall inequality, we have

$$\sup_{0\leq v\leq t}\mathbb{E}|x(v\wedge\sigma_k)|^p \leq K(t)e^{\varepsilon t}.$$

Consequently

$$P(\sigma_k \leq t)k^p \leq \mathbb{E}|x(t\wedge\sigma_k)|^p \leq \sup_{0\leq v\leq t}\mathbb{E}|x(v\wedge\sigma_k)|^p \leq K(t)e^{\varepsilon t}.$$

Let $k \to \infty$, we can see $P(\sigma_k \leq t) \to 0$, this implies $\sigma_\infty = \infty$ a.s.. With the previous analysis, we can obtain that SDDE (2.8) with any initial value has a unique global solution $x(t)$ on $t \geq -\tau_0$.

Step 3. Finally, we will illustrate the asymptotic boundedness of the pth moment of the solution. Set $t_k = k\tau$ for $k = 0, 1, 2\cdots$. For $t \in [t_k, t_{k+1}]$, by the Itô formula

$$\begin{aligned}e^t\mathbb{E}\bar{U}(x(t)) =& e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + \mathbb{E}\int_{t_k}^t e^s[\bar{U}(x(s))\\ &+ L\bar{U}(x(s), x(s-\tau_0), x(s-\varphi(s)), r(s), s)]ds.\end{aligned}$$

Appling (2.16), we can compute

$$\begin{aligned}&e^t\mathbb{E}\bar{U}(x(t))\\ \leq& e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + \mathbb{E}\int_{t_k}^t e^s[C_1 - |x(s)|^{q_2+p-2}\\ &+ \varepsilon|x(s-\tau_0)|^{q_2+p-2} - |x(s)|^p + \varepsilon|x(s-\tau_0)|^p + \varepsilon|x(s-\varphi(s))|^p]ds\\ =& e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + (e^t - e^{t_k})(C_1 + \varepsilon\mathbb{E}\bar{U}(x(t_k)))\\ &- \mathbb{E}\int_{t_k}^t e^s\Psi(x(s))ds + \varepsilon e^{\tau_0}\mathbb{E}\int_{t_k-\tau_0}^{t-\tau_0} e^s\Psi(x(s))ds, \quad (2.18)\end{aligned}$$

where

$$\Psi(x(s)) := |x(s)|^{q_2+p-2} + |x(s)|^p.$$

In particular,

$$\begin{aligned}&e^{t_{k+1}}\mathbb{E}\bar{U}(x(t_{k+1})) \leq e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + (e^{t_{k+1}} - e^{t_k})[C_1 + \varepsilon\mathbb{E}\bar{U}(x(t_k))]\\ &\quad - \mathbb{E}\int_{t_k}^{t_{k+1}} e^s\Psi(x(s))ds + \varepsilon e^{\tau_0}\mathbb{E}\int_{t_k-\tau_0}^{t_{k+1}-\tau_0} e^s\Psi(x(s))ds.\end{aligned}$$

This implies

$$\begin{aligned}&\mathbb{E}\bar{U}(x(t_{k+1}))\\ &\leq e^{-\tau}\mathbb{E}\bar{U}(x(t_k)) + (1-e^{-\tau})[C_1 + \varepsilon\mathbb{E}\bar{U}(x(t_k))]\\ &\quad - e^{-\tau}\mathbb{E}\int_{t_k}^{t_{k+1}}\Psi(x(s))ds + \varepsilon\mathbb{E}\int_{t_k-\tau_0}^{t_{k+1}-\tau_0}\Psi(x(s))ds\\ &\leq C_1 + a\mathbb{E}\bar{U}(x(t_k)) - e^{-\tau}\mathbb{E}\int_{t_k}^{t_{k+1}}\Psi(x(s))ds\\ &\quad + \varepsilon\mathbb{E}\int_{t_k-\tau_0}^{t_{k+1}-\tau_0}\Psi(x(s))ds, \quad (2.19)\end{aligned}$$

where $a := e^{-\tau} + \varepsilon(1 - e^{-\tau}) < 0.5(1 + e^{-\tau}) < 1$.

Furthermore, by $t \in [t_k, t_{k+1}]$, it follows from (2.18) that

$$
\begin{aligned}
&\mathbb{E}\bar{U}(x(t)) \\
&\le e^{t_k - t}\mathbb{E}\bar{U}(x(t_k)) + (1 - e^{t_k - t})(C_1 + \varepsilon\bar{U}(x(t_k))) \\
&\quad - e^{-t}\mathbb{E}\int_{t_k}^{t} e^s\Psi(x(s))ds + \varepsilon e^{\tau_0 - t}\mathbb{E}\int_{t_k-\tau_0}^{t-\tau_0} e^s\Psi(x(s))ds \\
&\le C_1(1 - e^{-\tau}) + (\varepsilon + (1 - \varepsilon)e^{t_k - t})\mathbb{E}\bar{U}(x(t_k)) \\
&\quad - e^{-\tau}\mathbb{E}\int_{t_k}^{t}\Psi(x(s))ds + \varepsilon\mathbb{E}\int_{t_k-\tau_0}^{t-\tau_0}\Psi(x(s))ds \\
&\le C_1 + \mathbb{E}\bar{U}(x(t_k)) - e^{-\tau}\mathbb{E}\int_{t_k}^{t}\Psi(x(s))ds + \varepsilon\mathbb{E}\int_{t_k-\tau_0}^{t}\Psi(x(s))ds.
\end{aligned}
$$

Combining this with (2.19), we deduce that

$$
\begin{aligned}
&\mathbb{E}\bar{U}(x(t)) \\
&\le C_1 + \Big[C_1 + a\mathbb{E}\bar{U}(x(t_{k-1})) - e^{-\tau}\mathbb{E}\int_{t_{k-1}}^{t_k}\Psi(x(s))ds \\
&\quad + \varepsilon\mathbb{E}\int_{t_{k-1}-\tau_0}^{t_k-\tau_0}\Psi(x(s))ds\Big] - e^{-\tau}\mathbb{E}\int_{t_k}^{t}\Psi(x(s))ds \\
&\quad + \varepsilon\mathbb{E}\int_{t_k-\tau_0}^{t}\Psi(x(s))ds \\
&\le 2C_1 + a\Big[C_1 + a\mathbb{E}\bar{U}(x(t_{k-2})) - e^{-\tau}\mathbb{E}\int_{t_{k-2}}^{t_{k-1}}\Psi(x(s))ds \\
&\quad + \varepsilon\mathbb{E}\int_{t_{k-2}-\tau_0}^{t_{k-1}-\tau_0}\Psi(x(s))ds\Big] - e^{-\tau}\mathbb{E}\int_{t_{k-1}}^{t}\Psi(x(s))ds \\
&\quad + \varepsilon\mathbb{E}\int_{t_{k-1}-\tau_0}^{t}\Psi(x(s))ds \\
&\le 2C_1 + C_1 a + a^2\mathbb{E}\bar{U}(x(t_{k-2})) - ae^{-\tau}\mathbb{E}\int_{t_{k-2}}^{t}\Psi(x(s))ds \\
&\quad + \varepsilon\mathbb{E}\int_{t_{k-1}-\tau_0}^{t}\Psi(x(s))ds + a\varepsilon\mathbb{E}\int_{t_{k-2}-\tau_0}^{t_{k-1}-\tau_0}\Psi(x(s))ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}\bar{U}(x(t)) \\
&\le C_1 + C_1(1 + a + a^2 + \cdots + a^{\bar{k}}) + a^{\bar{k}+1}\mathbb{E}\bar{U}(x(t_{k-\bar{k}-1})) \\
&- a^{\bar{k}}e^{-\tau}\mathbb{E}\int_{t_{k-\bar{k}-1}}^{t}\Psi(x(s))ds + \varepsilon\mathbb{E}\int_{t_{k-1}-\tau_0}^{t}\Psi(x(s))ds \\
&+ a\varepsilon\mathbb{E}\int_{t_{k-2}-\tau_0}^{t_{k-1}-\tau_0}\Psi(x(s))ds + \cdots + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-\bar{k}}-\tau_0}\Psi(x(s))ds.
\end{aligned}
\tag{2.20}
$$

Noting $t_{k-2} - \tau_0 \le t_{k-\bar{k}-1} < t_{k-1} - \tau_0$ from $(\bar{k} - 1)\tau \le \tau_0 < \bar{k}\tau$, we have

$$
\begin{aligned}
&\varepsilon\mathbb{E}\int_{t_{k-1}-\tau_0}^{t}\Psi(x(s))ds + a\varepsilon\mathbb{E}\int_{t_{k-2}-\tau_0}^{t_{k-1}-\tau_0}\Psi(x(s))ds \\
&\quad + \cdots + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-\bar{k}-1}-\tau_0}\Psi(x(s))ds \\
&\le\varepsilon\mathbb{E}\int_{t_{k-1}-\tau_0}^{t}\Psi(x(s))ds + a\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}}^{t_{k-1}-\tau_0}\Psi(x(s))ds \\
&\quad + a\varepsilon\mathbb{E}\int_{t_{k-2}-\tau_0}^{t_{k-\bar{k}-1}}\Psi(x(s))ds + \cdots + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-2\bar{k}}}^{t_{k-\bar{k}}-\tau_0}\Psi(x(s))ds \\
&\quad + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-2\bar{k}}}\Psi(x(s))ds \\
&\le\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}}^{t}\Psi(x(s))ds + a\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}}^{t_{k-\bar{k}-1}}\Psi(x(s))ds + \cdots \\
&\quad + a^{\bar{k}-1}\varepsilon\mathbb{E}\int_{t_{k-2\bar{k}}}^{t_{k-\bar{k}}-\tau_0}\Psi(x(s))ds + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-2\bar{k}}}\Psi(x(s))ds.
\end{aligned}
$$

Substituting this into (2.20), then using (2.15) and (2.19) , we obtain that

$$
\begin{aligned}
&\mathbb{E}\bar{U}(x(t)) \\
&\le C_1 + C_1(1 + a + a^2 + \cdots + a^{\bar{k}}) + a^{\bar{k}+1}\mathbb{E}\bar{U}(x(t_{k-\bar{k}-1})) \\
&- (a^{\bar{k}}e^{-\tau} - \varepsilon)e^{-\tau}\mathbb{E}\int_{t_{k-\bar{k}-1}}^{t}\Psi(x(s))ds + a\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}}^{t_{k-\bar{k}-1}}\Psi(x(s))ds \\
&+ \cdots + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-2\bar{k}}}\Psi(x(s))ds \\
&\le C_1 + C_1(1 + a + a^2 + \cdots + a^{\bar{k}}) + a^{\bar{k}+1}\mathbb{E}\bar{U}(x(t_{k-\bar{k}-1})) \\
&+ a\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}}^{t_{k-\bar{k}-1}}\Psi(x(s))ds + \cdots + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-2\bar{k}}}\Psi(x(s))ds \\
&\le C_1 + C_1(1 + a + a^2 + \cdots + a^{\bar{k}}) + a^{\bar{k}+1}\Big[C_1 + a\mathbb{E}\bar{U}(x(t_{k-\bar{k}-2})) \\
&- e^{-\tau}\mathbb{E}\int_{t_{k-\bar{k}-2}}^{t_{k-\bar{k}-1}}\Psi(x(s))ds + \varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}-\tau_0}^{t_{k-\bar{k}-1}-\tau_0}\Psi(x(s))ds\Big] \\
&+ a\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}}^{t_{k-\bar{k}-1}}\Psi(x(s))ds + \cdots + a^{\bar{k}}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-1}-\tau_0}^{t_{k-2\bar{k}}}\Psi(x(s))ds \\
&\le C_1 + C_1(1 + a + a^2 + \cdots + a^{\bar{k}+1}) + a^{\bar{k}+2}\mathbb{E}\bar{U}(x(t_{k-\bar{k}-2})) \\
&- a(a^{\bar{k}}e^{-\tau} - \varepsilon)\mathbb{E}\int_{t_{k-\bar{k}-2}}^{t_{k-\bar{k}-1}}\Psi(x(s))ds + a^2\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-3}}^{t_{k-\bar{k}-2}}\Psi(x(s))ds \\
&+ \cdots + a^{\bar{k}+1}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}-\tau_0}^{t_{k-2\bar{k}-1}}\Psi(x(s))ds \\
&\le C_1 + C_1(1 + a + a^2 + \cdots + a^{\bar{k}+1}) + a^{\bar{k}+2}\mathbb{E}\bar{U}(x(t_{k-\bar{k}-2})) \\
&+ a^2\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-3}}^{t_{k-\bar{k}-2}}\Psi(x(s))ds + \cdots + a^{\bar{k}+1}\varepsilon\mathbb{E}\int_{t_{k-\bar{k}-2}-\tau_0}^{t_{k-2\bar{k}-1}}\Psi(x(s))ds.
\end{aligned}
$$

It is straightforward to see that

$$
\begin{aligned}
&\sup_{t_k \le t \le t_{k+1}}\mathbb{E}\bar{U}(x(t)) \le C_1 + C_1(1 + a + \cdots + a^{k-1}) \\
&\quad + a^k\mathbb{E}\bar{U}(x(0)) + \varepsilon a^{k-1}\mathbb{E}\int_{-\tau_0}^{0}\Psi(x(s)))ds \\
&\le C_1 + \frac{C_1}{1 - a} + |x(0)|^p + \int_{-\tau_0}^{0}(\|\xi\|^{q_2+p-2} + \|\xi\|^p)ds.
\end{aligned}
\tag{2.21}
$$

As this holds for any $k \ge 0$, we hence get the required assertion (2.11). The proof is therefore complete. □

Assumptions 2.1, 2.2, 2.3 and 2.4 will form our standing hypotheses in this paper. Let us emphasise that we will NOT explicitly mention Assumptions 2.1, 2.2, 2.3 and 2.4 in the next section in order for us to concentrate on our new assumptions to be imposed.

3. Asymptotic Stabilization

In the previous section, we showed that the controlled system (2.8) is bounded, but the system may still be unstable. Next, we will illustrate how to design the control function to ensure the asymptotic stability of the controlled system (2.8) under some conditions. Let's start with the first condition.

Assumption 3.1. *For each $i \in S$, design the control function $u : R^n \times S \times R_+ \to R^n$ so that we can find constants $\alpha_{ij} > 0$, $\bar{\alpha}_{ij} > 0$, $j = 1, 3, 4$ and $\alpha_{i2}, \bar{\alpha}_{i2} \in R$ for both*

$$x^T[f(x, y, i, t) + u(x, i, t)] + \frac{1}{2}|g(x, y, i, t)|^2$$
$$\leq -\alpha_{i1}|x|^{q_1} + \alpha_{i2}|x|^2 + \alpha_{i3}|y|^{q_1} + \alpha_{i4}|y|^2 \quad (3.1)$$

and

$$x^T[f(x, y, i, t) + u(x, i, t)] + \frac{p_1}{2}|g(x, y, i, t)|^2$$
$$\leq -\bar{\alpha}_{i1}|x|^{q_1} + \bar{\alpha}_{i2}|x|^2 + \bar{\alpha}_{i3}|y|^{q_1} + \bar{\alpha}_{i4}|y|^2 \quad (3.2)$$

to hold for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$ (where q_1 has been specified in condition (2.4)). In addition, both

$$\mathcal{A}_1 := -2\text{diag}(\alpha_{12}, \cdots, \alpha_{N2}) - \Gamma,$$
$$\mathcal{A}_2 := -(p_1 + 1)\text{diag}(\bar{\alpha}_{12}, \cdots, \bar{\alpha}_{N2}) - \Gamma \quad (3.3)$$

are nonsingular M-matrices.

For the theory of M-matrix please refer to (see, e.g., Mao, and Yuan (2006), section 2.6). In fact, many control functions u can meet both Assumption 2.4 and Assumption 3.1. For example, if the state $x(t)$ of the given SDDE (2.1) is observable in any mode $i \in S$, we could give the linear control function $u(x, i, t) = Ax$ (obviously satisfies Assumption 2.4), where A is a symmetric $n \times n$ real-valued matrix such that $\lambda_{\max}(A) \leq -2\alpha_2$. Then

$$x^T u(x, i, t) \leq -2\alpha_2|x|^2, \quad \forall (x, i, t) \in R^n \times S \times R_+.$$

By inequality

$$|v|^b \leq |v|^a + |v|^c, \qquad \forall 0 < a \leq b \leq c. \quad (3.4)$$

Recalling (2.4), we have

$$|y|^{q_2} \leq |y|^2 + |y|^{q_1}.$$

It then follow from (2.5) that

$$x^T[f(x, y, i, t) + u(x, i, t)] + \frac{1}{2}|g(x, y, i, t)|^2$$
$$\leq -\alpha_1|x|^{q_1} - \alpha_2|x|^2 + \alpha_3|y|^{q_1} + (\alpha_3 + \alpha_4)|y|^2$$

as well as

$$x^T[f(x, y, i, t) + u(x, i, t)] + \frac{p_1}{2}|g(x, y, i, t)|^2$$
$$\leq -\alpha_1|x|^{q_1} - \alpha_2|x|^2 + \alpha_3|y|^{q_1} + (\alpha_3 + \alpha_4)|y|^2$$

while

$$\mathcal{A}_1 = 2\text{diag}(\alpha_2, \cdots, \alpha_2) - \Gamma$$

$$\text{and } \mathcal{A}_2 = (p_1 + 1)\text{diag}(\alpha_2, \cdots, \alpha_2) - \Gamma$$

which are nonsingular M-matrices (see, e.g., Mao, and Yuan (2006), Theorem 2.10 on page 68). That is, the control function $u(x, i, t) = Ax$ meets Assumption 3.1.

In order to lead second condition, we set

$$(\theta_1, \cdots, \theta_N)^T := \mathcal{A}_1^{-1}(1, \cdots, 1)^T,$$
$$(\bar{\theta}_1, \cdots, \bar{\theta}_N)^T := \mathcal{A}_2^{-1}(1, \cdots, 1)^T. \quad (3.5)$$

As $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonsingular M-matrices, all θ_i and $\bar{\theta}_i$ are positive. Define a function $U : R^n \times S \to R_+$ by

$$U(x, i) = \theta_i|x|^2 + \bar{\theta}_i|x|^{p_1+1}, \quad (x, i) \in R^n \times S \quad (3.6)$$

while define a operator $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ by

$$\mathcal{L}U(x, y, i, t) = 2\theta_i\Big[x^T[f(x, y, i, t) + u(x, i, t)] + \frac{1}{2}|g(x, y, i, t)|^2\Big]$$
$$+ (p_1 + 1)\bar{\theta}_i|x|^{p_1-1}\Big[x^T[f(x, y, i, t) + u(x, i, t)]$$
$$+ \frac{p_1}{2}|g(x, y, i, t)|^2\Big] + \sum_{j=1}^{N}\gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{p_1+1}). \quad (3.7)$$

By (3.1), (3.2) and the Young inequality, we can calculate

$$\mathcal{L}U(x, y, i, t)$$
$$\leq 2\theta_i(-\alpha_{i1}|x|^{q_1} + \alpha_{i2}|x|^2 + \alpha_{i3}|y|^{q_1} + \alpha_{i4}|y|^2)$$
$$+ (p_1 + 1)\bar{\theta}_i|x|^{p_1-1}(-\bar{\alpha}_{i1}|x|^{q_1} + \bar{\alpha}_{i2}|x|^2 + \bar{\alpha}_{i3}|y|^{q_1} + \bar{\alpha}_{i4}|y|^2)$$
$$+ \sum_{j=1}^{N}\gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{p_1+1})$$
$$\leq -|x|^2 + 2\theta_i\alpha_{i4}|y|^2 - (1 - (p_1 - 1)\bar{\theta}_i\bar{\alpha}_{i4})|x|^{p_1+1}$$
$$+ 2\bar{\theta}_i\bar{\alpha}_{i4}|y|^{p_1+1} - 2\theta_i\alpha_{i1}|x|^{q_1} + 2\theta_i\alpha_{i3}|y|^{q_1} - \Big((p_1 + 1)\bar{\theta}_i\bar{\alpha}_{i1}$$
$$- \frac{(p_1^2 - 1)\bar{\theta}_i\bar{\alpha}_{i3}}{q_1 + p_1 - 1}\Big)|x|^{q_1+p_1-1} + \frac{q_1(p_1 + 1)\bar{\theta}_i\bar{\alpha}_{i3}}{q_1 + p_1 - 1}|y|^{q_1+p_1-1}. \quad (3.8)$$

This observation makes the following assumption possible.

Assumption 3.2. *Assume that there exists a function $H(x) \in C(R^n \times [-h, \infty); R_+)$, as well as nonnegative numbers $\beta_1, \beta_2, \beta_3$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ and κ, such that*

$$\gamma_1 > \gamma_2, \quad \gamma_3|x|^{q_1+p_1-1} \leq H(x) \leq \gamma_4 + \gamma_5|x|^{q_1+p_1-1}, \kappa < 1 \quad (3.9)$$

and

$$\mathcal{L}U(x, y, i, t) + \beta_1(2\theta_i|x| + (p_1 + 1)\bar{\theta}_i|x|^{p_1})^2$$
$$+ \beta_2|f(x, y, i, t)|^2 + \beta_3|g(x, y, i, t)|^2$$
$$\leq -\gamma_1|x|^2 + \gamma_2|y|^2 - H(x) + \kappa H(y) \quad (3.10)$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Let's go on to show that Assumption 3.2 can always be met. In fact, by Assumption 2.2 and (3.8), we then derive

$$\mathcal{L}U(x, y, i, t) + \beta_1(2\theta_i|x| + (p_1 + 1)\bar{\theta}_i|x|^{p_1})^2$$

$$
\begin{aligned}
&+\beta_2|f(x,y,i,t)|^2+\beta_3|g(x,y,i,t)|^2\\
\le& -|x|^2+2\theta_i\alpha_{i4}|y|^2-(1-(p_1-1)\bar\theta_i\bar\alpha_{i4})|x|^{p_1+1}\\
&+2\bar\theta_i\bar\alpha_{i4}|y|^{p_1+1}-2\theta_i\alpha_{i1}|x|^{q_1}+2\theta_i\alpha_{i3}|y|^{q_1}-\Big((p_1+1)\bar\theta_i\bar\alpha_{i1}\\
&-\frac{(p_1^2-1)\bar\theta_i\bar\alpha_{i3}}{q_1+p_1-1}\Big)|x|^{q_1+p_1-1}+\frac{q_1(p_1+1)\bar\theta_i\bar\alpha_{i3}}{q_1+p_1-1}|y|^{q_1+p_1-1}\\
&+4\beta_1\theta_i^2|x|^2+4\beta_1\theta_i\bar\theta_i(p_1+1)|x|^{p_1+1}\\
&+\beta_1(p_1+1)^2\bar\theta_i^2|x|^{2p_1}+4\beta_2L^2(|x|^2+|x|^{2p_1}+|y|^2+|y|^{2p_1})\\
&+4\beta_3L^2(|x|^2+|x|^{2p_2}+|y|^2+|y|^{2p_2}). \qquad (3.11)
\end{aligned}
$$

Recalling (2.4), then using inequality (3.4) again, we get

$$|v|^{2p_1}\vee|v|^{2p_2}\le|v|^2+|v|^{q_1+p_1-1}.$$

Substituting this into (3.11) yields

$$
\begin{aligned}
&\mathcal{L}U(x,y,i,t)+\beta_1(2\theta_i|x|+(p_1+1)\bar\theta_i|x|^{p_1})^2\\
&\quad+\beta_2|f(x,y,i,t)|^2+\beta_3|g(x,y,i,t)|^2\\
&\le-\bar\gamma_1|x|^2+\bar\gamma_2|y|^2-\bar\gamma_3|x|^{p_1+1}+\bar\gamma_4|y|^{p_1+1}\\
&\quad-\bar\gamma_5|x|^{q_1}+\bar\gamma_6|y|^{q_1}-\bar\gamma_7|x|^{q_1+p_1-1}+\bar\gamma_8|y|^{q_1+p_1-1}, \qquad (3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar\gamma_1&=1-4\beta_1\min_{i\in S}\theta_i^2-\beta_1(p_1+1)^2\min_{i\in S}\bar\theta_i^2-8L^2(\beta_2+\beta_3),\\
\bar\gamma_2&=2\max_{i\in S}\theta_i\alpha_{i4}+8L^2(\beta_2+\beta_3),\\
\bar\gamma_3&=1-(p_1-1)\min_{i\in S}\bar\theta_i\bar\alpha_{i4}-4\beta_1\min_{i\in S}\theta_i\min_{i\in S}\bar\theta_i(p_1+1)\\
\bar\gamma_4&=2\max_{i\in S}\bar\theta_i\bar\alpha_{i4},\bar\gamma_5=2\theta_i\min_{i\in S}\alpha_{i1},\bar\gamma_6=2\theta_i\max_{i\in S}\alpha_{i3},\\
\bar\gamma_7&=(p_1+1)\min_{i\in S}\bar\theta_i\bar\alpha_{i1}-\frac{(p_1^2-1)\min_{i\in S}\bar\theta_i\bar\alpha_{i3}}{q_1+p_1-1}\\
&\quad-\beta_1(p_1+1)^2\min_{i\in S}\bar\theta_i^2-4L^2(\beta_2+\beta_3),\\
\bar\gamma_8&=\frac{q_1(p_1+1)\max_{i\in S}\bar\theta_i\bar\alpha_{i3}}{q_1+p_1-1}+4L^2(\beta_2+\beta_3).
\end{aligned}
$$

If we can choose nonnegative constants β_1-β_3 sufficiently small for

$$\bar\gamma_1>\bar\gamma_2,\bar\gamma_3>\bar\gamma_4,\bar\gamma_5>\bar\gamma_6,\bar\gamma_7>\bar\gamma_8.$$

Set

$$H(x):=\bar\gamma_3|x|^{p_1+1}+\bar\gamma_5|x|^{q_1}+\bar\gamma_7|x|^{q_1+p_1-1}$$

and

$$\kappa:=\frac{\bar\gamma_4}{\bar\gamma_3}\vee\frac{\bar\gamma_6}{\bar\gamma_5}\vee\frac{\bar\gamma_8}{\bar\gamma_7}.$$

It is easy to see that $H(x)$ and κ is meet condition (3.9). Then,

$$
\begin{aligned}
&\mathcal{L}U(x,y,i,t)+\beta_1(2\theta_i|x|+(p_1+1)\bar\theta_i|x|^{p_1})^2\\
&+\beta_2|f(x,y,i,t)|^2+\beta_3|g(x,y,i,t)|^2\\
&\le-\bar\gamma_1|x|^2+\bar\gamma_2|y|^2-H(x)+\kappa H(y). \qquad (3.13)
\end{aligned}
$$

For the asymptotic stability of the controlled system (2.8), we will use the Lyapunov functional method. Define two segments $\hat x_t:=\{x(t+s):-2h\le s\le0\}$ and $\hat r_t:=\{r(t+s):-2h\le s\le0\}$ for $t\ge0$. For $\hat x_t$ and $\hat r_t$ to be well defined for $0\le t<2h$, we set $x(s)=x(-\tau_0)$ for $s\in[-2h,-\tau_0)$ and $r(s)=r_0$ for $s\in[-2h,0)$. The Lyapunov functional used in this paper will be of the form

$$
\begin{aligned}
&V(\hat x_t,\hat r_t,t)=U(x(t),r(t))\\
&+\varrho\int_{-\tau}^0\int_{t+s}^t\Big[\tau|f(x(v),y(v),r(v),v)+u(x(\eta_v),r(v),v)|^2\\
&+|g(x(v),y(v),r(v),v)|^2\Big]dvds \qquad (3.14)
\end{aligned}
$$

for $t\ge0$, where U has been defined by (3.6) and ϱ is a positive constant to be determined later while we set

$$
\begin{aligned}
f(x,y,i,v)=f(x,y,i,0),\ g(x,y,i,v)=g(x,y,i,0),\\
u(x,i,v)=u(x,i,0)
\end{aligned}
$$

for $(x,y,i,v)\in R^n\times R^n\times S\times[-2h,0)$. By the generalized Itô formula (see, e.g., Mao, and Yuan (2006), Lemma 1.9 on page 49), we get

$$dU(x(t),r(t))=LU(x(t),x(t-\tau_0),x(\eta_t),r(t),t)dt+dM(t) \qquad (3.15)$$

for $t\ge0$, where $M(t)$ is a continuous local martingale with $M(0)=0$ (see e.g., Mao, and Yuan (2006), Theorem 1.45 on page 48) and $LU:R^n\times R^n\times R^n\times S\times R_+\to R$ is defined by

$$
\begin{aligned}
&LU(x,y,z,i,t)\\
&=2\theta_i\Big[x^T[f(x,y,i,t)+u(z,i,t)]+\frac{1}{2}|g(x,y,i,t)|^2\Big]\\
&+(p_1+1)\bar\theta_i|x|^{p_1-1}\Big[x^T[f(x,y,i,t)+u(z,i,t)]+\frac{1}{2}|g(x,y,i,t)|^2\Big]\\
&+\frac{(p_1^2-1)}{2}\bar\theta_i|x|^{p_1-3}|x^Tg(x,y,i,t)|^2+\sum_{j=1}^N\gamma_{ij}(\theta_j|x|^2+\bar\theta_j|x|^{p_1+1}).
\end{aligned}
$$

On the other hand, by the basic differential calculation

$$
\begin{aligned}
&d\Big(\varrho\int_{-\tau}^0\int_{t+s}^t\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)+u(x(\eta_v),r(v),v)|^2\\
&\quad+|g(x(v),x(v-\tau_0)r(v),v)|^2\Big]dvds\Big)\\
&=\Big(\varrho\tau\Big[\tau|f(x(t),x(t-\tau_0),r(t),t)+u(x(\eta_v),r(v),v)|^2\\
&\quad+|g(x(t),x(t-\tau_0),r(t),t)|^2\Big]\\
&\quad-\varrho\int_{t-\tau}^t\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)+u(x(\eta_v),r(v),v)|^2\\
&\quad+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv\Big)dt. \qquad (3.16)
\end{aligned}
$$

Combining (3.15) with (3.16), we get

$$
\begin{aligned}
&dV(\hat x_t,\hat r_t,t)=LU(x(t),x(t-\tau_0),x(\eta_t),r(t),t)dt+dM(t)\\
&\quad+\Big(\varrho\tau\Big[\tau|f(x(t),x(t-\tau_0),r(t),t)+u(x(\eta_v),r(v),v)|^2\\
&\quad+|g(x(t),x(t-\tau_0),r(t),t)|^2\Big]\\
&\quad-\varrho\int_{t-\tau}^t\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)+u(x(\eta_v),r(v),v)|^2\\
&\quad+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv\Big)dt. \qquad (3.17)
\end{aligned}
$$

Furthermore, by Assumption 2.4, we can compute

$$\begin{aligned}
&LU(x,y,z,i,t)\\
&\le \mathcal{L}U(x,y,z,i,t) - [2\theta_i + (p_i+1)\bar{\theta}_i|x|^{p_1-1}]x^T[u(x,i,t)-u(z,i,t)]\\
&\le \mathcal{L}U(x,y,z,i,t) + \beta_1[2\theta_i|x| + (p_1+1)\bar{\theta}_i|x|^{p_1}]^2 + \frac{\varpi^2}{4\beta_1}|x-z|^2,
\end{aligned}$$

where the function $\mathcal{L}U$ has been defined by (3.7). It then follows from (3.17) that

$$dV(\hat{x}_t,\hat{r}_t,t) \le \mathbb{L}V(\hat{x}_t,\hat{r}_t,t)dt + dM(t), \tag{3.18}$$

where

$$\begin{aligned}
&\mathbb{L}V(\hat{x}_t,\hat{r}_t,t) = \mathcal{L}U(x(t),x(t-\tau_0),x(\eta_t),r(t),t)\\
&\quad + \beta_1[2\theta_i|x(t)| + (p_1+1)\bar{\theta}_i|x(t)|^{p_1}]^2 + \frac{\varpi^2}{4\beta_1}|x(t)-x(\eta_t)|^2\\
&\quad + \varrho\tau\Big[\tau|f(x(t),x(t-\tau_0),r(t),t) + u(x(\eta_t),r(t),t)|^2\\
&\quad + |g(x(t),x(t-\tau_0),r(t),t)|^2\Big]\\
&\quad - \varrho\int_{t-\tau}^{t}\Big[\tau|f(x(v),x(v-\tau_0),r(v),v) + u(x(\eta_v),r(v),v)|^2\\
&\quad + |g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv.
\end{aligned}\tag{3.19}$$

We can now state our first stabilization result.

Theorem 3.3. *Let Assumption 3.1 and Assumption 3.2 hold. Assume also that exist a positive constant τ is sufficiently small for*

$$\tau < \frac{\sqrt{\beta_1(\gamma_1-\gamma_2)}}{2\varpi^2} \quad and \quad \tau \le \frac{\sqrt{\beta_1\beta_2}}{\sqrt{2}\varpi} \wedge \frac{\beta_1\beta_3}{\varpi^2} \wedge \frac{1}{4\varpi}, \tag{3.20}$$

then the solution of the controlled system (2.8) obeys

$$\int_0^\infty \mathbb{E}|x(t,\xi)|^{\tilde{p}}dt < \infty \tag{3.21}$$

for any $\tilde{p} \in [2, q_1+p_1-1]$ and any initial value ξ.

Proof. Fix the initial value ξ arbitrarily. Use the same stopping time σ_k as Theorem 2.5. It is easy to show that σ_k is increasing to infinity with probability 1 as $k \to \infty$. Using the generalized Itô formula and theory of stopping time, we then derive from (3.18) that

$$\begin{aligned}
0 &\le \mathbb{E}V(\hat{x}_{t\wedge\sigma_k},\hat{r}_{t\wedge\sigma_k},t\wedge\sigma_k)\\
&\le V(\hat{x}_0,\hat{r}_0,0) + \mathbb{E}\int_0^{t\wedge\sigma_k}\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)ds
\end{aligned}\tag{3.22}$$

for any $t \ge 0$ and $k \ge k_0$.

Let $\varrho = \varpi^2/\beta_1$. (Please recall that ϱ is a free parameter in the definition of the Lyapunov functional). Using condition (3.20), it is easy to show that

$$2\varrho\tau^2 \le \beta_2 \quad \text{and} \quad \varrho\tau \le \beta_3. \tag{3.23}$$

Substituting (3.23) into (3.19), then using condition (2.10) and (3.10), we obtain that

$$\begin{aligned}
&\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)\\
&\le \mathcal{L}U(x(s),x(s-\tau_0),r(s),s) + \beta_1[2\theta_i|x(s)| + (p_1+1)\bar{\theta}_i|x(s)|^{p_1}]^2\\
&\quad + \beta_2|f(x(s),x(s-\tau_0),r(s),s)|^2 + \beta_3|g(x(s),x(s-\tau_0),r(s),s)|^2\\
&\quad + \frac{2\tau^2\varpi^2}{\beta_1}|u(x(\eta_v),r(v),v)|^2 + \frac{\varpi^2}{4\beta_1}|x(s)-x(\eta_s)|^2\\
&\quad - \frac{\varpi^2}{\beta_1}\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\tau_0),,r(v),v) + u(x(\eta_v),r(v),v)|^2\\
&\quad + |g(x(v),x(v-\tau_0),,r(v),v)|^2\Big]dv\\
&\le -\gamma_1|x(s)^2 + \gamma_2|x(s-\tau_0)|^2 - H(x(s)) + \kappa H(x(s-\tau_0))\\
&\quad + \frac{2\tau^2\varpi^4}{\beta_1}|x(\eta_s)|^2 + \frac{\varpi^2}{4\beta_1}|x(s)-x(\eta_s)|^2\\
&\quad - \frac{\varpi^2}{\beta_1}\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(s-\tau_0),r(v),v) + u(x(\eta_v),r(v),v)|^2\\
&\quad + |g(x(v),x(s-\tau_0),r(v),v)|^2\Big]dv.
\end{aligned}$$

We note from condition (3.20) that $\varpi\tau \le 1/4$, consequently

$$\frac{2\tau^2\varpi^4}{\beta_1}|x(\eta_s)|^2 \le \frac{4\tau^2\varpi^4}{\beta_1}|x(s)|^2 + \frac{\varpi^2}{4\beta_1}|x(s)-x(\eta_s)|^2.$$

It is easy to see that

$$\begin{aligned}
&\mathbb{L}V(\hat{x}_s,\hat{r}_s,s) \le -\Big(\gamma_1 - \frac{4\tau^2\varpi^4}{\beta_1}\Big)|x(s)|^2 + \gamma_2|x(s-\tau_0)|^2\\
&\quad - H(x(s)) + \kappa H(x(s-\tau_0)) + \frac{\varpi^2}{2\beta_1}|x(s)-x(\eta_s)|^2\\
&\quad - \frac{\varpi^2}{\beta_1}\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\tau_0),r(v),v) + u(x(\delta_v),r(v),v)|^2\\
&\quad + |g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv.
\end{aligned}\tag{3.24}$$

Substituting (3.24) into (3.22) yields

$$\begin{aligned}
0 &\le \mathbb{E}V(\hat{x}_{t\wedge\sigma_k},\hat{r}_{t\wedge\sigma_k},t\wedge\sigma_k)\\
&\le V(\hat{x}_0,\hat{r}_0,0) + I_1 + I_2 + I_3 - I_4,
\end{aligned}\tag{3.25}$$

where

$$\begin{aligned}
I_1 &= \mathbb{E}\int_0^{t\wedge\sigma_k}\Big[-\Big(\gamma_1 - \frac{4\tau^2\varpi^4}{\beta_1}\Big)|x(s)|^2 + \gamma_2|x(s-\tau_0)|^2\Big]ds,\\
I_2 &= \mathbb{E}\int_0^{t\wedge\sigma_k}\big[-H(x(s)) + \kappa H(x(s-\tau_0))\big]ds,\\
I_3 &= \frac{\varpi^2}{2\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k}|x(s)-x(\eta_s)|^2ds,\\
I_4 &= \frac{\varpi^2}{\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k}\Big(\int_{s-\tau}^{s}[\tau|f(x(v),x(v-\tau_0),r(v),v)\\
&\quad + u(x(\eta_v),r(v),v)|^2 + |g(x(v),x(v-\tau_0),r(v),v)|^2]dv\Big)ds.
\end{aligned}$$

By the substitution technique, we deduce that

$$\int_0^{t\wedge\sigma_k}|x(s-\tau_0)|^2ds \le \int_{-\tau_0}^{t\wedge\sigma_k-\tau_0}|x(s)|^2ds \le \int_{-\tau_0}^{t\wedge\sigma_k}|x(s)|^2ds,$$

then

$$I_1 \leq \gamma_2 \int_{-\tau_0}^{0} |x(s)|^2 ds - b\mathbb{E}\int_0^{t\wedge\sigma_k} |x(s)|^2 ds, \tag{3.26}$$

where $b = \gamma_1 - \frac{4\tau^2\varpi^4}{\beta_1} - \gamma_2 > 0$ by condition (3.20). Similarly, we can show

$$\begin{aligned} I_2 \leq & \kappa \int_{-\tau_0}^{0} H(x(s))ds - (1-\kappa)\mathbb{E}\int_0^{t\wedge\sigma_k} H(x(s))ds \\ \leq & \gamma_4\kappa + \gamma_5\kappa \int_{-\tau_0}^{0} |x(s)|^{q_1+p_1-1} ds \\ & - \gamma_3(1-\kappa)\mathbb{E}\int_0^{t\wedge\sigma_k} |x(s)|^{q_1+p_1-1} ds. \end{aligned} \tag{3.27}$$

Plugging (3.26) and (3.27) into (3.25), we have

$$\begin{aligned} \mathbb{E}V(\hat{x}_{t\wedge\sigma_k}, \hat{r}_{t\wedge\sigma_k}, t\wedge\sigma_k) \leq & C_2 + I_3 - I_4 - b\mathbb{E}\int_0^{t\wedge\sigma_k} |x(s)|^2 ds \\ & - \gamma_3(1-\kappa)\mathbb{E}\int_0^{t\wedge\sigma_k} |x(s)|^{q_1+p_1-1} ds, \end{aligned} \tag{3.28}$$

where C_2 is a constant defined by

$$\begin{aligned} C_2 = & V(\hat{x}_0, \hat{r}_0, 0) + \gamma_2 \int_{-\tau_0}^{0} \mathbb{E}|x(s)|^2 ds \\ & + \gamma_4\kappa + \gamma_5\kappa \int_{-\tau_0}^{0} |x(s)|^{q_1+p_1-1} ds. \end{aligned}$$

We can let $k \to \infty$ and apply the Fatou lemma to get

$$b\mathbb{E}\int_0^t |x(s)|^2 ds + \gamma_3(1-\kappa)\mathbb{E}\int_0^t |x(s)|^{q_1+p_1-1} ds \leq C_2 + \bar{I}_3 - \bar{I}_4, \tag{3.29}$$

where

$$\begin{aligned} \bar{I}_3 = & \frac{\varpi^2}{2\beta_1}\mathbb{E}\int_0^t |x(s) - x(\eta_s)|^2 ds, \\ \bar{I}_4 = & \frac{\varpi^2}{\beta_1}\mathbb{E}\int_0^t \Big(\int_{s-\tau}^{s} [\tau|f(x(v), x(v-\tau_0), r(v), v) \\ & + u(x(\eta_v), r(v), v)|^2 + |g(x(v), x(v-\tau_0), r(v), v)|^2]dv\Big)ds. \end{aligned}$$

On the other hand, by the Fubini theorem,

$$\bar{I}_3 = \frac{\varpi^2}{2\beta_1}\int_0^t \mathbb{E}|x(s) - x(\eta_s)|^2 ds. \tag{3.30}$$

Using the Hölder inequality and the Itô isometry, we derive that

$$\begin{aligned} & \mathbb{E}|x(s) - x(\eta_s)|^2 \\ \leq & 2\mathbb{E}\int_{\eta_s}^{s} \Big(\tau|f(x(v), x(v-\tau_0), r(v), v) + u(x(\eta_v), r(v), v)|^2 \\ & + |g(x(v), x(v-\tau_0), r(v), v)|^2\Big)dv \\ \leq & 2\mathbb{E}\int_{s-\tau}^{s} \Big(\tau|f(x(v), x(v-\tau_0), r(v), v) + u(x(\eta_v), r(v), v)|^2 \\ & + |g(x(v), x(v-\tau_0), r(v), v)|^2\Big)dv. \end{aligned} \tag{3.31}$$

This implies

$$\bar{I}_3 \leq \bar{I}_4. \tag{3.32}$$

Substituting (3.32) into (3.29), we have

$$b\mathbb{E}\int_0^t |x(s)|^2 ds + \gamma_3(1-\kappa)\mathbb{E}\int_0^t |x(s)|^{q_1+p_1-1} ds \leq C_2. \tag{3.33}$$

It is straightforward to show that

$$\int_0^\infty \mathbb{E}(|x(t)|^2 + |x(t)|^{q_1+p_1-1})dt < \infty.$$

Using the inequality (3.4), we can derive that

$$|x(t)|^{\tilde{p}} \leq |x(t)|^2 + |x(t)|^{q_1+p_1-1}$$

for any $\tilde{p} \in [2, q_1 + p_1 - 1]$. We hence get the required assertion (3.37). The proof is therefore complete. □

The theorem 3.3 shows that it is possible to design the control function for the controlled system (2.8) is H_∞-stable in $L^{\tilde{p}}$ for any $\tilde{p} \in [2, q_1 + p_1 - 1]$. In general, it does not follow from H_∞-stable in $L^{\tilde{p}}$ that $\lim_{t\to\infty}\mathbb{E}|x(t)|^2 = 0$. But, in our case, this is possible. In fact, we can show a stronger result that $\lim_{t\to\infty}\mathbb{E}|x(t)|^{\bar{p}} = 0$ for any $\bar{p} \in [2, p)$. Let's state the second theorem in this section.

Theorem 3.4. *Under the same Assumptions of Theorem 3.3, the solution of the controlled hybrid SDDE (2.8) satisfies*

$$\lim_{t\to\infty}\mathbb{E}|x(t,\xi)|^{\bar{p}} = 0 \tag{3.34}$$

for any $\bar{p} \in [2, p)$ *and any initial value* ξ.

Proof. Fix the initial value ξ arbitrarily. Using the Itô formula, we derive that

$$\begin{aligned} & \mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2 \\ = & \mathbb{E}\int_{t_1}^{t_2} \Big(2x^T(t)[f(x(t), x(t-\tau_0), r(t), t) + u(x(\eta_t), r(t), t)] \\ & + |g(x(t), x(t-\tau_0), r(t), t)|^2\Big)dt \end{aligned}$$

for any $0 \leq t_1 < t_2 < \infty$. Combining (2.3), (2.4) and (2.10), we further get

$$\begin{aligned} & \Big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\Big| \\ & \leq \mathbb{E}\int_{t_1}^{t_2} \Big(2|x(t)|[L(|x(t)| + |x(t)|^{p_1} + |x(t-\tau_0)| + |x(t-\tau_0)|^{p_1}) \\ & + \varpi|x(\eta_t)|] + L^2[|x(t)| + |x(t)|^{p_2} + |x(t-\tau_0)| + |x(t-\tau_0)|^{p_2}]^2\Big)dt \\ & \leq \int_{t_1}^{t_2} C(1 + \mathbb{E}|x(t)|^p + \mathbb{E}|x(t-\tau_0)|^p + \mathbb{E}|x(\eta_t)|^p)dt. \end{aligned}$$

It then follows from the Theorem 2.5 that,

$$\Big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\Big| \leq C(1 + 3C_3)(t_2 - t_1),$$

where

$$C_3 := \sup_{0\leq t<\infty} \mathbb{E}|x(t)|^p < \infty.$$

Thus, $\mathbb{E}|x(t)|^2$ is uniformly continuous in t on R_+. Recalling (3.21), we therefore obtain

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0. \quad (3.35)$$

That is, the assertion (3.34) holds when $\bar{p} = 2$. Let us now fix any $\bar{p} \in (2, p)$. By the Hölder inequality, because of $(p-\bar{p})/(p-2) \in (0,1)$, we can calculate

$$\begin{aligned}\mathbb{E}|x(t)|^{\bar{p}} &\le (\mathbb{E}|x(t)|^2)^{(p-\bar{p})/(p-2)}(\mathbb{E}|x(t)|^p)^{(\bar{p}-2)/(p-2)} \\ &\le C_3^{(\bar{p}-2)/(p-2)}(\mathbb{E}|x(t)|^2)^{(p-\bar{p})/(p-2)}. \quad (3.36)\end{aligned}$$

It follows for (3.35) and (3.36) that implies the required assertion (3.34). □

At the end of this section, we will show that it is almost surely stable ($\lim_{t\to\infty}|x(t)| = 0$ a.s.). In fact, we can get theorem 3.3 directly without stopping time. However, in order to obtain almost surely stability and better estimate of p in condition (2.4), we still use stopping time method to prove it. Let's state with the third result.

Theorem 3.5. *Under the same assumptions of Theorem 3.3, then the solution of the controlled system (2.8) satisfies*

$$\lim_{t\to\infty} x(t,\xi) = 0 \quad a.s. \quad (3.37)$$

for any initial value ξ.

Proof. Again fix the initial value ξ arbitrarily. It follows from (3.33) in Theorem 3.3 and the well-known Fubini theorem that

$$\mathbb{E}\int_0^\infty |x(t)|^2 ds \le C_2/b < \infty,$$

which implies

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \quad (3.38)$$

We now claim that (3.37). If this is false, we can find a positive number ε sufficiently small, such that

$$\mathbb{P}(|x(t)|^2 \ge 2\varepsilon) \ge 4\varepsilon. \quad (3.39)$$

For each $k > \|\xi\|$, we use the same stopping time σ_k as in the proof of Theorem 3.3. Recalling (3.28), we further obtain

$$k^2\mathbb{P}(\sigma_k < t) \le C_2 + I_3 - I_4.$$

Combining this and (3.32), we get

$$\limsup_{k\to\infty} k^2\mathbb{P}(\sigma_k < t) \le C_2.$$

As this holds for any $t \ge 0$, we must have

$$k^2\mathbb{P}(\sigma_k < \infty) \le C_2 + 1.$$

Choosing a sufficiently large positive integer k_1, for

$$\mathbb{P}(\sigma_{k_1} < \infty) \le (C_2+1)/k_1^2 < \varepsilon.$$

This means

$$\mathbb{P}(|x(t)| < k_1) \ge 1-\varepsilon \quad \text{for all} \quad t \ge -\tau_0. \quad (3.40)$$

Set $\Omega_1 = \{|x(t)|^2 \ge 2\varepsilon\} \cap \{|x(t)| < k_1\}$. Summing (3.39) and (3.40) yields

$$\mathbb{P}(\Omega_1) > 3\varepsilon. \quad (3.41)$$

Let's define a boundedness process $\tilde{x}(t) = x(t \wedge \sigma_{k_1})$, then

$$d\tilde{x}(t) = \tilde{F}(t)dt + \tilde{G}(t)dw(t), \quad (3.42)$$

where

$$\begin{aligned}\tilde{F}(t) &= [f(x(t), x(t-\tau_0), r(t), t) + u(x(\eta_t), r(t), t)]I_{[0,\sigma_{k_1})}(t) \\ &\text{and} \quad \tilde{G}(t) = g(x(t), x(t-\tau_0), r(t), t)I_{[0,\sigma_{k_1})}(t).\end{aligned}$$

Clearly, $\tilde{x}(t)$ is an Itô process. By Assumption 2.2 and Assumption 2.4, we get

$$|\tilde{F}(t)| \vee |\tilde{G}(t)| \le L_{k_1} \quad a.s. \quad (3.43)$$

Define a sequence of stopping times:

$$\begin{aligned}\zeta_1 &= \inf\{t > 0, |x(t)|^2 \ge 2\varepsilon\}, \\ \zeta_{2n} &= \inf\{t > \zeta_{2n-1}, |x(t)|^2 \le \varepsilon\}, \\ \zeta_{2n+1} &= \inf\{t > \zeta_{2n}, |x(t)|^2 \ge 2\varepsilon\}.\end{aligned}$$

By (3.38) and the definition of Ω_1, we have $\omega \in \Omega_1$, then $\zeta_n \le \infty, n = 1, 2, \cdots, n, \cdots$. This implies

$$\Omega_1 \subset \{\zeta_n < \infty\}. \quad (3.44)$$

Set $\theta = \varepsilon/2k_1$. It is easy to see that

$$||x|^2 - |y|^2| \le \varepsilon \quad \text{whenever } |x-y| \le \theta, |x| \wedge |y| \le k_1. \quad (3.45)$$

Choose a sufficiently small constant $\mu > 0$ and a sufficiently large positive integer N such that

$$2\mu L_{k_1}(\mu+4)/\theta^2 < \varepsilon \quad \text{and} \quad C_2/b \le \varepsilon^2\mu N. \quad (3.46)$$

Using (3.41) and (3.44), there exists a sufficiently large constant T, we have

$$\mathbb{P}\{\zeta_{2N} < T\} \ge 2\varepsilon. \quad (3.47)$$

In particular, if $\zeta_{2N} < T$, then $\tilde{x}(\zeta_{2N}) = \varepsilon$, $\zeta_{2N} < \sigma_{k_1}$ (otherwise $\tilde{x}(\zeta_{2N}) = \tilde{x}(\sigma_{k_1}) = k_1$, a contradiction). Namely,

$$\tilde{x}(t,\omega) = x(t,\omega) \quad \text{for } \omega \in \{\zeta_{2N} < T\} \text{ and } t \in [0, \zeta_{2N}]. \quad (3.48)$$

Applying the Hölder inequality and the Doob martingale inequality, we then derive from (3.43) that, for any $\mu > 0$ and $n < N$,

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le t\le\mu} |\tilde{x}(\zeta_{2n-1}\wedge T + t) - \tilde{x}(\zeta_{2n-1}\wedge T)|^2\Big) \\ &= \mathbb{E}\Big(\sup_{0\le t\le\mu}\Big|\int_{\zeta_{2n-1}\wedge T}^{\zeta_{2n-1}\wedge T+t} \tilde{F}(s)ds + \int_{\zeta_{2n-1}\wedge T}^{\zeta_{2n-1}\wedge T+t} \tilde{G}(s)ds\Big|^2\Big)\end{aligned}$$

$$\leq 2\mu\mathbb{E}\Big(\sup_{0\leq t\leq\mu}\int_{\zeta_{2n-1}\wedge T}^{\zeta_{2n-1}\wedge T+t}|\tilde{F}(s)|^2ds\Big)+8\mathbb{E}\Big(\int_{\zeta_{2n-1}\wedge T}^{\zeta_{2n-1}\wedge T+\upsilon}|\tilde{G}(s)|^2ds\Big)$$
$$\leq 2\mu L_{k_1}(\mu+4).$$

By (3.46) and Markov inequality, we deduce that

$$\mathbb{P}\Big(\sup_{0\leq t\leq\mu}|\tilde{x}(\zeta_{2n-1}\wedge T+t)-\tilde{x}(\zeta_{2n-1}\wedge T)|\geq\theta\Big)\leq\varepsilon.$$

This, together with (3.48), we obtain

$$\mathbb{P}\Big(\{\zeta_{2N}<T\}\cap\Big\{\sup_{0\leq t\leq\mu}|x(\zeta_{2n-1}+t)-x(\zeta_{2n-1})|<\theta\Big\}\Big)$$
$$=\mathbb{P}(\{\zeta_{2N}<T\})-\mathbb{P}(\{\zeta_{2N}<T\}\cap\Big\{\sup_{0\leq t\leq\mu}|\tilde{x}(\zeta_{2n-1}+t)-\tilde{x}(\zeta_{2n-1})|\geq\theta\Big\})$$
$$\geq\mathbb{P}(\{\zeta_{2N}<T\})-\mathbb{P}(\Big\{\sup_{0\leq t\leq\mu}|\tilde{x}(\zeta_{2n-1}\wedge T+t)-\tilde{x}(\zeta_{2n-1}\wedge T)|\geq\theta\Big\})$$
$$\geq\varepsilon.$$

It then follows from (3.45) immediately that

$$\mathbb{P}\Big(\{\zeta_{2N}<T\}\cap\Big\{\sup_{0\leq t\leq\mu}|x^2(\zeta_{2n-1}+t)-x^2(\zeta_{2n-1})|<\varepsilon\Big\}\Big)$$
$$\geq\mathbb{P}\Big(\{\zeta_{2N}<T\}\cap\Big\{\sup_{0\leq t\leq\mu}|x(\zeta_{2n-1}+t)-x(\zeta_{2n-1})|<\theta\Big\}\Big)$$
$$\geq\varepsilon.$$

It is easy to show that

$$\zeta_{2n}-\zeta_{2n-1}>\mu$$
$$\text{for}\quad\omega\in\{\zeta_{2N}<T\}\cap\Big\{\sup_{0\leq t\leq\mu}|x^2(\zeta_{2n-1}+t)-x^2(\zeta_{2n-1})|<\varepsilon\Big\}.$$

Combining this with (3.44), we finally derive that

$$C_2/b\geq\mathbb{E}\int_0^\infty|x(t)|^2dt$$
$$\geq\sum_{n=1}^N\mathbb{E}\Big(\{I_{\zeta_2N\leq T}\}\int_{\zeta_{2n-1}}^{\zeta_{2n}}|x(t)|^2dt\}\Big)$$
$$\geq\varepsilon\sum_{n=1}^N\mathbb{E}\Big(\{I_{\zeta_2N\leq T}\}(\zeta_{2n}-\zeta_{2n-1})\}\Big)$$
$$\geq\varepsilon\mu\sum_{n=1}^N P\Big(\{\zeta_{2N}<T\}\cap\Big\{\sup_{0\leq t\leq\mu}|x^2(\zeta_{2n-1}+t)-x^2(\zeta_{2n-1})|<\varepsilon\Big\}\Big)$$
$$\geq\varepsilon^2\mu N.$$

But this contradicts the second inequality in (3.46). Therefore the required assertion must hold. The proof is therefore complete. □

4. Exponential Stabilization

In the previous part, we introduced how to design a feedback control based on discrete-time state observation step by step to make the controlled system (2.8) become H_∞-stable in $L^{\bar{p}}$ ($\bar{p}\in[2,q_1+p_1-1]$, asymptotic stable in $L^{\bar{p}}$ ($\bar{p}\in[2,p)$) and almost surely stable. Consequently, we will illustrate how to design a feedback control based on the discrete-time state observations to make the controlled system (2.8) become exponentially stable either in $L^{\bar{p}}$ ($\bar{p}\in[2,p)$) or almost surely.

For the purpose of the exponentially stable, we need to replace condition (3.20) by stronger condition.

Assumption 4.1. *Make sure the duration between the two consecutive state observations satisfies*

$$\tau<\frac{\sqrt{\beta_1(\gamma_1-\gamma_2)}}{2\varpi^2}\quad\text{and}\quad\tau\leq\frac{\sqrt{\beta_1\beta_2}}{\sqrt{2}\varpi}\wedge\frac{\beta_1\beta_3}{\varpi^2}\wedge\frac{1}{4\sqrt{2}\varpi}.\tag{4.1}$$

We should point out that the last term $\tau\leq 1/4\varpi$ in (3.20) is now replaced by $\tau\leq 1/4\sqrt{2}\varpi$ in (4.1) so the bound on τ here could be smaller than before.

Theorem 4.2. *Let Assumption 3.1 and Assumption 3.2 hold. If we can choose $\tau>0$ sufficiently small for Assumption 4.1 to hold, then the solution of the controlled system (2.8) satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t,\xi)|^{\bar{p}})<0\tag{4.2}$$

for any $\bar{p}\in[2,p)$ and any initial value ξ.

Proof. We will use the same Lyapunov functional $V(\hat{x}_t,\hat{r}_t,t)$ as defined by (3.14) with the same $\varrho=\varpi^2/\beta_1$. Similar to Step 3 of the proof of Theorem 3.3, by the generalized Itô formula, we can compute

$$e^{\varepsilon_1 t}\mathbb{E}V(\hat{x}_t,\hat{r}_t,t)\leq V(\hat{x}_0,\hat{r}_0,0)$$
$$+\int_0^t e^{\varepsilon_1 s}\mathbb{E}\Big(\varepsilon V(\hat{x}_s,\hat{r}_s,s)+\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)\Big)ds\tag{4.3}$$

for all $t\geq 0$, where ε_1 is a sufficiently small positive number to be determined later. Recalling the structure of V, we then have

$$b_1e^{\varepsilon_1 t}\mathbb{E}|x(t)|^2\leq V(\hat{x}_0,\hat{r}_0,0)+\frac{\varepsilon_1\varpi^2}{\beta_1}\Phi_1(t)+\int_0^t e^{\varepsilon_1 s}\Big(\varepsilon_1 b_2\mathbb{E}|x(s)|^2$$
$$+\varepsilon_1 b_3\mathbb{E}|x(s)|^{p_1+1}+\mathbb{E}\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)\Big)ds,\tag{4.4}$$

where $b_1=\min_{i\in S}\theta_i$, $b_2=\max_{i\in S}\theta_i$, $b_3=\max_{i\in S}\bar{\theta}_i$, and

$$\Phi_1(t)=E\int_0^t e^{\varepsilon_1 s}\Big(\int_{-\tau}^0\int_{s+u}^s\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)$$
$$+u(x(\eta_v),r(v),v)|^2+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dvdu\Big)ds.$$

Noting $4\sqrt{2}\varpi\tau\leq 1$, we can rewrite (3.24) as

$$\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)\leq-\Big(\gamma_1-\frac{4\tau^2\varpi^4}{\beta_1}\Big)|x(s)|^2+\gamma_2|x(s-\tau_0)|^2$$
$$-H(x(s))+\kappa H(x(s-\tau_0))+\frac{3\varpi^2}{8\beta_1}|x(s)-x(\eta_s)|^2$$
$$-\frac{\varpi^2}{\beta_1}\int_{s-\tau}^s\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)+u(x(\eta_v),r(v),v)|^2$$
$$+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv.\tag{4.5}$$

By (3.31), we then derive

$$
\begin{aligned}
&\mathbb{E}\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)\\
&\le -\Big(\gamma_1-\frac{4\tau^2\varpi^4}{\beta_1}\Big)\mathbb{E}|x(s)|^2+\gamma_2\mathbb{E}|x(s-\tau_0)|^2-H(x(s))\\
&\quad+\kappa H(x(s-\tau_0))-\frac{\varpi^2}{4\beta_1}\mathbb{E}\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)\\
&\quad+u(x(\eta_v),r(v),v)|^2+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv. \quad (4.6)
\end{aligned}
$$

Recalling (3.4), we get $\mathbb{E}|x(s)|^{p_1+1}\le\mathbb{E}|x(s)|^2+\mathbb{E}|x(s)|^{q_1+p_1-1}$. Substituting this and (4.6) into (4.4), by similar techniques in (3.26) yields

$$
\begin{aligned}
b_1e^{\varepsilon_1 t}\mathbb{E}|x(t)|^2&\le C+\frac{\varepsilon_1\varpi^2}{\beta_1}\Phi_1(t)-\frac{\varpi^2}{4\beta_1}\Phi_2(t)\\
&\quad-\Big(\gamma_1-\gamma_2-\frac{4\tau^2\varpi^4}{\beta_1}-\varepsilon_1b_2-\varepsilon_1b_3\Big)\int_0^t e^{\varepsilon_1 s}\mathbb{E}|x(s)|^2ds\\
&\quad-[\gamma_3(1-\kappa)-\varepsilon_1b_3]\int_0^t e^{\varepsilon_1 s}\mathbb{E}|x(s)|^{q_1+p_1-1}ds, \quad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_2(t)=\mathbb{E}\int_0^t e^{\varepsilon_1 s}\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)\\
+u(x(\eta_v),r(v),v)|^2+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv.
\end{aligned}
$$

On the other hand, it is straightforward to show that

$$
\begin{aligned}
&\Phi_1(t)\\
\le&\mathbb{E}\int_0^t e^{\varepsilon_1 s}\Big(\tau\int_{s-\tau}^{s}\Big[\tau|f(x(v),x(v-\tau_0),r(v),v)+u(x(\eta_v),r(v),v)|^2\\
&+|g(x(v),x(v-\tau_0),r(v),v)|^2\Big]dv\Big)ds=\tau\Phi_2(t).
\end{aligned}
$$

We further make sure $\varepsilon_1>0$ to be sufficiently small for

$$
\varepsilon_1\tau\le\frac{1}{4},\ \ \varepsilon_1(b_2+b_3)\le\gamma_1-\gamma_2-\frac{4\tau^2\varpi^4}{\beta_1},\ \ \varepsilon_1b_3\le\gamma_3(1-\kappa).
$$

It then follows from (4.7) immediately that

$$
\mathbb{E}|x(t)|^2\le\frac{C}{b_1}e^{-\varepsilon_1 t},\ \ \forall t\ge 0. \quad (4.8)
$$

Finally, by (3.36) and (4.8), and applying the Hölder inequality, we obtain

$$
\mathbb{E}|x(t)|^{\bar p}\le C_3^{(\bar p-2)/(p-2)}(C/b_1)^{(p-\bar p)/(p-2)}e^{-\varepsilon_1 t(p-\bar p)/(p-2)} \quad (4.9)
$$

for any $\bar p\in[2,p)$. This implies the required assertion (4.2). □

In general, it is not possible to imply the almost surely exponential stability from the $\bar p$th moment exponential stability. However, in our situation, this is possible as described in the following theorem.

Theorem 4.3. *Under the same Assumptions of Theorem 4.2 . Then the solution of the controlled system (2.8) satisfies*

$$
\limsup_{t\to\infty}\frac{1}{t}\log(|x(t,\xi)|)<0\ \ a.s. \quad (4.10)
$$

for any initial value ξ.

Proof. Fix the initial value ξ arbitrarily. Applying the Itô formula and the Burkholder-Davis-Gundy inequality (see, e.g., Mao, and Yuan (2006), Theorem 2.13 on page 70), we have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\\
&\le\mathbb{E}|x(t_k)|^2+\mathbb{E}\int_{t_k}^{t_{k+1}}\Big(2|x(t)||f(x(t),x(t-\tau_0),r(t),t)\\
&\quad+u(x(\eta_t),r(t),t)|+|g(x(t),x(t-\tau_0),r(t),t)|^2\Big)dt\\
&\quad+8\sqrt{2}\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}|x(t)|^2|g(x(t),x(t-\tau_0),r(t),t)|^2dt\Big)^{1/2}.
\end{aligned}
$$

By the Young inequality

$$
\begin{aligned}
&8\sqrt{2}\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}|x(t)|^2|g(x(t),x(t-\tau_0),r(t),t)|^2dt\Big)^{1/2}\\
\le&8\sqrt{2}\mathbb{E}\Big[\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|\Big)\Big(\int_{t_k}^{t_{k+1}}|g(x(t),x(t-\tau_0),r(t),t)|^2dt\Big)^{1/2}\Big]\\
\le&0.5\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)+64\mathbb{E}\int_{t_k}^{t_{k+1}}|g(x(t),x(t-\tau_0),r(t),t)|^2dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le 2\mathbb{E}|x(t_k)|^2\\
&+\mathbb{E}\int_{t_k}^{t_{k+1}}\Big(4|x(t)||f(x(t),x(t-\tau_0),r(t),t)+u(x(\eta_t),r(t),t)|\\
&\quad+130|g(x(t),x(t-\tau_0),r(t),t)|^2\Big)dt. \quad (4.11)
\end{aligned}
$$

Using Assumption 2.1, we then derive from (4.11) that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le 2\mathbb{E}|x(t_k)|^2+C\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}|x(t)|^2\\
&\quad+\mathbb{E}|x(t-\tau_0)|^2+\mathbb{E}|x(\eta_t)|^2+\mathbb{E}|x(t))|^{\hat p}+\mathbb{E}|x(t-\tau_0)|^{\hat p}\Big)dt,
\end{aligned}
$$

where $\hat p=(p_1+1)\vee(2p_2)$. From (2.4) and $p_1>1$, it is easy to show that $\hat p\in[2,p)$. Consequently, combining (4.8) with (4.9) yields

$$
\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le Ce^{-\hat\varepsilon t_k},
$$

where $\hat\varepsilon=\varepsilon_1(p-\hat p)/(p-2)$. We can then easily show that

$$
\sum_{k=0}^{\infty}\mathbb{P}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|>e^{-0.25\hat\varepsilon t_k}\Big)\le\sum_{k=0}^{\infty}Ce^{-0.5\hat\varepsilon t_k}<\infty.
$$

In view of the well-known Borel-Cantelli lemma (see, e.g., Mao, and Yuan (2006), Lemma 1.2 on page 10), we see that for almost all $\omega\in\Omega$, there is positive integer $k_0=k_0(\omega)$ such that

$$
\sup_{t_k\le t\le t_{k+1}}|x(t)|\le e^{-0.25\hat\varepsilon t_k},\forall\, k\ge k_0.
$$

Hence, for almost all $\omega\in\Omega$,

$$
\frac{1}{t}\log(|x(t)|)\le-\frac{0.25\hat\varepsilon\tau k}{t}\le-\frac{0.25\hat\varepsilon\tau k}{\tau(k+1)},\ \ t\in[t_k,t_{k+1}],\ k\ge k_0.
$$

Let $k\to\infty$, we get

$$
\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)\le-0.25\hat\varepsilon<0\ \ a.s.
$$

which is the assertion. Thus the proof is complete. □

5. Examples

We will illustrate our results with an example. In order to maintain the coherence of the article, we will take the hybrid SDDE (1.3) as an example. Let's recall that the coefficients f and g in SDDE (1.3) are defined by (1.4), where $w(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on $S = \{1, 2\}$ with the generator Γ defined by (1.5). Through computer numerical simulation (we set $\tau_0 = 0.05$ and the initial value $x(t) = 2 + sin(t)$ on $t \in [-0.05, 0]$ and $r(0) = 1$), we can find that hybrid SDDE (1.3) is unstable. This result can be referred to in Fig. 5.1.

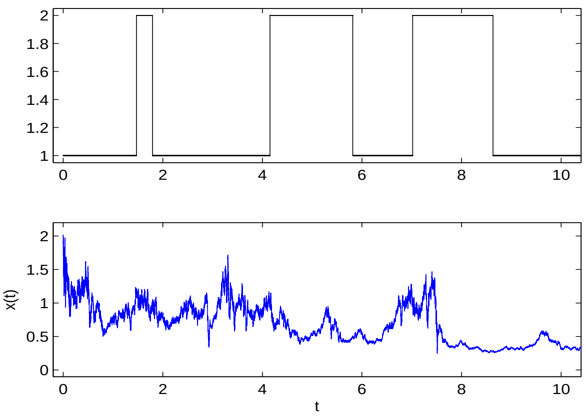

Figure 5.1: The computer simulation of the sample paths of the Markov chain and the SDDE (1.3) with $\tau_0 = 0.05$ using the Euler–Maruyama method with step size 10^{-4}.

We will choose the control function $u : R \times S \times R_+ \to R$ define by

$$u(x, 1, t) = -2x, \quad u(x, 2, t) = -3x. \tag{5.1}$$

We see easily that Assumption 2.4 hold with $\varpi = 3$. It follows from Theorem 2.5 immediately that the controlled system

$$\begin{aligned} dx(t) = [f(x(t), x(t-\tau_0), r(t), t) + u(x(\eta_t), r(t), t)]dt \\ + g(x(t), x(t-\tau_0), r(t), t)dw(t) \end{aligned} \tag{5.2}$$

has a unique global solution on $t \geq 0$ for any initial value $\xi \in C([-h, 0]; R^n)$ and the solution satisfies that

$$\sup_{0 \leq t < \infty} \mathbb{E}|x(t, \xi)|^p < \infty \quad \forall p > 2. \tag{5.3}$$

By simple calculation, for $(x, y, i, t) \in R \times R \times S \times R_+$, we get

$$\begin{aligned} & x[f(x, y, i, t) + u(x, t, i)] + \frac{1}{2}|g(x, y, t, i)|^2 \\ & \leq \begin{cases} -2x^6 - x^2 + 0.3125y^6 + 0.0625y^2 & \text{if } i = 1, \\ -3x^6 - x^2 + 0.75y^6 + 0.25y^2 & \text{if } i = 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & x[f(x, y, i, t) + u(x, t, i)] + \frac{p_1}{2}|g(x, y, t, i)|^2 \\ & \leq \begin{cases} -2x^6 - x^2 + 0.5625y^6 + 0.3125y^2 & \text{if } i = 1, \\ -3x^6 - x^2 + 1.75y^6 + 1.25y^2 & \text{if } i = 2. \end{cases} \end{aligned}$$

It is easy to see that

$$\alpha_{12} = \alpha_{22} = \bar{\alpha}_{12} = \bar{\alpha}_{22} = -1.$$

Hence,

$$\mathcal{A}_1 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \text{ and } \mathcal{A}_2 = \begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}$$

are both M-matrices. By (3.5), we then have

$$\theta_1 = \theta_2 = 0.5,$$

$$\bar{\theta}_1 = \bar{\theta}_2 = 0.1666667,$$

and while the Assumption 3.1 is satisfied. The function U defined by (3.6) becomes

$$U(x, i) = \begin{cases} 0.5x^2 + 0.1666667x^6 & \text{if } i = 1, \\ 0.5x^2 + 0.1666667x^6 & \text{if } i = 2. \end{cases}$$

Recalling (3.8), we can compute that

$$LU(x, i, t) \leq \begin{cases} -x^2 + 0.0625y^2 - 2.7916667x^6 & \\ +0.4166667y^6 - 1.775x^{10} + 0.3375y^{10} & \text{if } i = 1, \\ -x^2 + 0.25y^2 - 3.1666667x^6 & \\ +1.1666667y^6 - 2.3x^{10} + 1.05y^{10} & \text{if } i = 2. \end{cases}$$

To verify Assumption 3.2, we let $\beta_1 = 0.2$, $\beta_2 = 0.04$ and $\beta_3 = 0.8$. Noting

$$\begin{aligned} & LU(x, i, t) + \beta_1(2\theta_i|x| + (p_1 + 1)\bar{\theta}_i|x|^{p_1})^2 \\ & + \beta_2|f(x, i, t)|^2 + \beta_3|g(x, i, t)|^2 \\ & \leq -0.8x^2 + 0.45y^2 - H(x) + \kappa H(y), \end{aligned} \tag{5.4}$$

where $H(x) = 2.3916667x^6 + 1.255x^{10}$ and $\kappa = 0.9561$. That is, condition (3.10) is also met. After calculation, Assumption 4.1 becomes $\tau < 0.0146986$. By Theorems 4.2 and 4.3, we can therefore conclude that the controlled system (5.2) with the control function (5.1) is not only exponentially stable in $L^{\bar{p}}$ for any $\bar{p} \geq 2$ but also almost surely provided $\tau < 0.0146986$.

To perform a computer simulation, we set $\tau = 0.01, \tau_0 = 0.05$ and the initial value $x(t) = 2 + sin(t)$ on $t \in [-0.05, 0]$ and $r(0) = 1$. The sample paths of the Markov chain and the solution of the SDDE (5.2) are plotted in Fig. 5.2. The simulation supports our theoretical results clearly.

6. Conclusion

In this paper we have discussed the stabilization of highly nonlinear hybrid SDDEs by the feedback controls based on the discrete-time observations of the states. It should be noted that the results of stabilization of existing nonlinear stochastic systems mainly depend on linear growth conditions, and do not take into account the existence of delay in the system itself. There is hence a need to develop a new theory on the stabilization for the highly nonlinear SDDE models. In this paper we

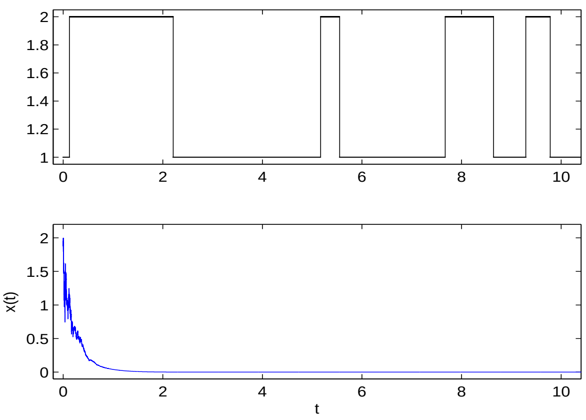

Figure 5.2: The computer simulation of the sample paths of the Markov chain and the SDDE (5.2) with the control function (5.1) and $\tau = 0.01, \tau_0 = 0.05$ using the Euler–Maruyama method with step size 10^{-4}.

consider a class of hybrid SDDEs which are not stable but their solutions are bounded in pth moment. We use a new technique to show that the controlled SDDE can maintain moment boundedness as long as the control function satisfies Lipschitz condition. We then show how to design the control functions more wisely so that the controlled SDDEs become stable. The stability discussed in this paper include the H_∞-stable in $\tilde{p}$, asymptottic stability in $\bar{p}$th moment,almost surely stability, pth moment exponential stability and almost surely exponential stability. The key technique used is this paper is the method of Lyapunov functionals. An examples and two computer simulations have been used to illustrate our theory.

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References

Ahlborn, A. and Parlitz, U.(2004). Stabilizing unstable steady states using multiple delay feedback control, *Phys. Rev. Lett.* **93**, 264101.

Cao, J., Li, H., and Ho, D.W.C.(2005). Synchronization criteria of Lur's systems with time-delay feedback control, *Chaos, Soliton. Fract.* **23**, 1285–1298.

Fei, C., Fei, W., Mao, X., Xia, D. and Yan, L.(2019) Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, *IEEE Trans. Automat. Control*, conditionly accepted.

Fei, C., Shen, M., Fei, W., Mao, X. and Yan, L.(2019). Stability of highly nonlinear hybrid stochastic integro-differential delay equations, *Nonlinear Analysis: Hybrid Systems* **31**,180–199.

Fei, W., Hu, L., Mao, X. and Shen, M.(2017). Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica* **82(8)**, 65–70.

Fei, W., Hu, L., Mao, X. and Shen, M.(2018). Structured robust stability and boundedness of nonlinear hybrid delay systems, *SIAM J. Control Optim.* **56(4)**, 2662–2689.

Hu, L., Mao, X. and Shen, Y.(2013). Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Syst. Control Lett.* **62**, 178–187.

Ladde, G.S. and Lakshmikantham, V.(1980). *Random Differential Inequalities*, Academic Press.

Ji, Y. and Chizeck, H.J.(1990). Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automat. Control* **35(7)**, 777–788.

Kolmanovskii, V.B. and Nosov, V.R.(1986). *Stability of Functional Differential Equations*, Academic Press.

Lewis A.L.(2000). *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press.

Mao, X.(1994). *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker.

Mao, X.(2007). *Stochastic Differential Equations and Their Applications*, 2nd Edition, Chichester: Horwood Pub.

Mao, X.(1999). Stability of stochastic differential equations with Markovian switching, *Sto. Proc. Appl.* **79**, 45–67.

Mao, X.(2002). Exponential stability of stochastic delay interval systems with Markovian switching, *IEEE Trans. Automat. Control* **47(10)**, 1604–1612.

Mao, X.(2013). Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, *Automatica* **49(12)**, 3677–3681.

Mao, X.(2016). Almost sure exponential stabilization by discrete-time stochastic feedback control, *IEEE Trans. Automat. Control* **61(6)**, 1619–1624.

Mao, X., Lam, J. and Huang, L.(2008). Stabilisation of hybrid stochastic differential equations by delay feedback control, *Syst. Control Lett.* **57**, 927–935.

Mao, X., Matasov, A. and Piunovskiy, A.B.(2000). Stochastic differential delay equations with Markovian switching, *Bernoulli* **6(1)**, 73–90.

Mao, X. and Yuan, C.(2006). *Stochastic Differential Equations with Markovian Switching*, Imperial College Press.

Mariton, M.(1990). *Jump Linear Systems in Automatic Control*, Marcel Dekker.

Mohammed, S.-E.A.(1984). *Stochastic Functional Differential Equations*, Longman Scientific and Technical.

Pyragas, K.(1995). Control of chaos via extended delay feedback, *Phys. Lett. A* **206(5-6)**, 323–330.

Qiu, Q., Liu, W., Hu, L., Mao, X. and You, S.(2016). Stabilisation of stochastic differential equations with Markovian switching by feedback control based on discrete-time state observation with a time delay, *Stat. Probabil. Lett.* **115**, 16–26.

Shaikhet, L.(1996). Stability of stochastic hereditary systems with Markov switching, *Theory of Stochastic Processes* **2(18)**, 180–184.

Shao, J.(2017). Stabilization of regime-switching processes by feedback control based on discrete time observations, *SIAM J. Control Optim.* **55(2)** ,724–740.

Shen, M., Fei, W., Mao, X., and Liang, Y.(2018). Stability of highly nonlinear neutral stochastic differential delay equations, *Syst. Control. Lett.* **115**, 1–8.

Shi, P., Mahmoud, M.S., Yi, J. and Ismail, A.(2006). Worst case control of uncertain jumping systems with multi-state and input delay information, *Information Sciences* **176(2)**, 186–200.

Sun, M., Lam, J., Xu, S. and Zou, Y.(2007). Robust exponential stabilization for Markovian jump systems with mode-dependent input delay, *Automatica* **43(10)**, 1799–1807.

Wei, G., Wang, Z., Shu, H. and Fang, J. (2006). Robust H_∞ control of stochastic time-delay jumping systems with nonlinear disturbances, *Optim. Control Appl. Meth.* **27(5)**, 255–271.

You, S., Liu, W., Lu, J., Mao, X. and Qiu, Q.(2015). Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM J. Control Optim.* **53(2)**, 905–925.

Yuan, C. Mao, X. and Lygeros, J.(2009). Stochastic hybrid delay population dynamics: Well-posed models and extinction, *J. Biol. Dynam.* **3(1)**, 1–21.

Yue, D. and Han, Q.(2005). Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, *IEEE Trans. Automat. Control* **50(2)**, 217–222.